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PHILOSOPHICAL
TRANSACTIONS
OF THE
ROYAL SOCIETY
OF
LONDON.

(A.)

FOR THE YEAR MDCCCXCIII.

VOL. 184.

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MDCCCXCIV.

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions,* that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

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upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

1894.

List of Institutions entitled to receive the Philosophical Transactions or Proceedings of the Royal Society.

Institutions marked A are entitled to receive Philosophical Transactions, Series A, and Proceedings.
„ „ B „ „ „ „ Series B, and Proceedings.
„ „ AB „ „ „ „ Series A and B, and Proceedings.
„ „ *p* „ „ Proceedings only.

America (Central).
Mexico.
p. Sociedad Científica "Antonio Alzate."

America (North). (See United States and Canada.)

America (South).
Buenos Ayres.
AB. Museo Nacional.
Caracas.
B. University Library.
Cordova.
AB. Academia Nacional de Ciencias.
Demerara.
p. Royal Agricultural and Commercial Society, British Guiana.
La Plata.
p. Museo de La Plata.
Rio de Janeiro.
p. Observatorio.

Australia.
Adelaide.
p. Royal Society of South Australia.
Brisbane.
p. Royal Society of Queensland.
Melbourne.
p. Observatory.
p. Royal Society of Victoria.
AB. University Library.
Sydney.
p. Geological Survey.
p. Linnean Society of New South Wales.
AB. Royal Society of New South Wales.
AB. University Library.

Austria.
Agram.
p. Jugoslavenska Akademija Znanosti i Umjetnosti.
p. Societas Historico-Naturalis Croatica.
Brünn.
AB. Naturforschender Verein.

Austria (continued).
Gratz.
AB. Naturwissenschaftlicher Verein für Steiermark.
Hermannstadt.
p. Siebenbürgischer Verein für die Naturwissenschaften.
Innsbruck.
AB. Das Ferdinandeum.
p. Naturwissenschaftlich - Medicinischer Verein.
Klausenburg.
AB. Az Erdélyi Muzeum. Das Siebenbürgische Museum.
Prague.
AB. Königliche Böhmische Gesellschaft der Wissenschaften.
Trieste.
B. Museo di Storia Naturale.
p. Società Adriatica di Scienze Naturali.
Vienna.
p. Anthropologische Gesellschaft.
AB. Kaiserliche Akademie der Wissenschaften.
p. K.K. Geographische Gesellschaft.
AB. K.K. Geologische Reichsanstalt.
B. K.K. Zoologisch-Botanische Gesellschaft.
B. K.K. Naturhistorisches Hof-Museum.
p. Œsterreichische Gesellschaft für Meteorologie.
A. Von Kuffner'sche Sternwarte.

Belgium.
Brussels.
B. Académie Royale de Médecine.
AB. Académie Royale des Sciences.
B. Musée Royal d'Histoire Naturelle de Belgique.
p. Observatoire Royal.
p. Société Malacologique de Belgique.

Belgium (continued).
- Ghent.
  - AB. University.
- Liége.
  - AB. Société des Sciences.
  - *p.* Société Géologique de Belgique.
- Louvain.
  - B. Laboratoire de Microscopie et de Biologie Cellulaire.
  - AB. Université.

Canada.
- Hamilton.
  - *p.* Hamilton Association.
- Montreal.
  - AB. McGill University.
  - *p.* Natural History Society.
- Ottawa.
  - AB. Geological Survey of Canada.
  - AB. Royal Society of Canada.
- Toronto.
  - *p.* Astronomical and Physical Society.
  - *p.* Canadian Institute.
  - AB. University.

Cape of Good Hope.
- A. Observatory.
- AB. South African Library.

Ceylon.
- Colombo.
  - B. Museum.

China.
- Shanghai.
  - *p.* China Branch of the Royal Asiatic Society.

Denmark.
- Copenhagen.
  - AB. Kongelige Danske Videnskabernes Selskab.

Egypt.
- Alexandria.
  - AB. Bibliothèque Municipale.

England and Wales.
- Aberystwith.
  - AB. University College.
- Bangor.
  - AB. University College of North Wales.
- Birmingham.
  - AB. Free Central Library.
  - AB. Mason College.
  - *p.* Philosophical Society.
- Bolton.
  - *p.* Public Library.
- Bristol.
  - *p.* Merchant Venturers' School.
- Cambridge.
  - AB. Philosophical Society.
  - *p.* Union Society.

England and Wales (continued).
- Cooper's Hill.
  - AB. Royal Indian Engineering College.
- Dudley.
  - *p.* Dudley and Midland Geological and Scientific Society.
- Essex.
  - *p.* Essex Field Club.
- Greenwich.
  - A. Royal Observatory.
- Kew.
  - B. Royal Gardens.
- Leeds.
  - *p.* Philosophical Society.
  - AB. Yorkshire College.
- Liverpool.
  - AB. Free Public Library.
  - *p.* Literary and Philosophical Society.
  - A. Observatory.
  - AB. University College.
- London.
  - AB. Admiralty.
  - *p.* Anthropological Institute.
  - B. British Museum (Nat. Hist.).
  - AB. Chemical Society.
  - A. City and Guilds of London Institute.
  - *p.* "Electrician," Editor of the.
  - B. Entomological Society.
  - AB. Geological Society.
  - AB. Geological Survey of Great Britain.
  - *p.* Geologists' Association.
  - AB. Guildhall Library.
  - A. Institution of Civil Engineers.
  - *p.* Institution of Electrical Engineers.
  - A. Institution of Mechanical Engineers.
  - A. Institution of Naval Architects.
  - *p.* Iron and Steel Institute.
  - AB. King's College.
  - B. Linnean Society.
  - AB. London Institution.
  - *p.* London Library.
  - A. Mathematical Society.
  - *p.* Meteorological Office.
  - *p.* Odontological Society.
  - *p.* Pharmaceutical Society.
  - *p.* Physical Society.
  - *p.* Quekett Microscopical Club.
  - *p.* Royal Agricultural Society.
  - *p.* Royal Asiatic Society.
  - A. Royal Astronomical Society.
  - B. Royal College of Physicians.
  - B. Royal College of Surgeons.
  - *p.* Royal Engineers (for Libraries abroad, six copies).

England and Wales (continued).

London (continued).
AB. Royal Engineers. Head Quarters Library.
p. Royal Geographical Society.
p. Royal Horticultural Society.
p. Royal Institute of British Architects.
AB. Royal Institution of Great Britain.
B. Royal Medical and Chirurgical Society.
p. Royal Meteorological Society.
p. Royal Microscopical Society.
p. Royal Statistical Society.
AB. Royal United Service Institution.
AB. Society of Arts.
p. Society of Biblical Archæology.
p. Society of Chemical Industry (London Section).
p. Standard Weights and Measures Department.
AB. The Queen's Library.
AB. The War Office.
AB. University College.
p. Victoria Institute.
B. Zoological Society.

Manchester.
AB. Free Library.
AB. Literary and Philosophical Society.
p. Geological Society.
AB. Owens College.

Netley.
p. Royal Victoria Hospital.

Newcastle.
AB. Free Library.
p. North of England Institute of Mining and Mechanical Engineers.
p. Society of Chemical Industry (Newcastle Section).

Norwich.
p. Norfolk and Norwich Literary Institution.

Oxford.
p. Ashmolean Society.
AB. Radcliffe Library.
A. Radcliffe Observatory.

Penzance.
p. Geological Society of Cornwall.

Plymouth.
B. Marine Biological Association.
p. Plymouth Institution.

Richmond.
A. "Kew" Observatory.

Salford.
p. Royal Museum and Library.

Stonyhurst.
p. The College.

England and Wales (continued).

Swansea.
AB. Royal Institution.

Woolwich.
AB. Royal Artillery Library.

Finland.

Helsingfors.
p. Societas pro Fauna et Flora Fennica.
AB. Société des Sciences.

France.

Bordeaux.
p. Académie des Sciences.
p. Faculté des Sciences.
p. Société de Médecine et de Chirurgie.
p. Société des Sciences Physiques et Naturelles.

Cherbourg.
p. Société des Sciences Naturelles.

Dijon.
p. Académie des Sciences.

Lille.
p. Faculté des Sciences.

Lyons.
AB. Académie des Sciences, Belles-Lettres et Arts.
p. Université.

Marseilles.
p. Faculté des Sciences.

Montpellier.
AB. Académie des Sciences et Lettres.
B. Faculté de Médecine.

Paris.
AB. Académie des Sciences de l'Institut.
p. Association Française pour l'Avancement des Sciences.
p. Bureau des Longitudes.
A. Bureau International des Poids et Mesures.
p. Commission des Annales des Ponts et Chaussées.
p. Conservatoire des Arts et Métiers.
p. Cosmos (M. L'ABBÉ VALETTE).
AB. Dépôt de la Marine.
AB. École des Mines.
AB. École Normale Supérieure.
AB. École Polytechnique.
AB. Faculté des Sciences de la Sorbonne.
AB. Jardin des Plantes.
p. L'Électricien.
A. L'Observatoire.
p. Revue Scientifique (Mons. H. DE VARIGNY).
p. Société de Biologie.
AB. Société d'Encouragement pour l'Industrie Nationale.
AB. Société de Géographie.
p. Société de Physique.
B. Société Entomologique.

France (continued).

- Paris (continued).
  - AB. Société Géologique.
  - *p.* Société Mathématique.
  - *p.* Société Météorologique de France.
- Toulouse.
  - AB. Académie des Sciences.
  - A. Faculté des Sciences.

Germany.

- Berlin.
  - A. Deutsche Chemische Gesellschaft.
  - A. Die Sternwarte.
  - *p.* Gesellschaft für Erdkunde.
  - AB. Königliche Preussische Akademie der Wissenschaften.
  - A. Physikalische Gesellschaft.
- Bonn.
  - AB. Universität.
- Bremen.
  - *p.* Naturwissenschaftlicher Verein.
- Breslau.
  - *p.* Schlesische Gesellschaft für Vaterländische Kultur.
- Brunswick.
  - *p.* Verein für Naturwissenschaft.
- Carlsruhe. See Karlsruhe.
- Danzig.
  - AB. Naturforschende Gesellschaft.
- Dresden.
  - *p.* Verein für Erdkunde.
- Emden.
  - *p.* Naturforschende Gesellschaft.
- Erlangen.
  - AB. Physikalisch-Medicinische Societät.
- Frankfurt-am-Main.
  - AB. Senckenbergische Naturforschende Gesellschaft.
  - *p.* Zoologische Gesellschaft.
- Frankfurt-am-Oder.
  - *p.* Naturwissenschaftlicher Verein.
- Freiburg-im-Breisgau.
  - AB. Universität.
- Giessen.
  - AB. Grossherzogliche Universität.
- Görlitz.
  - *p.* Naturforschende Gesellschaft.
- Göttingen.
  - AB. Königliche Gesellschaft der Wissenschaften.
- Halle.
  - AB. Kaiserliche Leopoldino - Carolinische Deutsche Akademie der Naturforscher.

Germany (continued).

- Halle (continued).
  - *p.* Naturwissenschaftlicher Verein für Sachsen und Thüringen.
- Hamburg.
  - *p.* Naturhistorisches Museum.
  - AB. Naturwissenschaftlicher Verein.
- Heidelberg.
  - *p.* Naturhistorisch-Medizinischer Verein.
  - AB. Universität.
- Jena.
  - AB. Medicinisch-Naturwissenschaftliche Gesellschaft.
- Karlsruhe.
  - A. Grossherzogliche Sternwarte.
  - *p.* Technische Hochschule.
- Kiel.
  - *p.* Naturwissenschaftlicher Verein für Schleswig-Holstein.
  - A. Sternwarte.
  - AB. Universität.
- Königsberg.
  - AB. Königliche Physikalisch - Ökonomische Gesellschaft.
- Leipsic.
  - *p.* Annalen der Physik und Chemie.
  - A. Astronomische Gesellschaft.
  - AB. Königliche Sächsische Gesellschaft der Wissenschaften.
- Magdeburg.
  - *p.* Naturwissenschaftlicher Verein.
- Marburg.
  - AB. Universität.
- Munich.
  - AB. Königliche Bayerische Akademie der Wissenschaften.
  - *p.* Zeitschrift für Biologie.
- Münster.
  - AB. Königliche Theologische und Philosophische Akademie.
- Potsdam.
  - A. Astrophysikalisches Observatorium.
- Rostock.
  - AB. Universität.
- Strasburg.
  - AB. Universität.
- Tübingen.
  - AB. Universität.
- Würzburg.
  - AB. Physikalisch-Medicinische Gesellschaft.

Greece.

- Athens.
  - A. National Observatory.

Holland. (See NETHERLANDS.)

Hungary.
- Pesth.
  - *p.* Königl. Ungarische Geologische Anstalt.
  - AB. Á Magyar Tudós Társaság. Die Ungarische Akademie der Wissenschaften.
- Schemnitz.
  - *p.* K. Ungarische Berg- und Forst-Akademie.

India.
- Bombay.
  - AB. Elphinstone College.
  - *p.* Royal Asiatic Society (Bombay Branch).
- Calcutta.
  - AB. Asiatic Society of Bengal.
  - AB. Geological Museum.
  - *p.* Great Trigonometrical Survey of India.
  - AB. Indian Museum.
  - *p.* The Meteorological Reporter to the Government of India.
- Madras.
  - B. Central Museum.
  - A. Observatory.
- Roorkee.
  - *p.* Roorkee College.

Ireland.
- Armagh.
  - A. Observatory.
- Belfast.
  - AB. Queen's College.
- Cork.
  - *p.* Philosophical Society.
  - AB. Queen's College.
- Dublin.
  - A. Observatory.
  - AB. National Library of Ireland.
  - B. Royal College of Surgeons in Ireland.
  - AB. Royal Dublin Society.
  - AB. Royal Irish Academy.
- Galway.
  - AB. Queen's College.

Italy.
- Bologna.
  - AB. Accademia delle Scienze dell' Istituto.
- Florence.
  - *p.* Biblioteca Nazionale Centrale.
  - AB. Museo Botanico.
  - *p.* Reale Istituto di Studi Superiori.
- Genoa.
  - *p.* Società Ligustica di Scienze Naturali e Geografiche.
- Milan.
  - AB. Reale Istituto Lombardo di Scienze, Lettere ed Arti.
  - AB. Società Italiana di Scienze Naturali.

Italy (continued).
- Naples.
  - *p.* Società di Naturalisti.
  - AB. Società Reale, Accademia delle Scienze.
  - B. Stazione Zoologica (Dr. DOHRN).
- Padua.
  - *p.* University.
- Pisa.
  - *p.* Società Toscana di Scienze Naturali.
- Rome.
  - *p.* Accademia Pontificia de' Nuovi Lincei.
  - *p.* Rassegna delle Scienze Geologiche in Italia.
  - A. Reale Ufficio Centrale di Meteorologia e di Geodinamica, Collegio Romano.
  - AB. Reale Accademia dei Lincei.
  - *p.* R. Comitato Geologico d' Italia.
  - A. Specula Vaticana.
  - AB. Società Italiana delle Scienze.
- Siena.
  - *p.* Reale Accademia dei Fisiocritici.
- Turin.
  - *p.* Laboratorio di Fisiologia.
  - AB. Reale Accademia delle Scienze.
- Venice.
  - *p.* Ateneo Veneto.
  - AB. Reale Istituto Veneto di Scienze, Lettere ed Arti.

Japan.
- Tokiô.
  - AB. Imperial University.
  - *p.* Asiatic Society of Japan.

Java.
- Buitenzorg.
  - *p.* Jardin Botanique.

Luxembourg.
- Luxembourg.
  - *p.* Société des Sciences Naturelles.

Malta.
- *p.* Public Library.

Mauritius.
- *p.* Royal Society of Arts and Sciences.

Netherlands.
- Amsterdam.
  - AB. Koninklijke Akademie van Wetenschappen.
  - *p.* K. Zoologisch Genootschap 'Natura Artis Magistra.'
- Delft.
  - *p.* École Polytechnique.
- Haarlem.
  - AB. Hollandsche Maatschappij der Wetenschappen.
  - *p.* Musée Teyler.
- Leyden.
  - AB. University.

Netherlands (continued).
Rotterdam.
AB. Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte.
Utrecht.
AB. Provinciaal Genootschap van Kunsten en Wetenschappen.

New Zealand.
Wellington.
AB. New Zealand Institute.

Norway.
Bergen.
AB. Bergenske Museum.
Christiania.
AB. Kongelige Norske Frederiks Universitet.
Tromsoe.
p. Museum.
Trondhjem.
AB. Kongelige Norske Videnskabers Selskab.

Nova Scotia.
Halifax.
p. Nova Scotian Institute of Science.
Windsor.
p. King's College Library.

Portugal.
Coimbra.
AB. Universidade.
Lisbon.
AB. Academia Real das Sciencias.
p. Secção dos Trabalhos Geologicos de Portugal.

Russia.
Dorpat.
AB. Université.
Irkutsk.
p. Société Impériale Russe de Géographie (Section de la Sibérie Orientale).
Kazan.
AB. Imperatorsky Kazansky Universitet.
Kharkoff.
p. Section Médicale de la Société des Sciences Expérimentales, Université de Kharkow.
Kieff.
p. Société des Naturalistes.
Moscow.
AB. Le Musée Public.
B. Société Impériale des Naturalistes.
Odessa.
p. Société des Naturalistes de la Nouvelle-Russie.
Pulkowa.
A. Nikolai Haupt-Sternwarte.
St. Petersburg.
AB. Académie Impériale des Sciences.

Russia (continued).
St. Petersburg (continued).
B. Archives des Sciences Biologiques.
AB. Comité Géologique.
p. Compass Observatory.
A. Observatoire Physique Central.

Scotland.
Aberdeen.
AB. University.
Edinburgh.
p. Geological Society.
p. Royal College of Physicians (Research Laboratory).
p. Royal Medical Society.
A. Royal Observatory.
p. Royal Physical Society.
p. Royal Scottish Society of Arts.
AB. Royal Society.
Glasgow.
AB. Mitchell Free Library.
p. Philosophical Society.

Servia.
Belgrade.
p. Académie Royale de Serbie.

Sicily.
Acireale.
p. Società Italiana dei Microscopisti.
Catania.
AB. Accademia Gioenia di Scienze Naturali.
Palermo.
A. Circolo Matematico.
AB. Consiglio di Perfezionamento (Società di Scienze Naturali ed Economiche).
A. Reale Osservatorio.

Spain.
Cadiz.
A. Instituto y Observatorio de Marina de San Fernando.
Madrid.
p. Comisión del Mapa Geológico de Espãna.
AB. Real Academia de Ciencias.

Sweden.
Gottenburg.
AB. Kongl. Vetenskaps och Vitterhets Samhälle.
Lund.
AB. Universitet.
Stockholm.
A. Acta Mathematica.
AB. Kongliga Svenska Vetenskaps-Akademie.
AB. Sveriges Geologiska Undersökning.
Upsala.
AB. Universitet.

Switzerland.
Basel.
- *p.* Naturforschende Gesellschaft.

Bern.
- AB. Allg. Schweizerische Gesellschaft.
- *p.* Naturforschende Gesellschaft.

Geneva.
- AB. Société de Physique et d'Histoire Naturelle.
- AB. Institut National Genevois.

Lausanne.
- *p.* Société Vaudoise des Sciences Naturelles.

Neuchâtel.
- *p.* Société des Sciences Naturelles.

Zürich.
- AB. Das Schweizerische Polytechnikum.
- *p.* Naturforschende Gesellschaft.
- *p.* Sternwarte.

Tasmania.
Hobart.
- *p.* Royal Society of Tasmania.

United States.
Albany.
- AB. New York State Library.

Annapolis.
- AB. Naval Academy.

Baltimore.
- AB. Johns Hopkins University.

Berkeley.
- *p.* University of California.

Boston.
- AB. American Academy of Sciences.
- B. Boston Society of Natural History.
- A. Technological Institute.

Brooklyn.
- AB. Brooklyn Library.

Cambridge.
- AB. Harvard University.
- B. Museum of Comparative Zoology.

Chapel Hill (N.C.).
- *p.* Elisha Mitchell Scientific Society.

Charleston.
- *p.* Elliott Society of Science and Art of South Carolina.

Chicago.
- AB. Academy of Sciences.
- *p.* Journal of Comparative Neurology.

United States (continued).
Davenport (Iowa).
- *p.* Academy of Natural Sciences.

Madison.
- *p.* Wisconsin Academy of Sciences.

Mount Hamilton (California).
- A. Lick Observatory.

New Haven (Conn.).
- AB. American Journal of Science.
- AB. Connecticut Academy of Arts and Sciences.

New York.
- *p.* American Geographical Society.
- *p.* American Museum of Natural History.
- *p.* New York Academy of Sciences.
- *p.* New York Medical Journal.
- *p.* School of Mines, Columbia College.

Philadelphia.
- AB. Academy of Natural Sciences.
- AB. American Philosophical Society.
- *p.* Franklin Institute.
- *p.* Wagner Free Institute of Science.

Rochester (N.Y.).
- *p.* Academy of Science.

St. Louis.
- *p.* Academy of Science.

Salem (Mass.).
- *p.* American Association for the Advancement of Science.
- AB. Essex Institute.

San Francisco.
- AB. California Academy of Sciences.

Washington.
- AB. Patent Office.
- AB. Smithsonian Institution.
- AB. United States Coast Survey.
- *p.* United States Commission of Fish and Fisheries.
- AB. United States Geological Survey.
- AB. United States Naval Observatory.
- *p.* United States Department of Agriculture.
- A. United States Department of Agriculture (Weather Bureau).

West Point (N.Y.)
- AB. United States Military Academy.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1893, by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to Sir GEORGE GABRIEL STOKES, Bart., F.R.S., for his Researches and Discoveries in Physical Science.

A ROYAL MEDAL to ARTHUR SCHUSTER, F.R.S., for his Spectroscopic Inquiries, and his Researches on Disruptive Discharge through Gases, and on Terrestrial Magnetism.

A ROYAL MEDAL to HARRY MARSHALL WARD, F.R.S., for his Researches into the Life History of Fungi and Schizomycetes.

The DAVY MEDAL to J. H. VAN 'T HOFF and J. A. LE BEL, in recognition of their Introduction of the Theory of Asymmetric Carbon, and its Use in Explaining the Constitution of Optically Active Carbon Compounds.

The Bakerian Lecture, "The Rate of Explosion in Gases," was delivered by Professor HAROLD B. DIXON, F.R.S.

The Croonian Lecture, "The Position of Pathology among Biological Studies," was delivered by Professor RUDOLF VIRCHOW, For.Mem.R.S.

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PHILOSOPHICAL TRANSACTIONS.

I. *Transmission of Sunlight through the Earth's Atmosphere.*

By CAPTAIN W. DE W. ABNEY, *C.B.*, *D.C.L.*, *F.R.S.*

Received April 7,—Read May 5, 1892.

PART II.—LOSS OF LIGHT AT DIFFERENT ALTITUDES.

Introductory.

XXII.—In a previous communication ('Phil. Trans.,' vol. 178, 1887, A., pp. 251–283) the results of the absorption of sunlight by different thicknesses of atmosphere near sea level were found from measurements of the luminosity of the entire spectrum. The method used was that devised by General FESTING and the Author, as detailed in the Bakerian Lecture in 1886. The absorption coefficient for the different thicknesses of atmosphere at sea level was found by measurements made principally at South Kensington, and these were compared with measures taken at the Riffel, above Zermatt, at a height of about 8000 feet. It was shown that when the air thickness is represented by x, the minimum intensity for each ray of the spectrum can be fairly represented by $I' = I\epsilon^{-\cdot 013x\lambda^{-4}}$, and the average intensity by $I' = I\epsilon^{-\cdot 017x\lambda^{-4}}$, I' and I being the transmitted and original intensities, and λ the wave length.

Further, it was shown that even if the absorption of each ray was very different from the above, the integral absorption was very accurately expressed by a^{-z}, z being the air thickness, and a a constant. This corresponds to $a^{-\sec\theta}$ corrected for refraction, when θ is the zenith distance.

This result was arrived at by taking the areas of the curves of luminosity as formed from the actual observations as a measure of the luminosity of the total white light which was decomposed into a spectrum. This was admissible, for in the paper already referred to, it was proved that the sum of luminosities of different rays is equal to their luminosity when compounded.

XXIII.—*Objections to the Use of the Formula.*

The formula involving λ^{-4} is the formula theoretically deduced by Lord RAYLEIGH for the scattering of light by small particles, and its adoption in this research is open to criticism—one being, that if it were absolutely applicable, the light of the sky should probably exhibit greater polarisation in a direction perpendicular to the sun's rays than it does. There is reason for believing, however, that the higher the station at which such observations are made the more complete is the polarisation. In any case, before this can be considered a valid objection, we have to know more than we do at present regarding the condition of the light reflected back from the earth and from the particles themselves.

It must also be admitted that grosser particles exist on some days, and that for these the loss of light must be in the form of $I' = I\epsilon^{-\mu x}$. They would prevent a certain portion of the total light from reaching the observer, whilst the smaller one would selectively reflect, and hence the law would not hold absolutely good, but the small number of gross particles, compared with the fine ones, would not appreciably alter the formula used except the coefficient κ.

Another objection which has been advanced by Mr. S. P. LANGLEY in a private letter against the adoption of the formula is, that if the spectrum were observed with a large dispersion, it will be seen that as the altitude of the sun diminishes the atmospheric lines increase in intensity, and that these must obey the laws of ordinary absorption. This is an objection which at first sight may seem fatal, not to the correctness of the observations, but to the adoption of the law above quoted; but it must be remembered that these special absorptions occupy a very limited area compared with the rest of the spectrum, and that they would practically disappear when the whole loss of light is under consideration, more especially as they would themselves obey the ordinary law of absorption. At any rate, the formula adopted appears to suit the case, and it must be borne in mind, that the results obtained by the integration of the spectrum luminosities bear out the formula which has been universally adopted by astronomers as representing the corrections to be made in star magnitudes when the stars are observed at different altitudes above the horizon.

XXIV.—*The Integration of the Spectrum Luminosities at different Solar Altitudes equivalent to the Luminosity of Monochromatic Light at the same Altitudes.*

In Section XVIII. of the paper of which this is a continuation it was shown that the areas of the curves obtained from the formula $I' = I\epsilon^{-\kappa x\lambda^{-4}}$ being capable of being represented by $I' = I\epsilon^{-\mu x}$ an important deduction could be made.

For the area of the curve is $\epsilon^{\kappa x}(a\epsilon^{\lambda_{,}^{-4}} + b\epsilon^{-\lambda_{,,}^{-4}} + c\epsilon^{\lambda_{,,,}^{-4}} + \&c.)$. *a*, *b*, *c*, &c., being the original luminosities of the different rays, must then be represented by $I = I\epsilon^{-\kappa x\Lambda^{-4}}$ where Λ is some one ray, that is $\mu = \kappa\Lambda^{-4}$ or $\frac{\mu}{\kappa} = \Lambda^{-4}$; $\frac{\mu}{\kappa}$ is a constant and was found from the observations to be 105 (on the scale used for Λ^{-4}), which is equivalent to λ 5770, or a ray near the place of maximum luminosity. Hence the visual observations of total sunlight at different zenith distances are equivalent to observing the alteration in intensity of one single ray of that wave length in its spectrum,* and, therefore, from such observations κ could be at once determined. It was similarly shown that when a photographic silver salt was employed to register the photographic spectrum it was equivalent to measuring the integrated spectrum with a coefficient of absorption μ', and that this was equivalent to observing the alteration in intensity of a ray λ 4540.

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* This, of course, is only true within certain limits, but it holds for any thickness of atmosphere through which observations were taken.

XXV.—*Application to the foregoing Results.*

This being the case, if by any photographic means we can measure the total intensity of white light affecting a photographic compound after passing through various air thicknesses, we ought to be able to find the value of μ_1 in the formula $\epsilon^{-\mu_1 s}$, and if we also observe optically the value of light transmitted through various air thicknesses we shall get the value of μ in the formula $\epsilon^{-\mu s}$. As already stated we know that $\mu = 105\kappa$. Similarly, $\mu' = \Lambda_{,}^{-4}.\kappa$, $\Lambda_{,}$ being the equivalent ray observed.

Therefore

$$\frac{\mu'}{\mu} = \frac{\Lambda_{,}^{-4}}{105} \quad \text{or} \quad \Lambda_{,}^{-4} = \frac{105\mu'}{\mu} \quad \text{and} \quad \kappa = \frac{\mu'}{\Lambda_{,}^{-4}},$$

that is, we can find the single ray which is the equivalent of the whole of the spectrum which is impressed on any photographic compound, and having thus found $\Lambda_{,}^{-4}$ we can at once deduce the value of κ—or the coefficient of scattering by the fine particles.

XXVI.—*Sensitive Compound employed.*

The question which presented itself was as to the best form of sensitive salt to use for convenience and accuracy. A process with a silver salt requiring development was almost impracticable. The exposure necessary to give to sunlight would have been so small that accuracy in timing it would have become a very difficult problem. The chloride of silver paper which darkens by sunlight was experimented with, but it had a very serious drawback. **Roscoe** and **Bunsen** have shown that chloride of silver paper may be prepared, which, when exposed to the same intensity of light for the same time, will always give the same blackness, but it is impossible to keep this paper for more than a few hours, and it would often be inconvenient to prepare it. Further it can never be predicted when a day will be suitable for making observations, and measurements of its blackness would be difficult except at a fixed observing station. There were also two other desiderata which had to be taken into account; one was that the record of the action of light should be as permanent as possible, and the other that it should be *easy to measure* the action produced by the light. Chloride of silver paper fulfilled neither of these desiderata, since fixing the chloride altered all measurement and the darkening was liable to fade even when the paper was fixed, and it was not easy to obtain accurate measures of the darkening owing to its ruddy colour.

In the comparatively new process of platinotype we have, however, a process which is in every way suitable for the purpose. It may be well to call to mind the principle on which it is based. A solution of ferric oxalate is mixed with a proportion of chloroplatinite of potassium and spread over a sized paper and dried. When light acts on this mixture it reduces the iron salt to the ferrous state proportionately to the intensity of the light and to the time during which it acts. When such an exposed paper is placed on a hot solution of potassium oxalate (neutral) the salt reduces the platinum salt, and metallic platinum (platinum black) is deposited. The black or grey produced by this means is most suitable for measuring in the manner which will subsequently be described.

The other desideratum is also found in this process. The prints are absolutely permanent, and the records obtained by it can be referred to at any time, and re-measured if necessary. Further, the paper keeps well before exposure, and after exposure and before development. When kept dry in an air-tight box with calcium chloride in contact with the air, in fact, it will keep unaltered in sensitiveness for several months. As an example of this it may be mentioned that two pieces of paper cut from the same sheet were each exposed to various and similar intensities of light for equal times. One was developed on the potassium oxalate solution immediately after exposure, and the other was not developed till about six weeks had elapsed; on measuring the greys on the two papers their darkness was found to be precisely the same, and the unexposed parts equally white in the two cases.

It was, for these reasons, determined to use the platinum process for the experiments detailed.

XXVII.—*Instrument used in the Observations.*

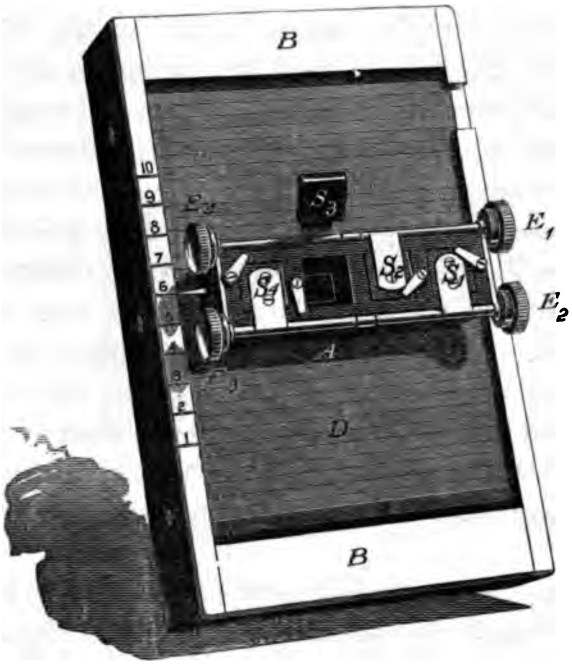
The instrument which was designed for use with this sensitive paper was of the simplest form ; one of the conditions which it had to fulfil was whilst exposing the paper it should allow but a small amount of light from the sky to reach it. The latter light is anything but negligible, for absolute measures, in some cases, show that if the photographic intensity of sunlight be called 100, that of the sky might be as much as 50.

To admit only a practical minimum of sky-light, and such as would be negligible in proportion to the sunlight, the instrument was constructed so that only about 10° of skylight fell upon the paper with the sunlight. The diagram, fig. 1, will give an idea of the instrument. B is a frame, in the back and front of the sides of which are cut deep continuous grooves. A wooden shutter, D, formed of narrow laths, glued on to a leather back, runs in these grooves also at the back of the box. This flexible shutter carries an oblong block A, pierced with four square apertures, as shown, which are closed by covers, S_1, S_2, S_3, S_4. The covers can be opened by turning the mill-headed buttons, E_1, E_2, E_3, E_4, which are connected by long pins with the covers.

In order to confine the photographically active spectrum to smaller limits than would be given if the light were unchecked, two thicknesses of cobalt-blue glass were placed in slots running through the length of A. The light had therefore to pass through these two glasses before it reached the paper beneath them.

Experiment has shown that the aperture can be completely uncovered in less than $\frac{1}{10}$th of a second, and closed in the same time; hence no appreciable error in exposure arises from this cause when the exposure lasts several seconds. The paper is placed on a grid, running the whole length of the box behind the flexible shutter, the bars falling between the aperture and different portions of the paper are exposed by moving the block A into different positions. These positions are indicated by a scale engraved on one side of the frame, a mark engraved on the shutter being moved successively to each division of the scale. As there are ten divisions of the scale, it will be seen that forty different exposures can be obtained on the same paper, ten through each aperture. The frame is closed by a wooden back, lined with velvet, which is pressed against the paper by means of a couple of brass springs working against buttons fixed to the frame.

Fig. 1.



XXVIII.—*Comparison of Results.*

In order to get a scale of blackness, by which to measure the intensity of light acting, it became necessary to have some means of exposing similar paper to light of different known intensities. From the blackness so produced a curve of blackness could be constructed, and the blackness produced by the sun with different exposures could be read off. Now, increase in time of exposure is equivalent to increase in intensity of light acting, at all events, when the exposures are as prolonged as those given, that is, if the unit time of exposure be doubled, the same result will be obtained if the intensity of light acting be doubled. This is not a mere theoretical assumption, but one which the writer has amply proved with very numerous experiments. To obtain such a scale, what is known as SPURGE'S sensitometer was employed. This is an instrument which was described in detail before the Photographic Society of Great Britain by the inventor. It is only needful to say that it consists of thirty small chambers, 2 centims. in height and 1 centim. square section, placed in six rows, of five chambers each. A brass plate covers one end of these chambers, and in it holes are pierced, of such dimensions that the area of each hole is exactly $\sqrt[3]{2}$ greater than that of the next to it in one row. One of these holes is pierced in the plate exactly over the centre of each chamber. Thus we have a series of graduated intensities of 1, $2^{\frac{1}{3}}$, $2^{\frac{2}{3}}$, 2, $2^{\frac{4}{3}}$, $2^{\frac{5}{3}}$, 4, &c., falling on the bottom of these chambers when the light falling on them comes from a large and equally illuminated surface.

Paper was exposed in this instrument to the light reflected from a large flat card, uniformly illuminated by sun and sky light, and after development the blackness was measured. The paper used was cut from sheets specially prepared by the kindness of Mr. W. WILLIS, of the Platinotype Company, and the sensitiveness of various parts was tried, and if found to be constant, the rest of the sheet was taken to be comparable with those tried. The care with which this paper is prepared is shown by the fact that with several varieties of paper, procured at intervals of several months, the sensitiveness showed no variation in gradation of tints. Latterly the formula seems to have been a little changed, and the sensitometer curve has slightly varied from that found at first; but this is unimportant, as the results are so reduced as to be comparable.

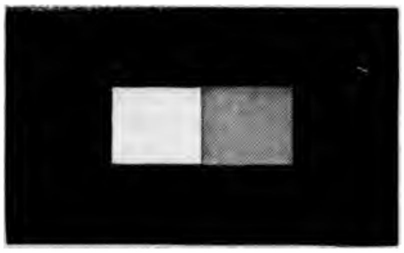
The term "developing" when applied to this paper may be a little misleading. It must not be confounded with the development of plates containing silver salts. In the case of this sensitive paper the amount of platinum deposited is exactly dependent on the amount of the ferric oxalate reduced by light, and if it be exposed to the developing solution after the ferrous salt has reduced the platinum salt in contact with it, no more platinum will be deposited. Hence the deposit of platinum is an exact equivalent of the iron salt reduced, and is a measure of the intensity of light multiplied by the time of exposure. With a silver salt the deposit of silver is increased according to the time during which the developer is in contact with it, and there would, therefore, be a necessity of exposing and developing on the same plate a scale of density together with that produced by the light to be measured.

It is evident that with the platinum process one scale will suffice for a batch of paper which is prepared at one time.

To measure the blackness of the paper a modified RUMFORD method of photometry was employed. A source of light illuminated a small square of white paper (the same as that used in the platinum process to hold the sensitive salt), and one of the blackened squares placed alongside it as in fig. 2.

A rod was so placed that its shadow fell on the white square. A mirror was placed at a little distance to one side of the source of light and the reflected beam was made to illuminate the shadow falling on the white square, and itself to cast a shadow on the dark square. Thus each square was illuminated by the same light, but coming from different points, and the two would be caused to be of equal darkness by placing rotating sectors opening and closing at pleasure in front of one or other of the beams. The amount of light cut off gave a measure of the darkness. Thus, in one case, when both squares were white the sectors had a total aperture of 80° to make them equally luminous; when a grey platinum square was substituted for one of them, the aperture of the sectors was 45°. The grey surface, therefore, only reflected $\frac{45}{80}$ of the light that the white surface reflected. The light to these squares was admitted through an aperture in the front of a closed box, the illumination being judged through an aperture fixed at one corner. The whole apparatus was placed in a darkened and blackened room in which the sole light was that used for the measures.

Fig. 2.

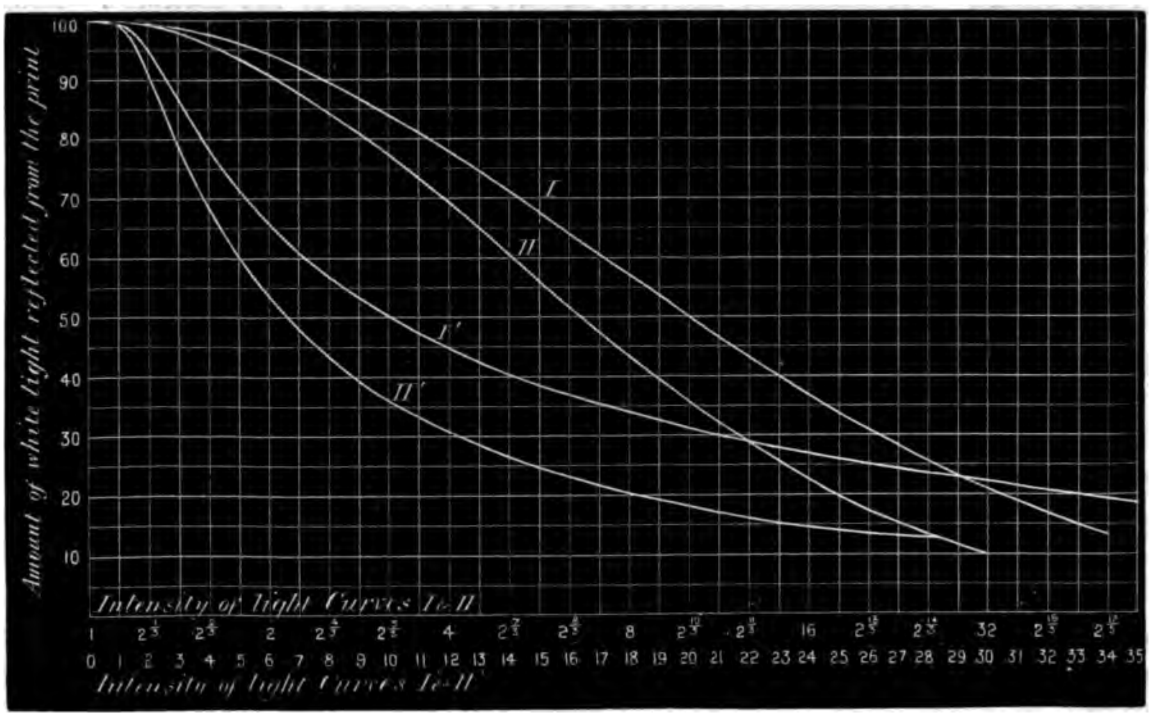


The two following scales of blackness, or, perhaps more accurately, of whiteness, were obtained by this plan, one or other of which will apply to all the measurements given in the paper:—

Comparative intensities of light.	Curve I.	Curve II.
	Amount of white reflected.	Amount of white reflected.
1	100	100
$2^{\frac{1}{3}}$	99·3	99
$2^{\frac{2}{3}}$	97·2	96
2	93·9	91
$2^{\frac{4}{3}}$	89·5	84·5
$2^{\frac{5}{3}}$	84·6	77
4	78	69
$2^{\frac{7}{3}}$	71	60·5
$2^{\frac{8}{3}}$	64	51·7
8	57	43·2
$2^{\frac{10}{3}}$	50	35·5
$2^{\frac{11}{3}}$	43	28·5
16	37	22·7
$2^{\frac{13}{3}}$	31	17·5
$2^{\frac{14}{3}}$	25·5	13·3
32	21	9·8
$2^{\frac{16}{3}}$	17	
$2^{\frac{17}{3}}$	13·5	
64	10·5	

In order to obtain from these measures a scale of the darkening obtained by arithmetical progression in the intensity of light, we have only to recollect that each hole admits $\sqrt[3]{2}$ times the light that the next smaller admits. By easy calculation this scale can be obtained, and is shown in the diagram.

Fig. 3.

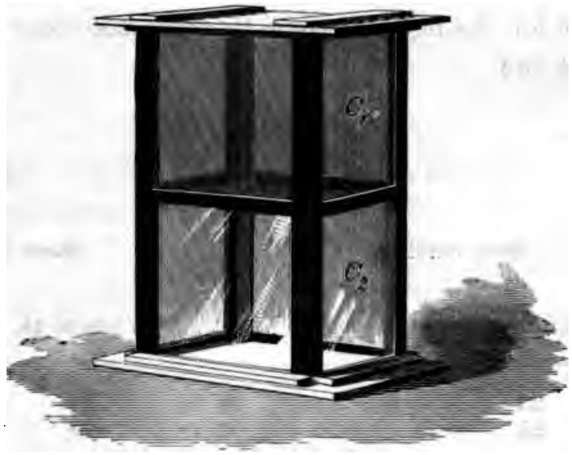


I have shown elsewhere that the curves of blackness on platinum paper can be represented by the formula, $A' = A\epsilon^{-\mu x^2}$ where A' is the amount of reflected white light, and A the amount reflected from the white paper, μ being a coefficient, and x any power of 2. In this case, for curve I., $\mu = {\cdot}00302$, and for curve II., $\mu = {\cdot}0103$.

XXIX.—*Method of finding κ.*

In the paper on Colour Photometry which appeared in the 'Phil. Trans.,' General **Festing** and myself proved that a turbid medium prepared by dropping a solution of mastic dissolved in alcohol into water, obeyed Lord **Rayleigh**'s law, as given above. If, therefore, one part of a piece of platinum paper were exposed for a certain time to sunlight after passing through a cell containing pure water, and a simultaneous exposure made on another portion of the paper with the same light after passing through turbid water prepared as above, the measures of the blackened paper would give the value of μ', the photographic coefficient of absorption in the formula $I' = I\epsilon^{-\mu'\gamma}$, where γ is the thickness of the turbid medium in any unit we please. Further, if the optical values of the light transmitted were compared when passing through the same media, μ could be found and the value of γ be reduced to atmospheric thicknesses. Now, as the ray is known which, if observed, would give the same alteration in luminosity as that of the whole spectrum, it follows that with these data the equivalent ray for the photographic light can be deduced as shown in § XXV.

Fig. 4.



To ensure accuracy several sets of double cells, C_1 C_2, were prepared, as shown in the figure (fig. 4). The top cell, C_1, contained turbid water, the turbidity being caused by suspended mastic. The mastic was very cautiously precipitated and was thus held in suspension. As a matter of fact turbid water of different turbidities was prepared nearly two years before it was employed; and thus any coarse particles had ample time to settle; for the vessels containing it were kept undisturbed during that period and the liquid was syphoned off as required. The bottom cell, C_2, contained pure water. Behind the cell the exposing box (fig. 1), already described, was placed, and in front of C_2 sectors which rotated by means of clock work. The sectors could be clamped at any desired aperture. The apparatus, as shown, was then placed on a stand at such an angle that the sun's rays fell directly on the platinum paper inside the box B, when the exposing apertures were open. The sectors were closed to the extent which it was judged should suffice to make the blackness of the two exposed squares of platinum paper approximately the same, though it will be seen by the experiments to be subsequently described that this was not always attained. Exposures for different lengths of time were given in each experiment, and consequently any small error in measurement, or from any other cause, became insignificant when the mean was taken.

XXX.—*Examples of the Method.*

The following are examples showing the accord between the different values found for Λ_1^{-4}.

Experiment I.—Sunlight—length of cell (inside measures), 3·15 inches; breadth of cell, 1·7 inches. When light passed through length of cell to the sensitive paper a double sector with an opening of 12° rotated in front of the clear cell. When light passed through the breadth of the cell, the sector opening was 36°·13.

For the cells placed lengthways, the value obtained for the light after passing through the clear cell had to be multiplied 180/12 = 15. When the cells were placed breadthways, the value for the light passing through the clear cell had to be multiplied by 180/36·13 = 4·982.

Cell.	1st exposure.			2nd exposure.		
	Mean reading.		Value from scale.	Mean reading.		Value from scale.
	White = 45.	White = 100.		White = 45.	White = 100.	
3·15 in., clear water	23	51	98	33	73·5	47
3·15 in., turbid „	25	55·5	84	35	78	40

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{98}{84} = 1{\cdot}166.$$

$$\text{2nd} \quad ,, \qquad ,, \qquad ,, \qquad = \frac{47}{40} = 1{\cdot}175.$$

$$\text{Mean} = \frac{1{\cdot}166 + 1{\cdot}175}{2} = 1{\cdot}17.$$

The photographic light passing through the clear water = 1·17 × 15 = 17·55 times that passing through the turbid water.

Using the formula $I' = I\epsilon^{-\mu_1\gamma}$

$$\log 17{\cdot}55 = 2{\cdot}86504 = 3{\cdot}15\mu_1$$
$$\mu_1 = {\cdot}911 \text{ for each inch of turbid water.}$$

Cell.	1st exposure.			2nd exposure.			3rd exposure.		
	Mean reading.		Value.	Mean reading.		Value.	Mean reading.		Value.
	White =45.	White = 100.		White = 45.	White = 100.		White = 45.	White = 100.	
1·7 in., clear water	37	72·5	147	32	63	172	26·5	52	204
1·7 in., turbid „	36	70·5	152	30·5	60	181	25	49	212

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{147}{152} = \frac{1}{1{\cdot}037}.$$

$$\text{2nd} \quad ,, \quad ,, \quad ,, \quad = \frac{172}{181} = \frac{1}{1{\cdot}052}.$$

$$\text{3rd} \quad ,, \quad ,, \quad ,, \quad = \frac{204}{212} = \frac{1}{1{\cdot}055}.$$

$$\text{Mean} = \frac{1}{1{\cdot}05}.$$

The photographic light passing through the clear water $= \frac{4{\cdot}982}{1{\cdot}05} = 4{\cdot}70$ times that passing through the turbid water.

Using the formula $I' = I\epsilon^{-\mu_1\gamma}$

$$\log 4{\cdot}7 = 1{\cdot}54756 = 1{\cdot}7\mu_1$$

$$\mu_1 = {\cdot}910 \text{ for each inch of turbid water.}$$

The optical measures of the light passing through the clear and turbid cells in the two directions were

3·15 inches clear water, 48·5 ; turbid water, 18·2.

1·7 ,, ,, 48·5 ; ,, 28·5.

Taking the measurement of the light passing through the 3·15-inch cell as lying on the curve in the formula $I' = I\epsilon^{-\mu\gamma}$,

$$\log 48{\cdot}5 - \log 18{\cdot}2 = \mu \times 3{\cdot}15,$$

$$\mu = 311.$$

If we take the measure of the 1·7-inch water, we have

$$\log 48{\cdot}5 - \log 28{\cdot}5 = \mu \times 1{\cdot}7,$$

$$\mu = {\cdot}313.$$

$$\text{Mean value} = {\cdot}312.$$

Now mean value of $\mu_1 = {\cdot}9105$. Now

$$\frac{\Lambda_1^{-4}}{\mu_1} = \frac{\Lambda^{-4}}{\mu} = \frac{105}{\mu},$$

therefore

$$\Lambda_1^{-4} = \frac{{\cdot}9015}{{\cdot}312} \times 105 = 306{\cdot}6.$$

Experiment II.—Sunlight, June, 1890, midday. Length of cell (inside measures), 3 inches; breadth, 2·1 inches.

When light passed through length of cell to the sensitive paper a double sector, with an opening 36°, rotated in front of clear cell.

When light passed through the breadth, the sector opening was 67°.

Thus, with cells placed lengthways, the value obtained had to be multiplied by $180°/36° = 5$. When the cells were placed breadthways, the light passing through the clear cell had to be multiplied by $180/67 = 2·686$.

Cells.	1st exposure.			2nd exposure.			3rd exposure.		
	Mean reading.		Value.	Mean reading.		Value.	Mean reading.		Value.
	White = 78.	White = 100.		White = 78.	White = 100.		White = 78.	White = 100.	
3 in., clear . .	70	89·5	21	68	87	23·5	49	63	46
3 in., turbid . .	55	70·5	39	51	65·5	44	32	41	85

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{21}{39} = ·538.$$

$$\text{2nd} \quad ,, \quad ,, \quad ,, \quad = \frac{23·5}{44} = ·534.$$

$$\text{3rd} \quad ,, \quad ,, \quad ,, \quad = \frac{46}{85} = ·541.$$

$$\text{Mean} = ·538.$$

Photographic light passing through the clear water $= ·538 \times 5 = 2·69$ times that passing through the turbid cell.

Using the formula as before,

$$\mu_1 = \frac{\log 2·69}{3} = \frac{·98954}{3} = ·330.$$

Cell.	1st exposure.			2nd exposure.			3rd exposure.		
	Mean reading.		Value.	Mean reading.		Value.	Mean reading.		Value.
	White = 78.	White = 100.		White = 78.	White = 100.		White = 78.	White = 100.	
2·1 in., clear .	57	73	36	41	52·5	62	27	34·5	106
2·1 in., turbid .	48	61·5	48	32	41	85	20	26	140

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{36}{48} = \cdot750.$$

$$\text{2nd} \quad ,, \quad ,, \quad ,, \quad = \frac{62}{85} = \cdot729.$$

$$\text{3rd} \quad ,, \quad ,, \quad ,, \quad = \frac{140}{106} = \cdot757.$$

$$\text{Mean} = \cdot745.$$

Photographic light passing through clear water = ·745 × 2·686 = 2·00 times that passing through turbid water.

Using formula as before,

$$\mu_1 = \frac{\log 2\cdot00}{2\cdot1} = \frac{\cdot69315}{2\cdot1} = \cdot330.$$

$$\text{Mean of } \mu_1 = \cdot330.$$

The optical measures of the clear and turbid cells in the direction of the length were—

3 in., clear water, 36·5 ; turbid water, 26.

$$\frac{\text{Clear water}}{\text{Turbid}} = 1\cdot4.$$

Using same formula

$$\mu = \frac{\log 1\cdot4}{3} = \frac{\cdot33647}{3} = \cdot1121.$$

The mean value of μ_1 is ·330, and as

$$\Lambda_1^{-4} = \frac{\mu_1}{\mu} \times 105,$$

therefore

$$\Lambda_1^{-4} = \frac{\cdot330}{\cdot1121} \times 105 = 309\cdot1.$$

Experiment III.—Arc electric light; light from crater of positive pole alone used —the arc being cut off by screens.

Length of cell, inside measure, 3 inches; breadth, 1·7 inch. When light passed through length of cells a double sector of 16½° rotated in front of the cell containing clear water. When passed through the breadth of the cell 28°.

For the cells placed lengthways the value of the light passing through the clear cell had to be multiplied by 180/16·5 = 10·91, when placed breadthways 180/28 = 6·43

Cell.	1st exposure.			2nd exposure.		
	Mean reading.		Value.	Mean reading.		Value.
	White = 78.	White = 100.		White = 78.	White = 100.	
3 in., clear . .	40	51·5	63	41	53	61
3 in., turbid .	54	69	40	53	70·5	39

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{63}{40} = 1\cdot575.$$

$$\text{2nd} \quad ,, \qquad ,, \qquad ,, \qquad = \frac{61}{39} = 1\cdot564.$$

$$\text{Mean} = 1\cdot57.$$

Photographic light passing through clear water = 1·57 × 10·91 = 17·1 times that passing through the turbid water.

Taking the formula as before

$$\mu = \frac{\log 17\cdot1}{3} = \frac{2\cdot8390}{3} = \cdot946.$$

Cells.	1st exposure.			2nd exposure.			3rd exposure.		
	Mean reading.		Value.	Mean reading.		Value.	Mean reading.		Value.
	White = 78.	White = 100.		White = 78.	White = 100.		White = 78.	White = 100.	
1·7 in., clear .	54	69	40	57	73	36	68	87	22·5
1·7 in., turbid .	47	60	50	50	64	45	64	82	28

$$\text{1st Exposure } \frac{\text{action through clear water}}{\text{action through turbid water}} = \frac{40}{50} = \cdot800.$$

$$\text{2nd} \quad ,, \qquad ,, \qquad ,, \qquad = \frac{36}{45} = \cdot800.$$

$$\text{3rd} \quad ,, \qquad ,, \qquad ,, \qquad = \frac{22\cdot5}{28} = \cdot800.$$

$$\text{Mean} = \cdot800.$$

Photographic light passing through clear water = ·8 × 6·43 = 5·058 times that passing through the turbid water.

Using formula as before

$$\mu_1 = \frac{\log 5{\cdot}06}{1{\cdot}7} = \frac{1{\cdot}62136}{1{\cdot}7} = {\cdot}953.$$

The optical measures of the clear and turbid cells—

3 inches, clear water 59, turbid, 22·5,
1·7 inch, „ 59, „ 34.

Taking the light through the 3-inch length of the cell as lying on the curve and using the formula

$$\mu = \frac{\log 59 - \log 22{\cdot}5}{3} = \frac{1{\cdot}77495 - {\cdot}81093}{3} = \frac{{\cdot}96402}{3}$$

$$= {\cdot}321.$$

Taking 1·7-inch breadth of the cell

$$\mu = \frac{\log 59 - \log 34}{1{\cdot}7} = \frac{1{\cdot}77495 - 1{\cdot}22377}{1{\cdot}7},$$

$$= \frac{{\cdot}55118}{1{\cdot}7} = {\cdot}324.$$

Mean of these values $= {\cdot}3225$.
Mean of $\mu^1 = {\cdot}950$.

$$\Lambda_1^{-4} = \frac{\mu_1}{\mu} \times 105 = \frac{{\cdot}950}{{\cdot}3225} \times 105,$$

$$= 309{\cdot}3.$$

From these three experiments we may derive a very good approximation to the true value of Λ_1. The following are the values obtained for Λ_1^{-4}—

1st experiment	306·6
2nd „	309·1
3rd „	309·3
Mean	308·3

Therefore

$$\Lambda = 4244 \text{ wave length.}$$

That is, the variation in the intensities of the spectrum, as photographed on platinum paper, is the same as would be observed if a ray of λ 4244 were isolated and the variation in its intensity were observed optically or otherwise.

Further, it must be observed that if we find μ_1 from the photographs, we can at once find μ for the visual intensity; for since

$$\frac{\mu_1}{\mu} = \frac{\Lambda_1^{-4}}{105} = \frac{308}{105},$$

therefore

$$\mu = \frac{105}{308}\mu' = \cdot 341\,\mu' \text{ or } \mu = \tfrac{1}{3}\mu' \text{ nearly.}$$

Now

$$\mu = \kappa\Lambda^{-4} \text{ in Lord Rayleigh's formula,}$$

and

$$\mu_1 = \kappa\Lambda_1^{-4},$$

therefore

$$\kappa = \frac{\mu}{105} = \frac{\mu'}{308}.$$

Thus by finding either μ or μ' the absorption of every ray of the spectrum can be determined.

XXXI.—*Absorption of the Blue Glass.*

As the platinum paper was exposed through blue glass, it becomes necessary to know the coefficient of absorption of the glass for the photographically active rays. Experiments have been made with this view, and exposures have been made simultaneously through the blue glass, and without any glass intervening. The results obtained were that the same blackness would be produced if exposure to light through blue glass to that without were as 2·15 to 1. In the tables of exposure, which are annexed, it follows that to obtain the true photographic intensity of total sunlight, the values given must be multiplied by 2·15.

Table of Sensitiveness of Platinum Paper to the different parts of the Solar Spectrum when the exposure was made through blue glass, as used in the apparatus with which the observations were made.

Scale number.	Relative sensitiveness.	Scale number.	Relative sensitiveness.
7	0	13	86
8	4	13·5	70
8·5	10	14	56
9	20	14·5	42
10	50	15	30
10·5	66	16	16
11	84	17	6
11·5	95	18	2
12	100	19	1
12·5	97	20	0

H is at 13·8; G at 11·1; Li at 9; and D at 4·3.
Scale No. 12 has a wave length of 4260.

A reference to the spectrum curve, fig. 5, will show that this wave length of 4244 is close to the maximum sensitiveness. This is what was to be expected when it is remembered that the equivalent ray for the optically observed spectrum was close to the place of maximum luminosity.

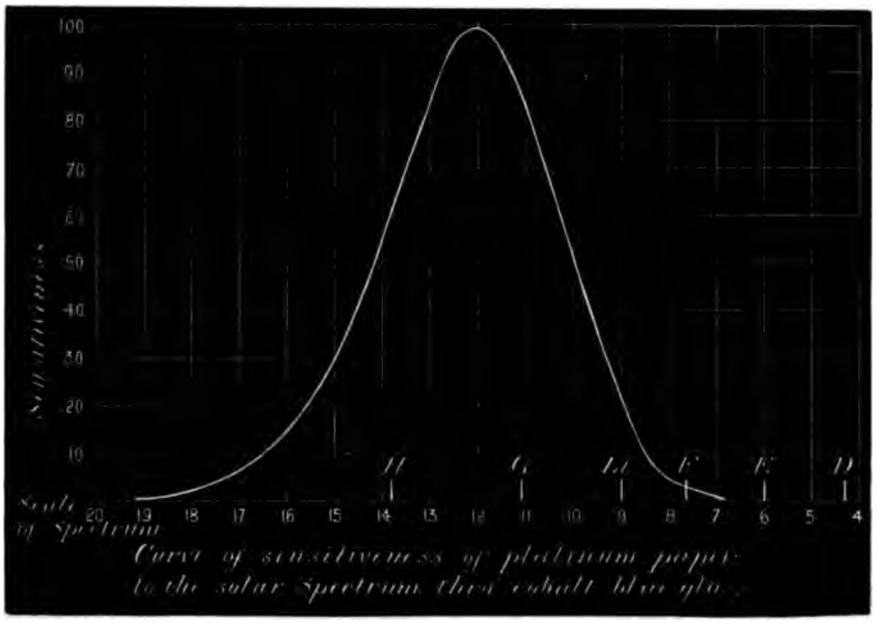
We are now in a position to reduce all the photographically measured coefficients of intensity to those which would have been observed optically.

XXXII.—*Places of Observation.*

It has been the object of these observations to determine, as far as possible, the different coefficients of absorption at different altitudes, and with this object in view the instrument has been taken to various places at different altitudes. Thus, South Kensington, Weybourne, Oxford, Derby, which may be taken as places at sea level. Grindelwald, in the valley, and on the various hills surrounding it, up to 8900 feet on the Faulhorn.

Como, Perarola, Cortina, and Interlaken, also Chamounix and Zermatt, with their surrounding mountains, have also been observing stations.

Fig. 5.



These stations are at various altitudes above the sea, and the results obtained are such as to show that the loss of light varies enormously, according as the altitude of the observing station is increased. This is a result which of course may be looked for. The observations of LANGLEY on Etna, and at Pike's Peak, show such to be the case; but it is necessarily more marked in the case of these observations owing to the part of the spectrum employed being that which suffers most loss by the scattering action of small particles. The blue end of the spectrum for this purpose has evidently a greater advantage over the red end and darker portions, more particularly over the latter, for in it the water absorptions play such an important *rôle* that the law of scattering could hardly be taken into account also. It will be noticed that observations have been made at all times of the year, and that the winter observations, when the surface of the ground at Grindelwald has been covered with snow and the air has been intensely cold (the thermometer sometimes showing as much as 40° F. of frost), give the coefficients of least absorption. The writer has held that most of the scattering particles are water in an extremely fine state of division, and that there is a probability that such is the case is shown by the fact that the most transparent atmosphere is that in which there is most warm aqueous vapour present. A change in the summer to what artists call an "atmospheric landscape" invariably shows a higher coefficient of absorption. Whatever these particles may be, there is no doubt that they are in the same volume fewer in number at high altitudes than they are at lower ones; as the coefficient of absorption is so much less in the former case than in the latter. Any one who has observed the sky at these higher altitudes will have observed that when the sun is brightest and highest, the sky is blackest, that is, that there are fewer scattering particles.

XXXIII.—*Standard of Light.*

It is somewhat difficult to know as to what standard of light to refer the light of the sun. The Author has made many comparisons between it and an amyl acetate lamp, and has come to the conclusion that at midday at sea level, and in the clearest atmosphere, the brightness of the sun exactly overhead would be close upon 7000 such standard lights at 1 foot distant from the screen, which would be equivalent to 5600 standard candles at the same distance. For the purpose of this paper, it is, however, unnecessary to refer the light to any particular standard, since all that was sought for was to obtain a comparison of the losses suffered on any one day by the light after passing through various thicknesses of atmosphere. It will be noticed in the various tables that the calculated brightness at the zenith varies very considerably even at the same altitudes. This must be laid down to one of three causes: (1) either an error in calculating the coefficients; (2) a slight haze of coarse particles intervening; or (3) owing to a varying sensitiveness in the paper used. Where the actual observed intensities throughout a day do not vary much more than those calculated, it may be presumed that the coefficient is not very far from the truth, and, consequently, the variation will in all probability be due to the second cause. As stated before, there was a change made in the paper which occurred in 1889, and it will be found that the intensities are practically the same before and after the above year.

XXXIV.—*Description, Tabulation, and Discussion of Results.*

The results which are tabulated do not show by any means all the observations made. They have been carried on for four years, and only those days are recorded in this paper when the sky has been practically cloudless whilst the observations have been taken. It need scarcely be said that the observations on three out of four days have been incomplete owing to atmospheric conditions not remaining constant. Every one of these has been tabulated, however, in the note books, and a check has thus been introduced in estimating the relative intensities of the light at the same altitudes of the sun above the horizon at the same observing station, for, in many instances, two or three observations under favourable conditions were made, the remainder being useless for the purpose of obtaining the coefficients of absorption.

Even the observations which are recorded are apparently so discordant, that it might appear difficult to arrange them in any order sufficient to get any empirical law which might connect barometric pressure with the exponential coefficient of transparency. As necessarily any approach to a law can only be an approximation, it has not been considered necessary to enter into any refinements such as the varying distance of the sun from the earth at different seasons, since any differences due to difference in atmospheric condition would more than hide any alteration due to that cause.

On certain days remarks are made that the atmosphere is exceptionally clear, and when we group such days together the results are not devoid of regularity.

Taking the observations at Faulhorn when the barometer was 21·5 inches, and when the day was noted as exceptionally fine, we get two values, ·231 and ·235 μ'. These are winter observations. We may take ·233 as the mean exponential coefficient at this barometric height in the clearest weather.

We also have on similar days—

	Bar.	μ_1.
At Derby	29·6	·437
Above Zweilutchinen . .	26·6	·356
„ Grindelwald . .	24·4	·307 and ·307
„ „ . .	23·3	·275 „ ·273
Faulhorn	21·5	·231 „ ·233

In addition to the observations given in the tables others were made at higher altitudes up to 12,000 feet at Zermatt on suitable days. It was not practicable to take a whole series of readings at these high altitudes, as it was not possible under ordinary circumstances to spend sufficient time at such elevations and that to get a difference in zenith distance sufficient to give a variation in air-thickness of such a magnitude as would give a reliable coefficient. As, however, the expeditions were made at such a time of the day as enabled a return to a station where the lapse of a few hours in the afternoon sufficed to obtained a large variation in the air-thickness, it became possible, by calculating first the coefficient of this station, and thence calculating the readings which would have been obtained at the time when the observations at the higher altitudes were made, to calculate the coefficient for the latter, assuming, of course, that no alteration in the general condition of the sky had taken place. Thus, at the Riffel house it was calculated that the coefficient at a barometric pressure of 21·2 inches, was ·222; and from observations made with the barometer at 18·9, it was calculated the coefficient would be ·182. Similarly, with the barometer at 19·9, it was calculated to be ·196.

It will be seen, if we take the Faulhorn observations as a standard, that we get what appears to be a very close approximation to the law that the exponential coefficient varies as the square of the barometric pressure.

The following is a table calculated on this assumption :—

30 inches	·453
29 „	·423
28 „	·394
27 „	·367
26 „	·340
25 „	·314
24 „	·289
23 „	·266
22 „	·244
21 „	·222
20 „	·201
19 „	·182

It will be seen that the coefficients obtained on these exceptionally clear days fall extremely close to the calculated values. The calculation for

29·6 is 441, 26·6 is 356, 24·4 is 299,
23·4 is 276, 21·5 is 232.

On the day when the Faulhorn observation gave a value to the coefficient of ·261, the sky was fairly good. If we take this as a starting point and calculate as before, we find the following table :—

30 inches	·509
29 „	·477
27 „	·412
26 „	·381
25 „	·353
24 „	·325
23 „	·297
21·5 „	·261

THE following coefficients were obtained from observations made on days which were classed as ordinarily bright.

Barometer.		Observed.	Calculated.
21·5	Faulhorn	·261	·261
23·4	Above Grindelwald . .	·290	·306
25·6	Montanvert	·384	·370
26·6	Grindelwald	·400	·400
29·7	Derby	·514	·497

It will be seen that on such days the same order of change in the coefficients is to be found.

Other coefficients were obtained on moderately bright days, which are higher than the above for the varying heights of the barometer. These observations are given to show the great variations which may occur owing to atmospheric conditions.

It thus appears that the light acting on platinum paper at any altitude up to those observed can be expressed by

$$I' = I\epsilon^{-\mu_1 h^2 x/30},$$

where h is the height of the barometer and μ_1 is the coefficient of absorption.

In the case of the most transparent atmosphere $\mu_1 = {\cdot}453$ at a barometer of 30 inches in height.

Therefore

$$I' = I\epsilon^{-{\cdot}0005033h^2}.$$

The coefficient for the visual absorption is $\cdot 341\mu'$.

$$A' = A\epsilon^{-{\cdot}0001716h^2},$$

where A and A′ are the original and transmitted lights.

We thus arrive at the following results :—

Barometer.	μ'	a'	μ	a
30	·453	·639	·154	·853
29	·423	·654	·144	·866
28	·344	·671	·134	·875
27	·367	·689	·124	·884
26	·340	·708	·115	·891
25	·314	·730	·107	·899
24	·289	·746	·098	·908
23	·266	·763	·090	·915
22	·244	·787	·083	·922
21	·222	·800	·075	·928
20	·201	·819	·068	·934
19	·182	·833	·062	·940

μ' and μ are the exponential coefficients as before. a' and a are the constants in the formula, $\mathrm{I}a^z = \mathrm{I}'$, z being the air thickness, and I and I′ the intensities before and after transmission.

We are now in a position to determine the constant κ in the formula

$$\mathrm{I}' = \mathrm{I}\epsilon^{-\kappa z \lambda^{-2}}$$

at sea-level with the barometer at 30 in. and in a very clear sky, for as

$$\kappa \Lambda^{-4} = \mu_1,$$

therefore

$$\kappa = \frac{\mu'}{\Lambda^{-4}} = \frac{\cdot 453}{308} = \cdot 00146\,;$$

in the case of fairly clear skies,

$$\kappa = \frac{\cdot 497}{\cdot 308} = \cdot 00161.$$

In Part I. of this paper the minimum value of κ was found to be ·0013, and a mean value about ·0017, so that these observations are fairly accordant.

κ may be taken to be a measure of the number of particles the rays encounter, and thence it may be concluded that the number of particles at any thin layer of the atmosphere is ah. The formula, therefore, for the scattering of a ray of any wave-length at any altitude becomes

$$\mathrm{I}' = \mathrm{I}\epsilon^{-ch^2 x \lambda^{-4}}$$

where c is a constant, h the height of the barometer, and x the air thickness, those of the zenith being "unity."

In comparing the comparative scattering of a ray at the same zenith distance, but at different altitudes, $\lambda a x$ are constants, and

$$\mathrm{I}' = \mathrm{I}\epsilon^{-mh^2}$$

where m is a constant and h the variable. This formula and that of the law of error are identical.

XXXV.—*Conclusions.*

In conclusion, it should be remarked that the loss of light as light from transmission through the atmosphere is, and must be, very different to that of the heating effect of the solar radiation. The latter is not principally dependent on scattering by small particles, but on the absorption of aqueous vapour, which is a very different matter. **Langley** has shown that the heating effect diminishes much more rapidly as the barometric pressure is diminished than is usually supposed, and this is not to be wondered at. For it must be recollected that the absorption by aqueous vapour takes place principally in the infra-red region of the spectrum, and, if it were entirely confined to that region, the presence or absence of the aqueous vapour would have no practical effect on the luminosity of the visible rays transmitted.* The absorption by this vapour where it takes place is large and follows the ordinary laws, as already stated, being principally in the infra-red of the spectrum, the heating effect will therefore diminish much more rapidly than the total illuminating value of sunlight as increased thickness of atmosphere is penetrated. In other words, the amount of light transmitted from the sun bears no comparison with that of its total heating effect.

It should be remarked that the portion of the spectrum used in these observations is almost free from any absorption by aqueous vapour, and, consequently, the scattering effect of the small particles in the atmosphere is probably almost entirely the cause of the loss of light from it, and enables a factor for such scattering to be arrived at without much difficulty. The measurement of the amount of light transmitted through different thicknesses of atmosphere has hitherto been almost entirely made on star and moon light, and, for reasons given in Part I. of this paper, the coefficients arrived at are probably slightly too large for the transmission of sunlight.

The results of these observations made at different altitudes, combined with others made in the laboratory, point to a probability that the particles which selectively scatter light are due to water. Their dimensions are probably very closely, if not exactly, the same, and the "mist" particles are large compared with them.

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* See a paper in No. 224 of the 'Proceedings of the Royal Society,' 1883, by General FESTING and the Author.

GRINDELWALD, 16th May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	—	6.48	26·4	2·49				100	97	μ = ·408. Fairly clear sky.
2	18				33·2	182	101			
3	40				16·5	400	100			
4	10	7.45		1·90	45	120	120	120	124	
5	20				29·7	210	105			
6	25				21·7	300	120			
7	5	9.55		1·26	62·5	67	134	164	162	
8	10				36·2	160	160			
9	9				39	146	162			
10	19				21	310	163			
11	5	P.M. 1.55		1·24	56·5	82	164	164	163	
12	10				36·2	168	168			
13	30				14·5	480	160			
14	15	3.16		1·50	30·5	204	136	132	146	
15	40				12·2	512	128			
16	10	6.0		4·04	68	57	57	51	51	
17	30				37·5	154	51			
18	60				16	310	51			
19	30				37·5	154	51			

ZWEILÜTCHINEN, 2nd May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	12.0	26·4	1·17	70·5	52	104	130	135	μ = ·356.
2	10				42·5	130	130			
3	20				—	—	—			
4	5	1.25		1·23	63·5	65	130	131	131	Very clear.
5	10				41·5	132	132			
6	30				89·2	360	120			
7	—									
8	5	2.18		1·35	76	43	86	83	126	
9	20				36·5	160	80			
10	20				—	—	—			
11	5	4.8		1·98	69	51	102	102	100	
12	10				50	102	102			
13	20				29·5	207	103			
14	—									
15	5	4.50		2·51	84·5	31	62	83	83	
16	10				56·5	82	82			
17	20				35	170	85			
18	5	5.30		3·53	88	27	54	52	56·7	
19	20				50·5	100	50			
20	20	6.25		7·8	90	24	12	12	12·6	

MONTANVERT, 10th August, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	12.55	25·6	1·19	68	45	45	48	52·7	$\mu = ·384$.
2	20				42	90	45			
3	20				—	96	48			
4	30				28·7	142	48			Fairly bright.
5	10				69	44	44			Sky clear.
6	15	3.0		1·5	50	72	48	47	47	
7	10				67·5	46	46			
8	24				35	112	46·6			
9	10				64	49	49			
10	22				37·5	106	47·2			
11	10	4.0		1·89	75	38	38	37·5	40·4	
12	15				60	55	37			
13	20				47·5	76	38			
14	20				48·5	74	37			
15	20	5.0		2·72	66	47	23·5	26·5	29·6	
16	30				47·5	76	25·3			
17	40				37	108	27			
18	45	5.56		4·8	55·5	61	13·5	13·2	13·2	
19	75				35·5	112	13			
20	120	6.5		7	44	78	6·5	6·5	5·7	

ZWEILÜTCHINEN, 2nd May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	12.0	26·4	1·17	70·5	52	104	130	135	$\mu = \cdot 356$.
2	10				42·5	130	130			
3	20				—	—	—			
4	5	1.25		1·23	63·5	65	130	131	131	Very clear.
5	10				41·5	132	132			
6	30				89·2	360	120			
7	—									
8	5	2.18		1·35	76	43	86	83	126	
9	20				36·5	160	80			
10	20				—	—	—			
11	5	4.8		1·98	69	51	102	102	100	
12	10				50	102	102			
13	20				29·5	207	103			
14	—									
15	5	4.50		2·51	84·5	31	62	83	83	
16	10				56·5	82	82			
17	20				35	170	85			
18	5	5.30		3·53	88	27	54	52	56·7	
19	20				50·5	100	50			
20	20	6.25		7·8	90	24	12	12	12·6	

MONTANVERT, 10th August, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	12.55	25·6	1·19	68	45	45	48	52·7	μ = ·384.
2	20				42	90	45			
3	20				—	96	48			
4	30				28·7	142	48			Fairly bright.
5	10				69	44	44			Sky clear.
6	15	3.0		1·5	50	72	48	47	47	
7	10				67·5	46	46			
8	24				35	112	46·6			
9	10				64	49	49			
10	22				37·5	106	47·2			
11	10	4.0		1·89	75	38	38	37·5	40·4	
12	15				60	55	37			
13	20				47·5	76	38			
14	20				48·5	74	37			
15	20	5.0		2·72	66	47	23·5	26·5	29·6	
16	30				47·5	76	25·3			
17	40				37	108	27			
18	45	5.56		4·8	55·5	61	13·5	13·2	13·2	
19	75				35·5	112	13			
20	120	6.5		7	44	78	6·5	6·5	5·7	

Above Grindelwald, 23rd December, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	1.15	23·4	3·23	91·3	19	19	17·5	17·5	μ = ·273. Very clear sky, sun very hot, and light excellent.
2	20				75	32	16			
3	30				53	53	17·6			
4	10	2.15		4·2	96			13·1	13·5	
5	20				82	27	13·5			
6	30				67·3	38	13			
7	10				94					
8	20				84	26	13			
9	30				70	37	12·6			
10										
11	10	2.40		5·03	96			10·8	10·7	
12	20				86	22	11			
13	30				77	32	10·6			

ABOVE Grindelwald, 2nd January, 1891.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	17	1.10	23·4	3·1	72·7	34	20	19·5	19·3	$\mu = {\cdot}290$.
2	30				47·4	60	20·			
3	45				35	88	19·6			
4	30				48·7	59	18·3			Sky deep.
5	17	2.45		5·11	93·5	18	10·6	11·1	10·7	Very good sun.
6	30				72	34	11·3			
7	45				56·2	48	10·6			
8	60				44	66	11			
9	80				35	88	11			
10	—									
11	120	3.14		7·04	38·6	76	6·3	6·2	6·2	
12	120				40	73	6·1			

FAULHORN, 13th January, 1888.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	20	9.50	24·56		33	91	45·5	54		$\mu = \cdot 235$.
2	30				32	162	54			
3	30				21	162	54			Splendid sky for depth, sun very bright.
4	20	12.13	21·6	2·6	23	144	72	72	72	
5	20				24·5	137	68·5			
6	30				15·5	216	72			
7	20	2.13	21·6	3·5	28	116	58	57	58	
8	20				28·5	114	57			
9	30				21·5	160	53·3			
10	30				19·5	177	59			
11	30				21	162	54			
12	30				18·5	185	61·6			
13	20	3.8	21·6	5	38	72	36	40	40	
14	30				27	122	41			
15	30				27·5	120	40			

FAULHORN, 18th January, 1888.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	8.55	25		60	24	24	30		μ = ·231.
2	20				43	58	28			
3	30				33	92	31			
4	30				33	91	30			Same as January 13.
5	10	12.0	21·9	2·5	33	91	91	92·5	92·5	
6	20				18·5	185	92·5			
7	20				18·5	185	92·5			
8	30				12·6	280	93·3			
9	20	1.45	21·7	3	21	162	81	81	82·3	
10	10				37	68	68			
11	20				23	146	73			
12	20				21	162	81			
13	30				14·5	243	81			
14	30				14·5	243	81			
15	10	3.10	21·7	4·9	44	52	52	53	53	
16	20				31	100	50			
17	30				21	164	55			

GRINDELWALD, 12th August, 1888.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	20	6.48	26·6	2·84	38	94	47	47	47	$\mu = 400$.
2	40				17·5	197	44			Fair sky.
3	30				27·5	132	44			
4	60				12	276	49			
5	5	7.40			56·5	25	50	48		
6	20				35	100	50			
7	30				28	130	43			
8	20				37	96	48			
9	25				28·5	128	51			
10	15	8.2			37·5	96	64	68	68	
11	15				32·7	102	68			
12	20				25	144	72			
13	10	8.58			42·5	81	81	81	80	
14	15				30	122	81			
15	25				19·7	182	73			
16	13	10.24			32	114	89	89	89	
17	20				20·5	172	86			
18	20				18·7	178	89			

GRINDELWALD, 23rd May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	11.40	26·6		59·5	74	74	67		$\mu = \cdot 381$.
2	20				41	136	68			
3	10				66	60	60			Very fair sky; good sun.
4	20				41	136	68			
5	10				63	66	66			
6	10	1.33		1·24	58·5	76	76	75	75	
7	20				39·7	143	71·5			
8	20				37·5	154	77			
9	10	2.48		1·41	62	68	68	70	70·4	
10	20				39·5	144	72			
11	10	4.30		2·02	74	46	46	57·5	56·5	
12	20				47	112	56			
13	30				34	178	59			
14	15	5.20		2·64	70·5	52	35	41	43·9	
15	30				45	119	40			
16	45				32	189	42			
17	40	5.52		3·31	39	146	36·5	35·8	34·7	
18	60				29	212	35·3			
19	60	6.18		4·46	42·5	130	22	22	22	
20	60				42·5	130	22			

GRINDELWALD, 7th July, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	—		26·6							$\mu = \cdot 71$.
2	10	12.40		1·075	43·5	126	126	125	125	
3	15				33	184	123			
4	5				66	60	120			
5	10	2.42		1·26	47·5	110	110	111	110	Poor sky; milky.
6	20				27·8	224	112			
7	5				80·5	37	37			
8	10	5.40		2·63	70·5	51	35	36·7	37·5	
9	20				51·5	96	48			
10	30				46·2	115	38			
11	15				79	39	26			
12	20	6.10		3·34	71·5	51	25·5	25	25	
13	30				59·5	74	24·7			
14	40				50	100	25			
15	20				86	29	14·5			
16	25	6.35		4·28	82·5	34	13·6	13·3	13·6	
17	40				72·5	49	12			
18	20				99	11	5·5			
19	40	7.0		5·5	93·5	21	5·2	5·5	5·4	
20	60				82	34	5·7			

COMO, 28th May, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	10.45	29·1	1·142	64	64	128	120	120	$\mu = ·55$.
2	10				46	115	115			
3	20				27	232	116			Blue sky but rather milky.
4	5	12.30			62	68	132	128	123	
5	10				41·5	134	134			
6	20				25·2	255	128			
7	5	3.23			75	50	100	100	100	
8	10				50	101	101			
9	20				31	100	100			

ROSENLAUI, 4th July, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	7.5	26·5	2·19	72	50	100	98	98	$\mu = \cdot 435$.
2	10				50	100	100			
3	20				32	190	95			Fairly good sky. Clouded over about 1.30.
4	5	8.50		1·4	53	92	184	143	143	
5	10				39	140	146			
6	20				23	280	140			
7	5	11.15		1·02	52	94	188	163	163	
8	10				36·5	160	160			
9	15				25·5	250	167			
10	20				19·2	326	163			

GRINDELWALD, 24th June, 1887.

No. of exposure.	Expo-sure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	5	12.0	26·2	1·09	77	34	68	68	68	$\mu = \cdot 707$.
2	10				49	68	68			
3	20				27	134	67			Milky sky, cloudy in morning, bright after 11.
4	5	1.10		1·125	76	33	66	65	66·5	
5	10				51·5	64	64			
6	20				28	130	65			
7	5	4.25		1·842	84	18	36	39·6	40	
8	10				69·5	40	40			
9	20				40	87	43			

South Kensington, 25th September, 1889.

No. of exposure.	Expo-sure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	12.40	29·6	1·75	75·5	34	34	34	34	μ = ·627.
2	20				48·5	68	34			
3	40				28·2	130	32·5			Poor sky, but same tint maintained.
4	10	2.0		1·91	85	25	25	26	30·5	
5	20				64	45	22·5			
6	40				35	104	26			
7	15	3.0		2·48	79·5	30	20	21·5	21·4	
8	30				56·5	57	19·5			
9	40				41·2	88	22			
10	10	3.55			98	14	14			
11	15				90	21	14			
12	30				71	38	12·7			
13	10	4.2		3·4	96	—		12	12	
14	20				86	24	12			
15	40				65·5	44	11			
16	10				98	14	14			
17	20				85	25	12·5			
18	40				65	44	11			

ABOVE Grindelwald towards Faulhorn, 25th December, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint, White, =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	15	12.0	24·4	2·92	82·5	26	17·3	17·7		$\mu = \cdot 307$.
2	30				53·3	53	17·6			Beautifully deep sky; sun very bright.
3	30				53·3	53	17·6			
4	45				36·7	82	18·2			
5	30	12.25		3	52	55	18·3	18·5	18·5	
6	15				80	28	18·6			
7	10	1.30		3·4	95	17	17	16·5	16·5	
8	20				73·5	33	16·5			
9	30				52·7	54	18			
10	20				72·5	32	16			
11	20	2.0		3·8	77·5	30	15·	14·5	14·5	
12	40				49·5	58	14·5			
13	60				36	86	14·5			
14										
15	30	2.40		5	79	29	9·7	9·7	9·7	
16	45				60	45	10			
17	60				50·7	57	9·5			
8	30	3.30		7·6	97	15	5	4·7	4·5	
19	60				85·5	28	4·7			
20	60				87	27	4·5			

ABOVE Grindelwald towards Faulhorn. 27th December, 1890.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	15	12·0		2·9	83	27	18	16·3		$\mu = \cdot 275$.
2	30				58·5	47	15·7			
3	45				42·5	68	15·1			
4	15	12.25	24·4	3	81	28	18·6	18·3	18·3	
5	30				52·5	54	18			
6	15	1.45	23·4	3·6	86·5	23	15·3	15·3	15·3	$\mu = \cdot 306$.
7	30				57	48	16			
8	—									
9	45				44	66	14·7			Beautiful black sky; sun very bright.
10	30	2.35	23·4	4·9	72	34	11·3	10·7	10·7	
11	60				47	61	10·2			
12	60	3.34	24·4		95·5	15	2·5	2·5	2·5	
13	120				84	26	2·2			

FAULHORN, 8th July, 1887.

No. of exposure.	Expo-sure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White = ·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
		A.M.								
1	15	5.10	21·4	6·7	68·5	56	42	40	40	$\mu = \cdot 261$.
2	40				38	151	38			
3	80				19	350	41			
4	10	5.40		4·55	65	62	62	70	70	Good sky, sun rose yellow.
5	20				44·5	122	61			
6	60				16·5	420	70			Sky good at top and at Grindelwald.
7	10	6.40		2·55	50	101	101	114	116	
8	20				29	210	105			
9	35				17	400	114			
10	11	8.10		1·6	36·2	161	146	147	147	
11	9				42	132	147			
12	15				28	222	147			
13	—									
14	5	9.45		1·24	54·5	87	174	160	163	
15	10				36·5	160	160			
16	20				20·5	320	160			
17	10	10.15		1·18	34	170	170	170	170	
18	15				25	255	170			
19	10	1.55		1·20	47	112	112	112	—	
20	15				34·5	172	113			

DERBY, 27th August, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec. exposure.	Calculated intensity.	Remarks.
1	10	3.17	29·6	1·90	65·4	61	61	56	56·1	μ = ·514.
2	20				47·2	109	54·5			
3	30				34·7	171	57			Sky rather milky but sun good.
4	20				48·6	104	52			
5	10	4.9		2·41	75	44	44	42	44·5	
6	20				54·2	86	43			
7	30				43	126	42			
8	40				34·7	165	41			
9	—									
10	20	4.46		3·00	66·7	58	29	32·3	31·9	
11	30				49·3	102	34			
12	10				82	34	34			
13	20	5.15		3·85	80·5	36	18	18·2	20·6	
14	40				61·1	69	17·3			
15	60				47·2	109	18·2			
16	30				75	44	14·3			
17	60	5.40		5·07	61·1	69	11·5	11	11	
18	90				51·4	94	10·4			
19	80	5.58		6·59	77·7	40	5	5	5	
20	120				68	56	4·6			

Derby, 28th August, 1887.

No. of exposure.	Exposure in seconds.	Time.	Barometer, pressure in inches.	Air thickness, that of ZD being 1.	Depth of tint. White =·100.	Equivalent in intensity.	Intensity for 10 sec. exposure.	Adopted mean intensity for 10 sec exposure.	Calculated intensity.	Remarks.
1	—		29·6							$\mu = \cdot437$.
2	10	12.25		1·37	54·5	87	87	86	87	Exceptionally clear. Saw the Malvern Hills distinctly. *The* one day of the year probably.
3	20				35·5	166	83			
4	15				42	132	88			
5	10	1.3		1·42	52·5	93	93	88	88	
6	15				42·5	130	86			
7	20				34·5	174	87			
8	10	2.4		1·53	48·5	107	107	83·5	83·8	
9	20				35·5	168	84			
10	12				50·5	99	83			
11	10	3.15		1·89	59	74	74	72	71·6	
12	20				40·5	139	70			
13	8	4.13		2·47	44·5	41	51	51	55·6	
14	20				51·5	99	49			
15	40				30·5	204	51			
16	20	4.45		3·04	54	88	44	44	43·3	
17	80	5.16		3·96	30·5	234	29	29	29	
18	40				46·5	114	28·5			

II. *The Potential of an Anchor Ring.*

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Introduction.

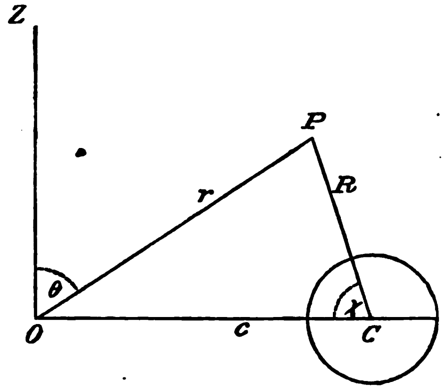
In this Paper I have developed a method of dealing with questions connected with Anchor Rings.

If r, θ, ϕ be the coordinates of any point outside an anchor ring, whose central circle is of radius c, then

$$\int_0^{\pi} \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr \sin\theta \cos\phi)}}$$

is a solution of Laplace's equation, finite at all external points and vanishing at infinity. Let this be called I. Then $d\text{I}/dz$ is another solution; and two sets of solutions may be found by differentiating I and $d\text{I}/dz$ any number of times with respect to c. These solutions are symmetrical with respect to the axis of the ring. In the first set z is involved only in even powers; in the second set in odd powers.

Take a section through the axis Oz of the ring and the point P, (r, θ) cutting the central circle of the ring in C.



Let CP = R and ∠ OCP = χ.

When R is less than c, the integral

$$\int_0^{\pi} \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr \sin\theta \cos\phi)}}$$

is expanded in ascending powers of R/c, and the expansions of the integrals, obtained by differentiating this with respect to c, are deduced. Section I. is devoted to the discussion of these functions, and some similar ones, needed in hydrodynamic applications.

In Section II. the potential of an anchor ring at all external points is found in a very convergent series of integrals. The expansions of Section I. are not needed; but the first few terms are reduced to elliptic integrals. The equipotential surfaces are drawn for the ratios $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, 1 of the thickness of the ring to its mean diameter.

In Section III. the potential of a conductor, in the form of an anchor ring, is found at external points; the surface density at any point of the ring and the charge are also determined. Section IV. consists of a discussion of the motion of an anchor ring in an infinite fluid; the velocity potential or stream line function for motion parallel to the axis, perpendicular to the axis, &c., being first determined. The kinetic energy is determined in the several cases; and in the case of the cyclic motion through the ring, the linear momentum. In this last case, the solid angle subtended by a circle at a point near a circumference, is incidentally found.

In Section V. the annular form of rotating fluid is discussed, when the thickness of the annulus is small compared with its mean radius.

It is shown that the form of the cross section may be represented by $R = a\,(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi +, \&c.)$, where β_2, β_3, &c., are of the second, third, &c., order in a/c. Their values are found as far as $(a/c)^4$.

To the second order

$$\frac{\omega^2}{\pi\rho} = \left(\frac{a}{c}\right)^2 \left(\log\frac{8c}{a} - \frac{5}{4}\right),$$

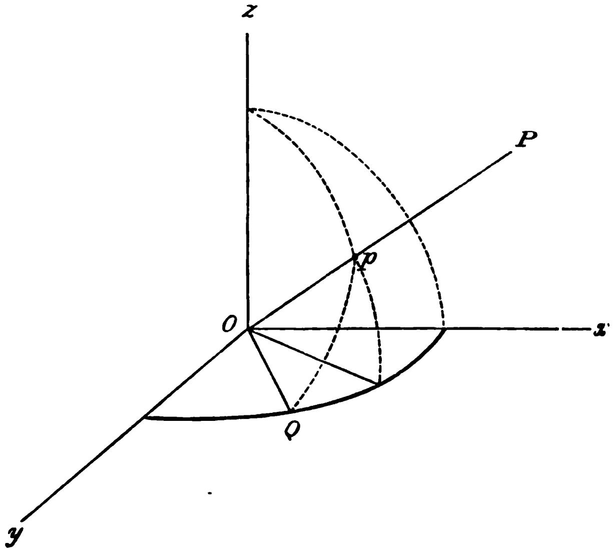
$$\beta_2 = \frac{5}{8}\left(\frac{a}{c}\right)^2 \left(\log\frac{8c}{a} - \frac{17}{12}\right).$$

The method employed throughout the paper has not the analytical elegance of Mr. HICKS' Toroidal Functions, but it has many advantages. The potential of an attracting ring takes a very simple form. The boundary conditions to be satisfied in hydrodynamical applications are very simple. The results are obtained in terms of R and χ, the quantities most obviously connected with a ring.

I am greatly indebted to Mr. HERMAN, Fellow and Assistant Tutor of Trinity College, Cambridge, for the careful manner in which he has read over much of the work, and the many errors he has corrected. In consequence, the paper will, I think, be found free from any serious mistakes.

Section I.—*Preliminary Analysis.*

§ 1. Take a fine ring in the plane of x, y, centre at the origin, and consisting of attracting matter of density $k \cos n\,\phi'$, where ϕ' is the azimuth of any point.



The potential of this ring at a point P, whose coordinates are r, θ, ϕ, is

$$\int_0^{2\pi} \frac{k \cos n\phi'\, c\, d\phi'}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos(\phi - \phi')\}}}.$$

Put $\phi' = \phi + \psi$, then the potential

$$= \int_0^{2\pi} \frac{ck\,(\cos n\phi \cos n\psi - \sin n\phi \sin n\psi)\, d\psi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\psi\}}}$$

$$= \cos n\phi \int_0^{2\pi} \frac{ck \cos n\psi\, d\psi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\psi\}}}$$

as the second integral vanishes between the limits.

Therefore

$$\cos n\phi \int_0^{\pi} \frac{\cos n\phi\, d\phi}{\sqrt{(r^2 + c^2 - 2cr \sin\theta \cos\phi)}}$$

or

$$\cos n\phi \int_0^{\pi} \frac{\cos n\phi\, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos\phi + \varpi^2)}}$$

is a solution of LAPLACE's equation.

Similarly

$$\sin n\phi \int_0^\pi \frac{\cos n\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}}$$

is a solution.

Let V stand for either of these integrals. Then

$$\frac{d^{p+q}\,\mathrm{V}}{dc^p \, dz^q}$$

is also a solution of LAPLACE'S equation.

For different values of p and q the solutions are not all independent: as it is easily seen that

$$\int_0^\pi \frac{\cos n\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}}$$

satisfies a linear partial differential equation of the second order in c and z.

But two independent sets of solutions are obtained by giving different values to p in $\frac{d^p\mathrm{V}}{dc^p}$ and $\left(\frac{d}{dc}\right)^p \cdot \left(\frac{d\mathrm{V}}{dz}\right)$.

The only cases considered in this paper are when $n = 0$ or $n = 1$. That is the solutions of the forms

$$\left(\frac{d}{dc}\right)^p \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}} \, ;$$

$$\left(\frac{d}{dc}\right)^p \frac{d}{dz} \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}} \, ;$$

$$\cos \phi \left(\frac{d}{dc}\right)^p \int_0^\pi \frac{\cos \phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + c^2)}} \, ;$$

and

$$\cos \phi \left(\frac{d}{dc}\right)^p \frac{d}{dz} \int_0^\pi \frac{\cos \phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}} \, ;$$

as these cover all the cases to which the functions are applied in this paper.

It is easily seen that

$$\varpi \int_0^\pi \frac{\cos \phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}}$$

and, consequently,

$$\varpi \left(\frac{d}{dc}\right)^p \int_0^\pi \frac{\cos \phi \, d\phi}{\sqrt{(z^2 + c^2 - 2c\varpi \cos \phi + \varpi^2)}}$$

are solutions of the equation

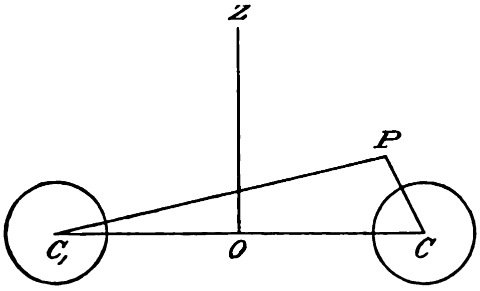
$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0.$$

For putting $\psi = \varpi\phi$, this equation becomes

$$\frac{d^2\phi}{dz^2} + \frac{d^2\phi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\Phi}{d\varpi} - \frac{1}{\varpi^2}\Phi = 0,$$

that is, $\Phi \cos \phi$ is a solution of LAPLACE's equation.

§ 2. *Expansion of the Functions.*—Let the plane through the axis of the ring and a point P, cut the ring in the two circles whose centres are C and C_1.



Let

$$CP = R\,; \quad C_1P = R_1\,; \quad OC = c\,;$$

and the angle

$$OCP = \chi.$$

Also, for convenience, put

$$s = \frac{R}{c},$$

and

$$l = \log\frac{8c}{R} - 2.$$

When R is less than c the above integrals may be expanded in ascending powers of R/c or s.

$$\int_0^{\pi} \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}}$$

$$= \frac{2}{R_1}\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{2}{1.2}\right) + \frac{1^2 3^2}{2^2 4^2}\frac{R^4}{R_1^4}\left(\log\frac{4R_1}{R} - \frac{2}{1.2} - \frac{2}{3.4}\right) + \ldots\right\}.$$

[CAYLEY, 'Ell. Func.,' p. 54.]

$$\int_0^{\pi} \frac{\cos\phi \, d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{(2\cos^2\phi - 1)\,d\phi}{\sqrt{\left\{1 - \dfrac{R_1^2 - R^2}{R_1^2}\cos^2\phi\right\}}}$$

$$= \frac{2}{R_1}\int_0^{\frac{\pi}{2}} \frac{2\left(\cos^2\phi - \dfrac{R_1^2}{R_1^2 - R^2}\right) - 1 + \dfrac{2R_1^2}{R_1^2 - R^2}}{\sqrt{\left\{1 - \dfrac{R_1^2 - R^2}{R_1^2}\cos^2\phi\right\}}}\,d\phi,$$

$$= \frac{2}{R_1}\left\{\frac{2}{1 - \dfrac{R^2}{R_1^2}}\left[\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \dfrac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}} - \int_0^{\frac{\pi}{2}} \sqrt{\left\{1 - \frac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}\,d\phi\right] \right.$$

$$\left. - \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \dfrac{R_1^2 - R^2}{R_1^2}\sin^2\phi\right\}}}\right\}$$

$$= \frac{4}{R_1\left(1 - \dfrac{R^2}{R_1^2}\right)}\left[\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{2}{1.2}\right) + \ldots\right\}\right.$$

$$\left. - \left\{1 + \frac{1}{2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{1}{1.2}\right) + \frac{1^2.3}{2^2.4}\frac{R^4}{R_1^4}\left(\log\frac{4R_1}{R} - \frac{2}{1.2} - \frac{1}{3.4}\right) + \ldots\right\}\right]$$

$$- \frac{2}{R'}\left\{\log\frac{4R_1}{R} + \frac{1^2}{2^2}\frac{R^2}{R_1^2}\left(\log\frac{4R_1}{R} - \frac{2}{1.2}\right) + \frac{1^2.3^2}{2^2.4^2}\frac{R^4}{R_1^4}\left(\log\frac{4R_1}{R} - \frac{2}{1.2} - \frac{2}{3.4}\right) + \ldots\right\}$$

[Cayley, 'Ell. Func.,' p. 54.]

$$= \frac{2}{R_1}\left\{\log\frac{4R_1}{R} - 2 + \frac{R^2}{R_1^2}\,\frac{5\log\dfrac{4R_1}{R} - 7}{4} + \frac{R^4}{R_1^4}\,\frac{162\log\dfrac{4R_1}{R} - 225}{128} + \&c.\right\}.$$

Now

$$R_1^2 = 4c^2 - 4cR\cos\chi + R^2$$

$$= 4c^2\left(1 - \frac{R}{c}\cos\chi + \frac{R^2}{4c^2}\right)$$

$$= 4c^2\left(1 - s\cos\chi + \frac{s^2}{4}\right).$$

Therefore

$$\log\frac{4R_1}{R} = \log\frac{8}{s} - \frac{s}{2}\cos\chi - \frac{s^2}{4}\frac{\cos 2\chi}{2} - \frac{s^3}{8}\frac{\cos 3\chi}{3} - \&c.$$

$$= l + 2 - \frac{s}{2}\cos\chi - \frac{s^2}{8}\cos 2\chi - \frac{s^3}{24}\cos 3\chi - \&c.$$

$$\frac{R^2}{4R_1^2} = \frac{s^2}{16} + \frac{s^3}{16}\cos\chi + \frac{s^4}{64}(1 + 2\cos 2\chi) + \&c.$$

Substituting in the above expansions

$$\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{2}{\mathrm{R}_1}\left\{l + 2 - s\frac{\cos\chi}{2} + s^2\left(\frac{l+1}{16} - \frac{\cos 2\chi}{8}\right) + s^3\left(\frac{2l+1}{32}\cos\chi - \frac{\cos 3\chi}{24}\right)\right.$$

$$+ s^4\left(\frac{50l+15}{2048} + \frac{4l+1}{128}\cos 2\chi - \tfrac{1}{64}\cos 4\chi\right) + \ldots$$

$$\int_0^\pi \frac{\cos\phi\, d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{2}{\mathrm{R}_1}\left\{l - s\frac{\cos\chi}{2} + s^2\left(\frac{5l+3}{16} - \frac{\cos 2\chi}{8}\right) + s^3\left(\frac{10l+1}{32}\cos\chi - \tfrac{1}{24}\cos 3\chi\right)\right.$$

$$\left. + s^4\left(\frac{322l+35}{2048} + \frac{20l-3}{128}\cos 2\chi - \tfrac{1}{64}\cos\chi\right) + \ldots\right\}.$$

Now

$$\frac{2}{\mathrm{R}_1} = \frac{1}{c}\left(1 - s\cos\chi + \frac{s^2}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{c}\left\{1 + \frac{s\cos\chi}{2} + s^2\left(\frac{1}{16} + \frac{3}{16}\cos 2\chi\right) + s^3\left(\frac{3}{64}\cos\chi + \frac{5}{64}\cos 3\chi\right)\right.$$

$$\left. + s^4\left(\frac{9}{1024} + \frac{20}{1024}\cos 2\chi + \frac{35}{1024}\cos 4\chi\right) + \ldots\right\}.$$

Therefore

$$\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{1}{c}\left\{l + 2 + s\frac{l+1}{2}\cos\chi + s^2\left(\frac{2l+1}{16} + \frac{3l+2}{16}\cos 2\chi\right)\right.$$

$$+ s^3\left(\frac{9l+3}{64}\cos\chi + \frac{15l+7}{192}\cos 3\chi\right)$$

$$\left. + s^4\left(\frac{105l+27}{2048} + \frac{60l+13}{768}\cos 2\chi + \frac{105l+34}{3072}\cos 4\chi\right) + \&c.\right\}.$$

And

$$\int_0^\pi \frac{\cos\phi d\phi}{\sqrt{\{r^2 + c^2 - 2cr \sin\theta \cos\phi\}}}$$

$$= \frac{1}{c}\left\{l + s\frac{l-1}{2}\cos\chi + s^2\left(\frac{6l+1}{16} + \frac{3l-4}{16}\cos 2\chi\right)\right.$$

$$+ s^3\left(\frac{33l+1}{64} + \frac{15l-23}{192}\cos 2\chi\right)$$

$$\left. + s^4\left(\frac{540l+27}{2048} + \frac{240l-23}{768}\cos 2\chi + \frac{105l-176}{3072}\cos 4\chi\right) + \&c.\right\}.$$

§ 3. To obtain the expansion of $\frac{d}{dc}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2+c^2-2cr\sin\theta\cos\phi\}}}$ in ascending powers of $\frac{\mathrm{R}}{c}$, the corresponding expansion of $\int_0^\pi \frac{d\phi}{\sqrt{\{r^2+c^2-2cr\sin\theta\cos\phi\}}}$ must be differentiated with respect to c, R and χ being considered as functions of r, θ, and c.

Now

$$\left.\begin{array}{l}\mathrm{R}\cos\chi = c - r\sin\theta = c - \varpi \\ \mathrm{R}\sin\chi = r\cos\theta = z\end{array}\right\}.$$

Therefore

$$\left.\begin{array}{l}\cos\chi\frac{\partial\mathrm{R}}{\partial c} - \mathrm{R}\sin\chi\frac{\partial\chi}{\partial c} = 1 \\ \sin\chi\frac{\partial\mathrm{R}}{\partial c} + \mathrm{R}\cos\chi\frac{\partial\chi}{\partial c} = 0\end{array}\right\}.$$

Solving these equations

$$\frac{\partial\mathrm{R}}{\partial c} = \cos\chi$$

$$\mathrm{R}\frac{\partial\chi}{\partial c} = -\sin\chi.$$

Again

$$\cos\chi\frac{\partial\mathrm{R}}{\partial z} - \mathrm{R}\sin\chi\frac{\partial\chi}{\partial z} = 0$$

and

$$\left.\sin\chi\frac{\partial\mathrm{R}}{\partial z} + \mathrm{R}\cos\chi\frac{\partial\chi}{\partial z} = 1\right\}.$$

Solving these equations

$$\frac{\partial\mathrm{R}}{\partial z} = \sin\chi\,;\quad \mathrm{R}\frac{\partial\chi}{\partial z} = \cos\chi.$$

From these formulæ it follows that

$$\frac{d}{dc}(\mathrm{R}^n\cos n\chi) = n\mathrm{R}^{n-1}\cos(n-1)\chi$$

and

$$\frac{d}{dc}\left(\frac{\cos n\chi}{\mathrm{R}^n}\right) = -n\frac{\cos(n+1)\chi}{\mathrm{R}^{n+1}}$$

and that

$$\frac{d}{dz}(\mathrm{R}^n\cos n\chi) = -n\mathrm{R}^{n-1}\sin(n-1)\chi$$

and

$$\frac{d}{dz}\left(\frac{\cos n\chi}{\mathrm{R}^n}\right) = -\frac{n\sin(n+1)\chi}{\mathrm{R}^{n+1}}$$

Let us write

$$\mathrm{I}_1 = \int_0^\pi \frac{d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}}$$

I_1 has been expanded as a function of R, χ, and c.

The expansion of dI_1/dc may therefore be found by differentiating this by the rule

$$\frac{d}{dc} = \frac{\partial}{\partial c} + \cos\chi \frac{\partial}{\partial R} - \frac{\sin\chi}{R}\frac{\partial}{\partial\chi}.$$

It is rather simpler to use the formulæ for d/dc ($R^n \cos n\chi$) proved above.

Take, for example, the term

$$\frac{1}{c}\frac{9l+3}{64}s^3 \cos\chi.$$

This

$$= \frac{9l+3}{64}\frac{R^2}{c^4} R \cos\chi.$$

$$\frac{d}{dc}\left(\frac{9l+3}{64}\frac{R^2}{c^4} R\cos\chi\right) = \frac{(9l+3)R^2}{64c^4}\frac{d}{dc}(R\cos\chi) + R\cos\chi\frac{d}{dc}\left\{\frac{(9l+3)R^2}{64c^4}\right\}$$

$$= \frac{(9l+3)R^2}{64c^4} + \frac{R\cos^2\chi}{64c^4}\frac{\partial}{\partial R}\{(9l+3)R^2\} + \frac{R^3\cos\chi}{64}\frac{\partial}{\partial c}\left(\frac{9l+3}{c^4}\right).$$

In performing the partial differentiations with respect to R and c, it must be remembered that $l = \log 8c/R - 2$, and is a function of R and c.

Let

$$I_1 = \frac{l+2}{c} + \frac{l+1}{2}\frac{R\cos\chi}{c^2} + \left\{\frac{(2l+1)R^2}{16c^3} + \frac{(3l+2)R^2\cos 2\chi}{16c^3}\right\}$$

$$+ \left\{\frac{(9l+3)R^3\cos\chi}{64c^4} + \frac{(15l+7)R^3\cos 3\chi}{192c^4}\right\} + \&c.$$

$$\frac{dI_1}{dc} = -\frac{\cos\chi}{cR} - \frac{1+\cos 2\chi}{4c^2} + \left\{\frac{(8l-3)R\cos\chi}{32c^3} - \frac{3R\cos 3\chi}{32c^3}\right\}$$

$$+ \left\{\frac{(18l-3)R^2}{128c^4} + \frac{(18l-8)R^2\cos 2\chi}{128c^4} - \frac{5R^2\cos 4\chi}{128c^4}\right\} + \ldots$$

$$+ \frac{l+1}{2c^2} + \frac{(3l+2)R\cos\chi}{8c^3} + \left\{\frac{(9l+3)R^2}{64c^4} + \frac{(15l+7)R^2\cos 2\chi}{64c^4}\right\} + \ldots$$

$$- \frac{l+1}{c^2} - \frac{(2l+1)R\cos\chi}{2c^3} - \left\{\frac{(6l+1)R^2}{16c^4} + \frac{(9l+3)R^2\cos 2\chi}{16c^4}\right\} + \ldots$$

$$= -\frac{\cos\chi}{cR} - \left(\frac{2l+3}{4c^2} + \frac{\cos 2\chi}{4c^2}\right) - \left\{\frac{(20l+11)R\cos\chi}{32c^3} + \frac{3R\cos 3\chi}{32c^3}\right\}$$

$$- \left\{\frac{(12l+5)R^2}{128c^4} + \frac{(12l+9)R^2\cos 2\chi}{64c^4} + \frac{5R^2\cos 4\chi}{128c^4}\right\} - \&c.$$

From this $d^2\mathrm{I}_1/dc^2$ might be obtained, and then $d^3\mathrm{I}_1/dc^3$, &c., but in the question of an attracting ring there is a special advantage in using functions of the form

$$\left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\mathrm{I}_1.$$

Call this I_n. Thus

$$\mathrm{I}_n = \left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2+c^2-2cr\sin\theta\cos\phi\}}}.$$

Also denote

$$\left(-\frac{1}{c}\frac{d}{dc}\right)^{n-1}\int_0^\pi \frac{\cos\phi\, d\phi}{\sqrt{\{r^2+c^2-2cr\sin\theta\cos\phi\}}}$$

by J_n.

The values of the functions are collected here for the sake of reference.

(A)

$$\begin{aligned}\mathrm{I}_1 = \frac{1}{c}\Big\{ & l + 2 + \frac{l+1}{2}\cos\chi\, s + \left(\frac{2l+1}{16} + \frac{3l+2}{16}\cos 2\chi\right)s^2 \\ & + \left(\frac{9l+3}{64}\cos\chi + \frac{15l+7}{192}\cos 3\chi\right)s^3 \\ & + \left(\frac{108l+27}{2048} + \frac{60l+13}{768}\cos 2\chi + \frac{105l+34}{3072}\cos 4\chi\right)s^4 + \dots\Big\}\end{aligned}$$

$$\begin{aligned}\mathrm{I}_2 = \frac{1}{c^2\mathrm{R}}\Big\{ & \cos\chi + \left(\frac{2l+3}{4} + \frac{\cos 2\chi}{4}\right)s + \left(\frac{12l+11}{32}\cos\chi + \tfrac{3}{32}\cos 3\chi\right)s^2 \\ & + \left(\frac{12l+5}{128} + \frac{12l+9}{64}\cos 2\chi + \tfrac{5}{128}\cos 4\chi\right)s^3 + \dots\Big\}\end{aligned}$$

$$\begin{aligned}\mathrm{I}_3 = \frac{1}{c^3\mathrm{R}^2}\Big\{ & \cos 2\chi + \left(\tfrac{9}{4}\cos\chi + \tfrac{1}{4}\cos 3\chi\right)s \\ & + \left(\frac{36l+51}{32} + \tfrac{3}{4}\cos 2\chi + \tfrac{3}{32}\cos 4\chi\right)s^2 + \dots\Big\}\end{aligned}$$

$$\mathrm{I}_4 = \frac{2!}{c^4\mathrm{R}^3}\left\{\cos 3\chi + \left(\tfrac{5}{2}\cos 2\chi + \tfrac{1}{4}\cos 4\chi\right)s + \dots\right\}$$

$$\mathrm{I}_5 = \frac{3!}{c^5\mathrm{R}^4}\left\{\cos 4\chi + \dots\right\}.$$

(B)

$$\begin{aligned}\frac{d\mathrm{I}_1}{dz} = -\frac{1}{c\mathrm{R}}\Big\{ & \sin\chi + \tfrac{1}{4}\sin 2\chi\, s + \left(\frac{4l+5}{32}\sin\chi + \tfrac{3}{32}\sin 3\chi\right)s^2 \\ & + \left(\frac{3l+3}{32}\sin 2\chi + \tfrac{5}{128}\sin 4\chi\right)s^3 \\ & + \left(\frac{6l+3}{256}\sin\chi + \frac{120l+101}{2048}\sin 3\chi + \tfrac{35}{2048}\sin 5\chi\right)s^4 + \dots\Big\}\end{aligned}$$

$$\frac{dI_2}{dz} = -\frac{1}{c^2R^2}\Big\{\sin 2\chi + \left(\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi\right)s + \left(\tfrac{3}{8}\sin 2\chi + \tfrac{3}{32}\sin 4\chi\right)s^2$$
$$+ \left(\frac{12l+13}{64}\sin\chi + \tfrac{27}{128}\sin 3\chi + \tfrac{5}{128}\sin 5\chi\right)s^3 + \ldots\Big\}$$

$$\frac{dI_3}{dz} = -\frac{2!}{c^3R^3}\{\sin 3\chi + \left(\tfrac{5}{4}\sin 2\chi + \tfrac{1}{4}\sin 4\chi\right)s$$
$$+ \left(\tfrac{15}{16}\sin\chi + \tfrac{15}{32}\sin 3\chi + \tfrac{3}{32}\sin 5\chi\right)s^2 + \ldots\}$$

$$\frac{dI_4}{dz} = -\frac{3!}{c^4R^4}\{\sin 4\chi + \left(\tfrac{7}{4}\sin 3\chi + \tfrac{1}{4}\sin 5\chi\right)s + \ldots\}$$

$$\frac{dI_5}{dz} = -\frac{4!}{c^5R^5}\{\sin 5\chi + \ldots\}.$$

(C)

$$J_1 = \frac{1}{c}\Big\{l + \frac{l-1}{2}\cos\chi\, s + \left(\frac{6l+1}{16} + \frac{3l-4}{16}\cos 2\chi\right)s^2$$
$$+ \left(\frac{33l+1}{64}\cos\chi\,\frac{15l-23}{192}\cos 3\chi\right)s^3$$
$$+ \left(\frac{540l+27}{2048} + \frac{240l-23}{768}\cos 2\chi + \frac{105l-176}{3072}\cos 4\chi\right)s^4 + \ldots\Big\}$$

$$J_2 = \frac{1}{c^2R}\Big\{\cos\chi + \left(\frac{2l-1}{4} + \tfrac{1}{4}\cos 2\chi\right)s + \left(-\frac{4l+21}{32}\cos\chi + \tfrac{3}{32}\cos 3\chi\right)s^2$$
$$+ \left(\frac{12l+5}{128} - \frac{12l+19}{64}\cos 2\chi + \tfrac{5}{128}\cos 4\chi\right)s^3$$
$$+ \left(\frac{18l+1}{256}\cos\chi - \frac{35l+32}{256}\cos 3\chi + \tfrac{35}{2048}\cos 5\chi\right)s^4 + \ldots\Big\}$$

$$J_3 = \frac{1}{c^3R^2}\Big\{\cos 2\chi + \left(\tfrac{9}{4}\cos\chi + \tfrac{1}{4}\cos 3\chi\right)s$$
$$+ \left(\frac{52l-21}{32} + \tfrac{1}{2}\cos 2\chi + \tfrac{3}{32}\cos 4\chi\right)s^2$$
$$+ \left(-\frac{20l+127}{64}\cos\chi + \tfrac{21}{128}\cos 3\chi + \tfrac{5}{128}\cos 5\chi\right)s^3 + \ldots\Big\}$$

$$J_4 = \frac{2!}{c^4R^3}\{\cos 3\chi + \left(\tfrac{5}{2}\cos 2\chi + \tfrac{1}{4}\cos 4\chi\right)s$$
$$+ \left(\tfrac{81}{16}\cos\chi + \tfrac{21}{32}\cos 3\chi + \tfrac{3}{32}\cos 5\chi\right)s^2 + \ldots\}$$

$$J_5 = \frac{3!}{c^5R^4}\{\cos 4\chi + \left(\tfrac{35}{12}\cos 3\chi + \tfrac{1}{4}\cos 5\chi\right)s + \ldots\}$$

$$J_6 = \frac{4!}{c^6R^5}\{\cos 5\chi + \ldots\}.$$

(D)

$$\frac{dJ_1}{dz} = -\frac{1}{cR}\left\{\sin\chi + \tfrac{1}{4}\sin 2\chi\, s + \left(-\frac{12l+11}{32}\sin\chi + \tfrac{3}{32}\sin 3\chi\right)s^2\right.$$
$$+\left(-\frac{9l+5}{32}\sin 2\chi + \tfrac{5}{128}\sin 4\chi\right)s^3$$
$$\left.+\left(-\frac{30l+9}{256}\sin\chi - \frac{360l+123}{2048}\sin 3\chi + \tfrac{35}{2048}\sin 5\chi\right)s^4 + \ldots\right\}$$

$$\frac{dJ_2}{dz} = -\frac{1}{c^2R^2}\left\{\sin 2\chi + (\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi)\, s + (\tfrac{1}{8}\sin 2\chi + \tfrac{3}{32}\sin 4\chi)\, s^2\right.$$
$$\left.+\left(-\frac{36l+31}{64}\sin\chi + \tfrac{3}{128}\sin 3\chi + \tfrac{5}{128}\sin 5\chi\right)s^3 + \ldots\right\}$$

$$\frac{dJ_3}{dz} = -\frac{2!}{c^3R^3}\{\sin 3\chi + (\tfrac{5}{4}\sin 2\chi + \tfrac{1}{4}\sin 4\chi)\, s$$
$$+ (\tfrac{17}{16}\sin\chi + \tfrac{11}{32}\sin 3\chi + \tfrac{3}{32}\sin 5\chi)\, s^2 + \ldots\}$$

$$\frac{dJ_4}{dz} = -\frac{3!}{c^4R^4}\{\sin 4\chi + (\tfrac{7}{4}\sin 3\chi + \tfrac{1}{4}\sin 5\chi)\, s + \ldots\}$$

$$\frac{dJ_5}{dz} = -\frac{4!}{c^5R^5}\{\sin 5\chi + \ldots\}.$$

The following formulæ are useful for questions involving the stream-line function. They are obtained by multiplying the expansions just given for J_1, J_2, &c., by ϖ or $c(1 - s\cos\chi)$.

(E)

$$J_1\varpi = l - \frac{l+1}{2}\cos\chi\, s + \left(\frac{2l+5}{16} - \frac{l}{16}\cos 2\chi\right)s^2 + \left(\frac{3l+5}{64}\cos\chi - \frac{3l-1}{192}\cos 3\chi\right)s^3$$
$$+\left(\frac{12l+11}{2048} + \frac{12l+17}{768}\cos 2\chi - \frac{15l-8}{3072}\cos 4\chi\right)s^4 + \ldots$$

$$J_2\varpi = \frac{1}{cR}\left\{\cos\chi + \left(\frac{2l-3}{4} - \tfrac{1}{4}\cos 2\chi\right)s + \left(-\frac{20l+17}{32}\cos\chi - \tfrac{1}{32}\cos 3\chi\right)s^2\right.$$
$$+\left(\frac{20l+47}{128} - \frac{8l+1}{64}\cos 2\chi - \tfrac{1}{128}\cos 4\chi\right)s^3$$
$$\left.+\left(\frac{18l+29}{256}\cos\chi - \frac{11l+1}{256}\cos 3\chi - \tfrac{5}{2048}\cos 5\chi\right)s^4 + \ldots\right\}$$

$$J_3\varpi = \frac{1}{c^2R^2}\left\{\cos 2\chi + (\tfrac{7}{4}\cos\chi - \tfrac{1}{4}\cos 3\chi)\, s + \left(\frac{52l-57}{32} - \tfrac{3}{4}\cos 2\chi - \tfrac{1}{32}\cos 4\chi\right)s^2\right.$$
$$\left.+\left(-\frac{124l+101}{64}\cos\chi - \tfrac{17}{128}\cos 3\chi - \tfrac{1}{128}\cos 5\chi\right)s^3 + \ldots\right\}$$

$$J_4\varpi = \frac{2!}{c^3R^3}\{\cos 3\chi + (2\cos 2\chi - \tfrac{1}{4}\cos 4\chi)\,s + (\tfrac{61}{16}\cos\chi - \tfrac{23}{32}\cos 3\chi - \tfrac{1}{32}\cos 5\chi)\,s^2 + \ldots\}$$

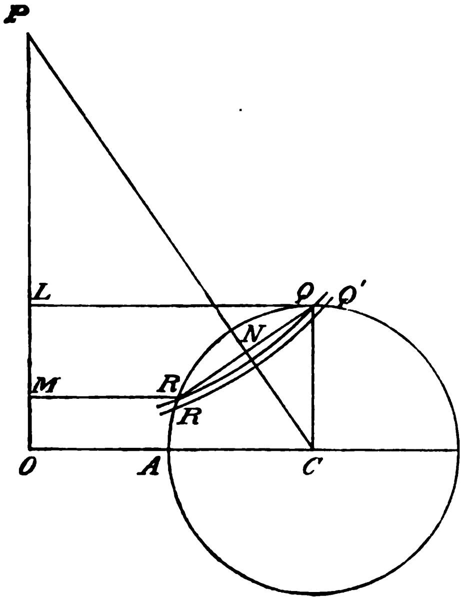
$$J_5\varpi = \frac{3!}{c^4R^4}\{\cos 4\chi + (\tfrac{29}{12}\cos 3\chi - \tfrac{1}{4}\cos 5\chi)\,s + \ldots\}$$

$$J_6\varpi = \frac{4!}{c^5R^5}\{\cos 5\chi + \ldots\}.$$

These five sets of formulæ will be referred to as (A), (B), (C), (D), (E).

Section II.—*The Potential of an Anchor Ring at an External Point.*

§ 4. The potential of an anchor ring at a point on its axis may be easily found in several ways. One simple method is to divide the ring into elements by spheres, having the given point as centre.



Let QAR be a circle which by revolution round OP generates an anchor ring. Let C be its centre, and let OC be perpendicular to OP.

With centre P describe circles, dividing the circle QAR into elements; let QR and Q′R′ be two of these circles.

By revolution of the figure round OP we obtain an anchor ring divided into elements.

Let

$$AC = a. \qquad OC = c. \qquad PC = R. \quad PQ = \rho.$$

$$\angle\, PCQ = \psi \qquad \angle\, OCP = \alpha.$$

Then

$$\rho^2 = \mathrm{R}^2 + a^2 - 2a\mathrm{R}\cos\psi.$$

Therefore

$$\rho\, d\rho = a\mathrm{R}\sin\psi\, d\psi.$$

The volume formed by the revolution of QRR′Q′ is

$$2\pi\rho\, d\rho . \mathrm{LM}.$$

The potential of this at P

$$= \frac{2\pi\rho\, d\rho . \mathrm{LM}}{\rho}$$

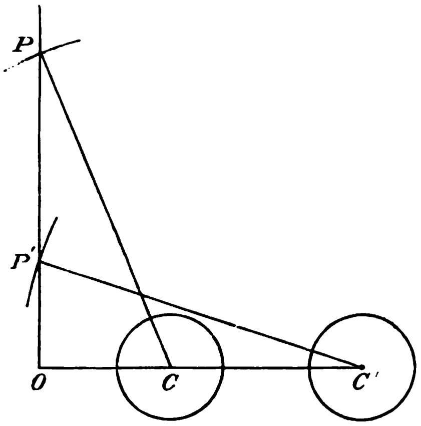
$$= 2\pi \frac{a\mathrm{R}\sin\psi . d\psi 2a\sin\psi . \cos\alpha}{\sqrt{\{\mathrm{R}^2 + a^2 - 2a\mathrm{R}\cos\psi\}}}$$

$$= 4\pi a^2 c \frac{\sin^2\psi\, d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}}.$$

Therefore the potential of the whole ring

$$= 2\frac{\mathrm{M}}{\pi}\int_0^\pi \frac{\sin^2\psi\, d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}},$$

where M is the mass of the ring.



Let there be two anchor rings of different radii, but of equal generating circles, having the same axis and centre.

Let C and C′ be the centres of the generating circles.

Let P and P′ be points on the axis such that CP = C′P′.

If the densities of the rings be inversely proportional to the distance of the centres of their generating circles from the axis of revolution, that is, if the densities be made such that the masses of the rings are equal, then the above formula shows that the potential of the C ring at P is equal to the potential of the C′ ring at P′.

§ 5. The integral

$$\int_0^\pi \frac{\sin^2\psi\, d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}}$$

may be reduced to elliptic functions.

Let it be called I. Then

$$\begin{aligned} \mathrm{I} &= -\int_0^\pi \frac{\sin\psi\, d(\cos\psi)}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}} \\ &= -\frac{1}{a\mathrm{R}}\int_0^\pi \cos\psi \sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}\, d\psi \\ &= -\frac{1}{a\mathrm{R}}\int_0^\pi \frac{(\mathrm{R}^2 + a^2)\cos\psi - 2a\mathrm{R}\,(1 - \sin^2\psi)}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}}\, d\psi. \end{aligned}$$

Therefore

$$3\mathrm{I}^{*} = 2\int_0^\pi \frac{d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}} - \left(\frac{\mathrm{R}}{a} + \frac{a}{\mathrm{R}}\right)\int_0^\pi \frac{\cos\psi\, d\psi}{\sqrt{\{\mathrm{R}^2 - 2a\mathrm{R}\cos\psi + a^2\}}}.$$

Writing α for a/R,

$$\mathrm{I} = \frac{2}{3\mathrm{R}}\int_0^\pi \frac{d\psi}{\sqrt{\{1 - 2\alpha\cos\psi + \alpha^2\}}} - \frac{1}{3\mathrm{R}}\left(\alpha + \frac{1}{\alpha}\right)\int_0^\pi \frac{\cos\psi\, d\psi}{\sqrt{\{1 - 2\alpha\cos\psi + \alpha^2\}}}.$$

These integrals may be transformed by putting

$$\sin(\phi - \psi) = \alpha\sin\phi.$$

This gives

$$\begin{aligned} d\psi &= d\phi\left\{1 - \frac{\alpha\cos\phi}{\sqrt{\cos(\phi - \psi)}}\right\} \\ &= d\phi\left\{1 - \frac{\alpha\cos\phi}{\sqrt{(1 - \alpha^2\sin^2\phi)}}\right\}. \end{aligned}$$

Also

$$\sin\phi\,(\cos\psi - \alpha) = \cos\phi\sin\psi.$$

Therefore

$$\sin^2\phi\,(\cos\psi - \alpha)^2 = \cos^2\phi\sin^2\psi,$$

or

$$\cos^2\psi - 2\alpha\cos\psi + \alpha^2 = \cot^2\phi - \cot^2\phi\cos^2\psi.$$

Therefore

$$\cos^2\psi\operatorname{cosec}^2\phi - 2\alpha\cos\psi + \alpha^2 = \cot^2\phi,$$

or

$$\begin{aligned} (\cos\psi - \alpha\sin^2\phi)^2 &= \cos^2\phi - \alpha^2\sin^2\phi + \alpha^2\sin^4\phi \\ &= \cos^2\phi - \alpha^2\sin^2\phi\cos^2\phi. \end{aligned}$$

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[*] This method of reduction was given by one of the Referees.

Therefore

$$\cos\psi = \alpha\sin^2\phi + \cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}.$$

Therefore

$$1 - 2\alpha\cos\psi + \alpha^2 = 1 + \alpha^2 - 2\alpha^2\sin^2\phi - 2\alpha\cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}$$
$$= [\sqrt{(1-\alpha^2\sin^2\phi)} - \alpha\cos\phi]^2.$$

Therefore

$$\sqrt{(1-2\alpha\cos\psi+\alpha^2)} = \{\sqrt{(1-\alpha^2\sin^2\phi)} - \alpha\cos\phi\}.$$

Thus

$$\frac{d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}}$$

and

$$\frac{\cos\psi\, d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = \frac{\alpha\sin\phi + \cos\phi\sqrt{(1-\alpha^2\sin^2\phi)}}{\sqrt{(1-\alpha^2\sin^2\phi)}}\, d\phi,$$

giving

$$\int_0^\pi \frac{d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = 2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} = 2\mathrm{F}$$

$$\int_0^\pi \frac{\cos\psi\, d\psi}{\sqrt{(1-2\alpha\cos\psi+\alpha^2)}} = 2\alpha\int_0^{\frac{\pi}{2}} \frac{\sin^2\phi\, d\phi}{\sqrt{(1-\alpha^2\sin^2\phi)}} = \frac{2}{\alpha}(\mathrm{F}-\mathrm{E}).$$

Therefore

$$\int_0 \frac{\sin^2\psi\, d\psi}{\sqrt{\{1-2\alpha\cos\psi+\alpha^2\}}} = \frac{2}{3}\left\{4\mathrm{F} - \frac{2}{\alpha}\left(\alpha+\frac{1}{\alpha}\right)(\mathrm{F}-\mathrm{E})\right\}$$
$$= \frac{4}{3}\left\{\mathrm{F}\left(1-\frac{1}{\alpha^2}\right) + \mathrm{E}\left(1+\frac{1}{\alpha^2}\right)\right\}.$$

Therefore the potential

$$\mathrm{V} = \frac{4\mathrm{M}}{3\pi\mathrm{R}}\left\{\left(1-\frac{1}{\alpha^2}\right)\mathrm{F} + \left(1+\frac{1}{\alpha^2}\right)\mathrm{E}\right\}.$$

Now

$$\mathrm{F} = \frac{\pi}{2}\left\{1 + \frac{1^2}{2^2}\alpha^2 + \frac{1^2.3^2}{2^2.4^2}\alpha^4 + \&\mathrm{c.}\right\}$$

and

$$\mathrm{E} = \frac{\pi}{2}\left\{1 - \frac{1.1}{2^2}\alpha^2 - \frac{1^2.3}{2^2.4^2}\alpha^4 - \&\mathrm{c.}\right\}.$$

Substituting, we find

$$\mathrm{V} = \frac{\mathrm{M}}{\mathrm{R}}\left\{1 - \frac{1}{8}\alpha^2 - \frac{1}{64}\alpha^4 - \frac{5}{1024}\alpha^6 - \&\mathrm{c.} - 2\,\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\alpha^{2n} - \&\mathrm{c.}\right\}$$
$$= \mathrm{M}\left\{\frac{1}{\mathrm{R}} - \frac{1}{8}\frac{a^2}{\mathrm{R}^3} - \frac{1}{64}\frac{a^4}{\mathrm{R}^5} - \frac{5}{1024}\frac{a^6}{\mathrm{R}^7} - \&\mathrm{c.} - 2\,\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\frac{a^{2n}}{\mathrm{R}^{2n+1}} - \&\mathrm{c.}\right\}.$$

This series is less than the series whose general term is

$$\frac{1}{2n^2}\frac{a^{2n}}{\mathrm{R}^{2n+1}},$$

and is therefore convergent if $a = \mathrm{R}$ or $a < \mathrm{R}$.

§ 6. To find the potential of an anchor ring at any external point.

When $a = 0$, that is when the generating circle of the ring is infinitesimal, the potential at the point r, θ is

$$\frac{\mathrm{M}}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}.$$

When $\theta = 0$, and $r = z$, *i.e.*, for a point on the axis, the expression reduces to $\mathrm{M}/\pi\ \pi/\mathrm{R}$, *i.e.*, to M/R.

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{\{r^2 + c^2 - 2cr\sin\theta\cos\phi\}}}$$

satisfies LAPLACE's equation, and at a point on the axis of the ring becomes

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(z^2 + c^2)}} = (-1)^n.\pi\, 1.\,3.\,5\,\ldots\,(2n-1)\frac{1}{(z^2+c^2)^{n+\frac{1}{2}}}$$
$$= (-1)^n.\pi\frac{1.\,3.\,5\,\ldots\,(2n-1)}{\mathrm{R}^{2n+1}}.$$

Now at any point on the axis of the ring

$$\mathrm{V} = \mathrm{M}\left\{\frac{1}{\mathrm{R}} - \frac{1}{8}\frac{a^2}{\mathrm{R}^3} - \frac{1}{64}\frac{a^4}{\mathrm{R}^5} - \frac{5}{1024}\frac{a^6}{\mathrm{R}^7} - \&\text{c.} - 2\,\frac{1^2.3^2\ldots(2n-3)^2.(2n-1)}{2^2.4^2\ldots(2n-2)^2.(2n)^2.(2n+2)}\frac{a^{2n}}{\mathrm{R}^{2n+1}} - \&\text{c.}\right\}.$$

Therefore, at any external point r, θ,

$$\mathrm{V} = \frac{\mathrm{M}}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \frac{a^2}{8}\frac{d}{c\,dc}\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}\right.$$
$$- \frac{a^4}{192}\left(\frac{d}{c\,dc}\right)^2\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \&\text{c.}$$
$$\left. + (-1)^{n+1}\frac{2a^{2n}}{2n+2}\,\frac{1.\,3.\,5\ldots(2n-3)}{2^2.4^2.6^2\ldots(2n)^2}\left(\frac{d}{c\,dc}\right)^n\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}} + \&\text{c.}\right\}.$$

For this series is finite at all external points, satisfies LAPLACE's equation, vanishes at infinity, and agrees with the value of V on the axis. The series is very convergent, as is seen by the next paragraph.

§ 7. The integral

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\sin\theta\cos\phi)}}$$

takes a simple form at points in the plane of the central circle. For, putting $\theta = \pi/2$, it becomes

$$\left(\frac{d}{c\,dc}\right)^n \int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\cos\phi)}}.$$

Now

$$\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - 2cr\cos\phi)}} = \int_0^\pi \frac{d\phi}{\sqrt{(r^2 - c^2\sin^2\phi)}}$$

when $r > c$, and

$$= \int_0^\pi \frac{d\phi}{\sqrt{(c^2 - r^2\sin^2\phi)}}$$

when $r < c$.

Therefore at points in the plane of x, y, further from the axis than the ring,

$$V = \frac{M}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(r^2 - c^2\sin^2\phi)}} + \frac{a^2}{8}\int_0^\pi \frac{\sin^2\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{\frac{3}{2}}} - \&c.\right.$$
$$\left. + (-1)^n \frac{1^2.3^2 \ldots (2n-3)^2\,(2n-1)\,2a^{2n}}{2^2.4^2 \ldots (2n-2)^2\,(2n)^2\,(2n+2)}\int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} + \&c.\right\}.$$

And at a point in the plane of xy between the axis and the ring

$$V = \frac{M}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(c^2 - r^2\sin^2\phi)}} - \frac{a^2}{8}\int_0^\pi \frac{d\phi}{(c^2 - r^2\sin^2\phi)^{\frac{3}{2}}} - \&c.\right.$$
$$\left. - \frac{1^2.3^2 \ldots (2n-3)^2\,(2n-1)\,2a^{2n}}{2^2.4^2 \ldots (2n-2)^2\,(2n)^2\,(2n+2)}\int_0^\pi \frac{d\phi}{(c^2 - r^2\sin^2\phi)^{n+\frac{1}{2}}} - \&c.\right\}.$$

[* The series in its general form must converge at about the same rate at which these two particular cases do. Their convergency is easily discussed. For

$$\int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} < \frac{\pi}{(r^2 - c^2)^{n+\frac{1}{2}}},$$

and r is $> a + c$.

Thus

$$\int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(r^2 - c^2\sin^2\phi)^{n+\frac{1}{2}}} < \frac{\pi}{a^{n+\frac{1}{2}}(2c + a)^{n+\frac{1}{2}}} < \frac{\pi}{a^{n+\frac{1}{2}}(2c)^{n+\frac{1}{2}}}.$$

Hence at all points in the plane of the central circle, beyond the ring, the series is more convergent than the series whose general term is

$$(-1)^n \frac{2a^{2n}}{2n(2n+2)}\,\frac{1}{2^n a^n c^n},$$

i.e., than the series whose n^{th} term is

$$\frac{1}{n(n+1)}\left(\frac{a}{2c}\right)^n.$$

Similarly at points within the ring, in the plane of the central circle, the series is more convergent than the series whose general term is

$$\frac{1}{n(n+1)}\left(\frac{a}{2c - a}\right)^n.]$$

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* This consideration of the convergency was inserted at the suggestion of one of the Referees, August, 1892.

§ 8. For the purposes of calculation, it is convenient to transform the integrals of § 6 by **Landen's** theorem.

$$\int_0^{\pi} \frac{d\phi}{\sqrt{\{r^2 - 2cr\sin\theta\cos\phi + c^2\}}} = 2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{\mathrm{R}_1^2 - (\mathrm{R}_1^2 - \mathrm{R}^2)\sin^2\phi\}}},$$

$$= \frac{2}{\mathrm{R}_1}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\left\{1 - \frac{\mathrm{R}_1^2 - \mathrm{R}^2}{\mathrm{R}_1^2}\sin\phi\right\}}},$$

$$= \frac{4\mathrm{F}}{\mathrm{R} + \mathrm{R}_1};$$

where

$$\mathrm{F} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \mu^2\sin^2\phi}},$$

and

$$\mu = \frac{\mathrm{R}_1 - \mathrm{R}}{\mathrm{R}_1 + \mathrm{R}},$$

R and R_1 being the least and greatest distances of the point r, θ, from the circular axis of the ring.

Now,

$$\frac{d\mathrm{F}}{c\,dc} = \frac{d\mathrm{F}}{d\mu}\frac{d\mu}{c\,dc} = \frac{\mathrm{E} - \mu'^2\mathrm{F}}{\mu\mu'^2}\frac{d\mu}{c\,dc}$$

and

$$\left.\frac{d\mathrm{E}}{c\,dc} = \frac{d\mathrm{E}}{d\mu}\frac{d\mu}{c\,dc} = \frac{\mathrm{E} - \mathrm{F}}{\mu}\frac{d\mu}{c\,dc}\right\}.$$

[Cayley, 'Ell. Func.,' p. 48.]

But

$$\mathrm{R}^2 = r^2 + c^2 - 2cr\sin\theta.$$

Therefore

$$\frac{d\mathrm{R}}{dc} = \frac{c - r\sin\theta}{\mathrm{R}} = \frac{4c^2 + \mathrm{R}^2 - \mathrm{R}_1^2}{4c\mathrm{R}}.$$

Similarly

$$\frac{d\mathrm{R}_1}{dc} = \frac{c + r\sin\theta}{\mathrm{R}} = \frac{4c^2 + \mathrm{R}_1^2 - \mathrm{R}^2}{4c\mathrm{R}}.$$

Therefore

$$\frac{d\mu}{dc} = \frac{(\mathrm{R}_1 + \mathrm{R})\left(\frac{d\mathrm{R}_1}{dc} - \frac{d\mathrm{R}}{dc}\right) - (\mathrm{R}_1 - \mathrm{R})\left(\frac{d\mathrm{R}_1}{dc} + \frac{d\mathrm{R}}{dc}\right)}{(\mathrm{R}_1 + \mathrm{R})^2},$$

$$= 2\,\frac{\mathrm{R}\frac{d\mathrm{R}_1}{dc} - \mathrm{R}_1\frac{d\mathrm{R}}{dc}}{(\mathrm{R}_1 + \mathrm{R})^2},$$

$$= \frac{\mathrm{R}_1 - \mathrm{R}}{\mathrm{R}_1 + \mathrm{R}}\,\frac{\mathrm{R}^2 + \mathrm{R}_1^2 - 4c^2}{2c\mathrm{R}\mathrm{R}_1},$$

$$= \frac{\mu}{c}\cos\psi;$$

where ψ is the angle between R and R_1.

Hence

$$\frac{d}{c\,dc}\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 - 2cr\sin\theta\cos\phi + c^2\}}} = 4\,\frac{d}{c\,dc}\left(\frac{F}{R + R_1}\right)$$

$$= \frac{1}{c^2}\left\{E\,\frac{R + R_1}{RR_1}\cos\psi - \frac{4F}{R + R_1}\cos^2\frac{\psi}{2}\right\}.$$

Differentiating again we shall find

$$\left(\frac{d}{c\,dc}\right)^2\int_0^\pi \frac{d\phi}{\sqrt{\{r^2 - 2cr\sin\theta\cos\phi + c^2\}}}$$

$$= \frac{E\,(R + R_1)}{c^4 RR_1}\left\{2\cos^3\psi - 4\cos\psi + \frac{2c^2}{RR_1}\cos 2\psi\right\}$$

$$+ \frac{F}{c^4(R + R_1)}\left\{5 + 8\cos\psi - \cos^2\psi - 4\cos^3\psi - \frac{4c^2}{RR_1}\cos 2\psi\right\}.$$

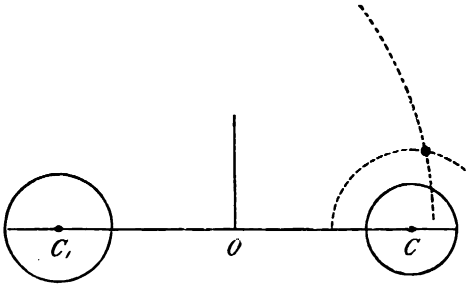
Therefore, at any external point,

$$V = \frac{4F}{R + R_1}\left\{1 - \tfrac{1}{8}\frac{a^2}{c^2}\cos^2\frac{\psi}{2} - \tfrac{1}{768}\frac{a^4}{c^4}\left[5 + 8\cos\psi - \cos^2\psi - 4\cos^3\psi - \frac{4c^2}{RR_1}\cos 2\psi\right] + \&c.\right\}$$

$$+ \frac{E\,(R + R_1)}{RR_1}\left\{\frac{a^2}{8c^2}\cos\psi - \tfrac{1}{192}\frac{a^4}{c^4}\left[2\cos^2\psi - 4\cos\psi + \frac{2c^2}{RR_1}\cos 2\psi\right] + \&c.\right\},$$

where ψ is the angle between R and R_1, and the modulus of the elliptic functions is

$$\frac{R_1 - R}{R_1 + R}.$$



I have calculated the value of V, retaining only the terms

$$\frac{4F}{R + R_1}\left\{1 - \tfrac{1}{8}\frac{a^2}{c^2}\cos^2\frac{\psi}{2}\right\} + \frac{E\,(R + R_1)}{RR_1}\frac{a^2}{8c^2}\cos\psi,$$

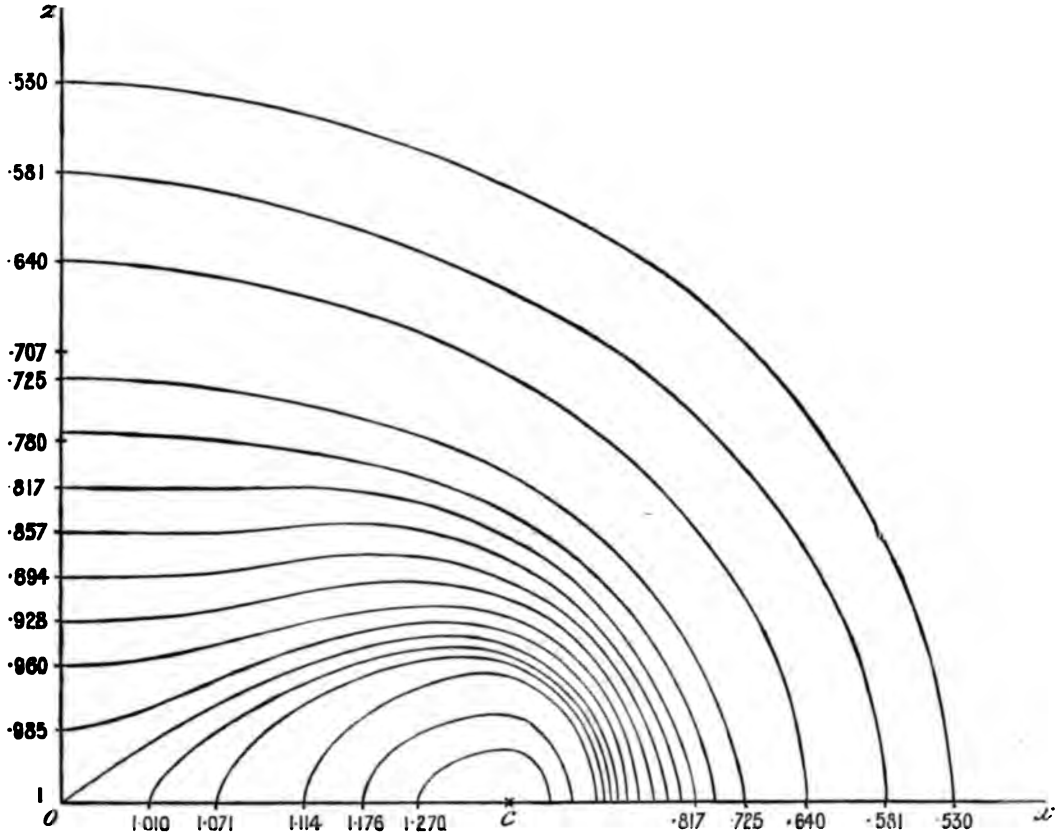
for the values

$$a = 0,\ \frac{c}{5},\ \frac{2c}{5},\ \frac{3c}{5},\ \frac{4c}{5},\ c,$$

and drawn the equipotential surfaces. The method consisted in drawing circles of known radii, with centres, C and C_1; and finding the value of V at their points of intersection.

In the most unfavourable case, where $a = c$, the value of the potential is not more than three per cent. in error at any point, the introduction of the terms in a^4/c^4 reduces the error to less than one and a half per cent. When $a < c$, the approximation is considerably nearer.

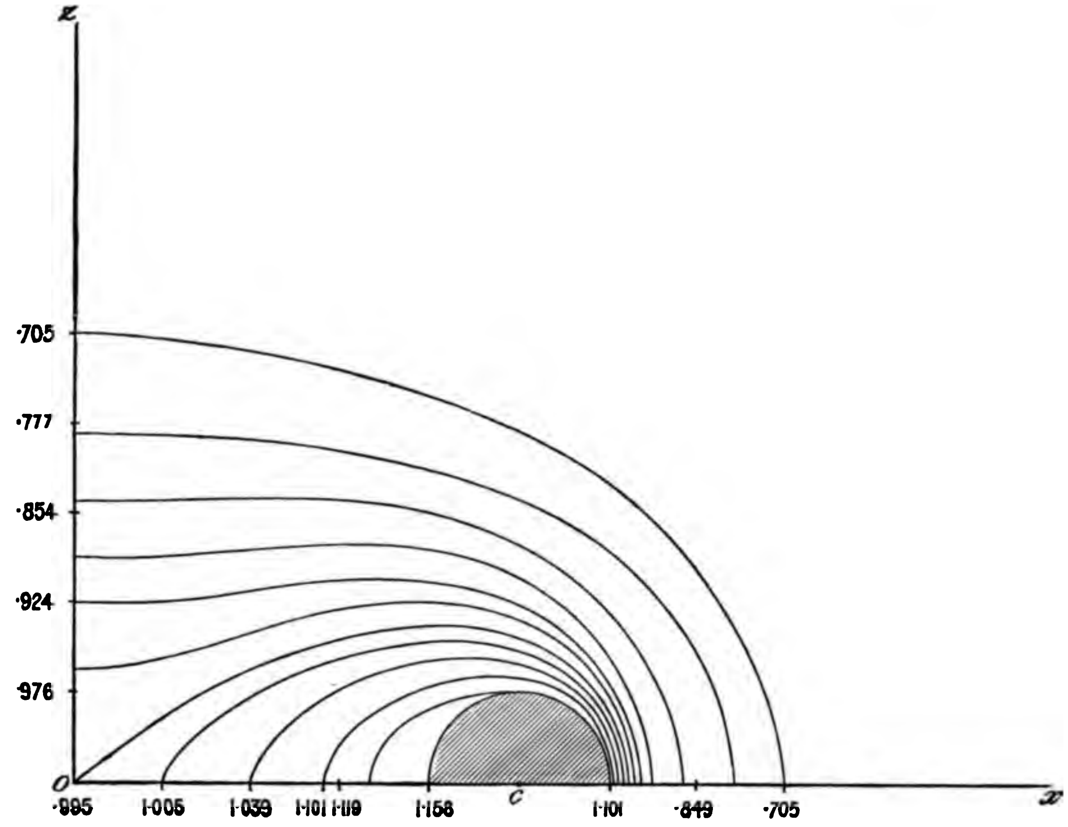
Fig. 1.



Equipotential surfaces of ring formed by the revolution of the point C about Oz.

$\left.\begin{array}{r}\text{If Mass} = \text{M}\\ \text{and OC} = c\end{array}\right\}$ Then $\frac{\text{M}}{c}$ is taken $= 1$.

Fig. 2.



Section of Equipotential surfaces of the ring formed by revolution of shaded ⊙ round Oz.
Mass = M. OC = c.

and $\frac{M}{c}$ is taken = 1.

Fig. 3.

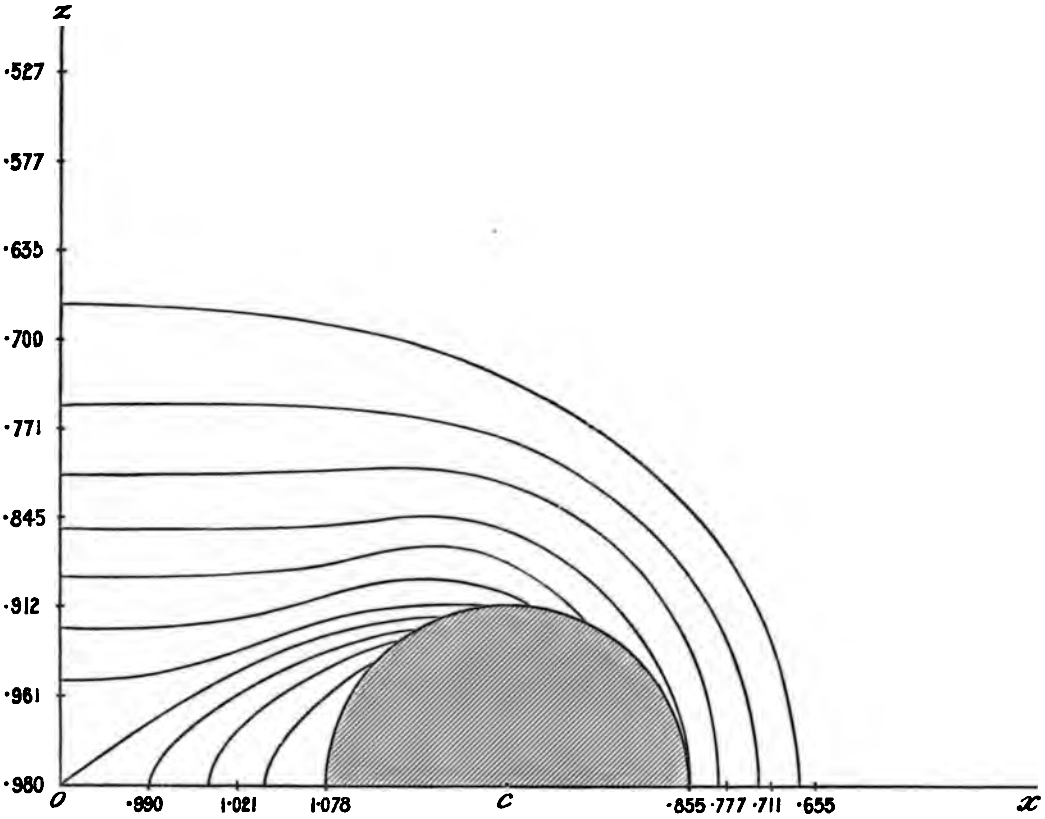

z
·527
·577
·635
·700
·771
·845
·912
·961
·980
O
·990
1·021
1·078
C
·855
·777
·711
·655
x

Fig. 4.

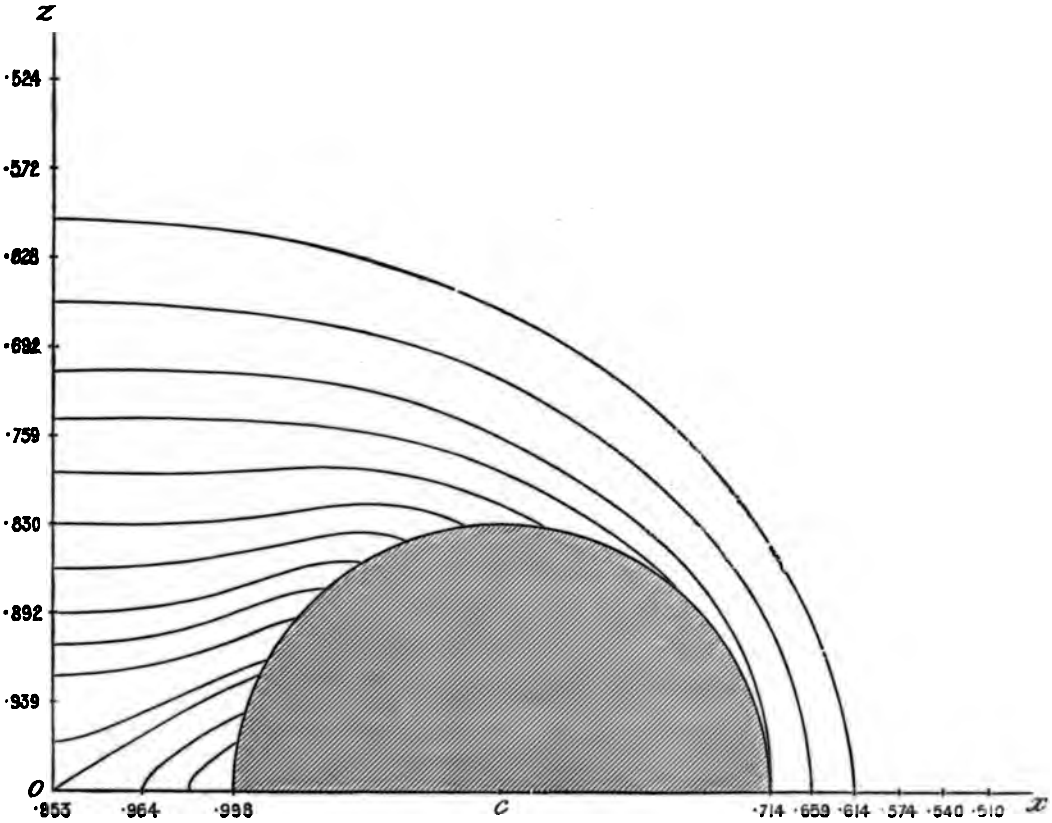

z
·524
·572
·628
·692
·759
·830
·892
·939
O
·953
·964
·998
C
·714
·659
·614
·574
·540
·510
x

Fig. 5.

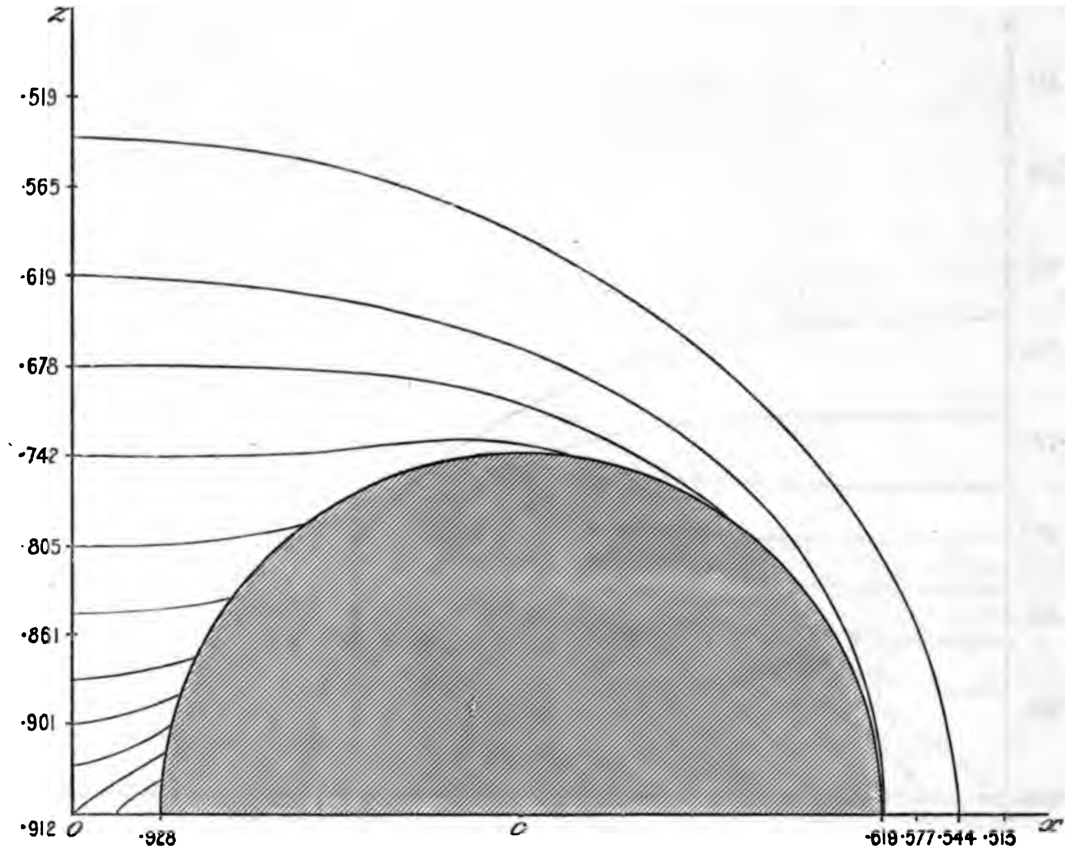
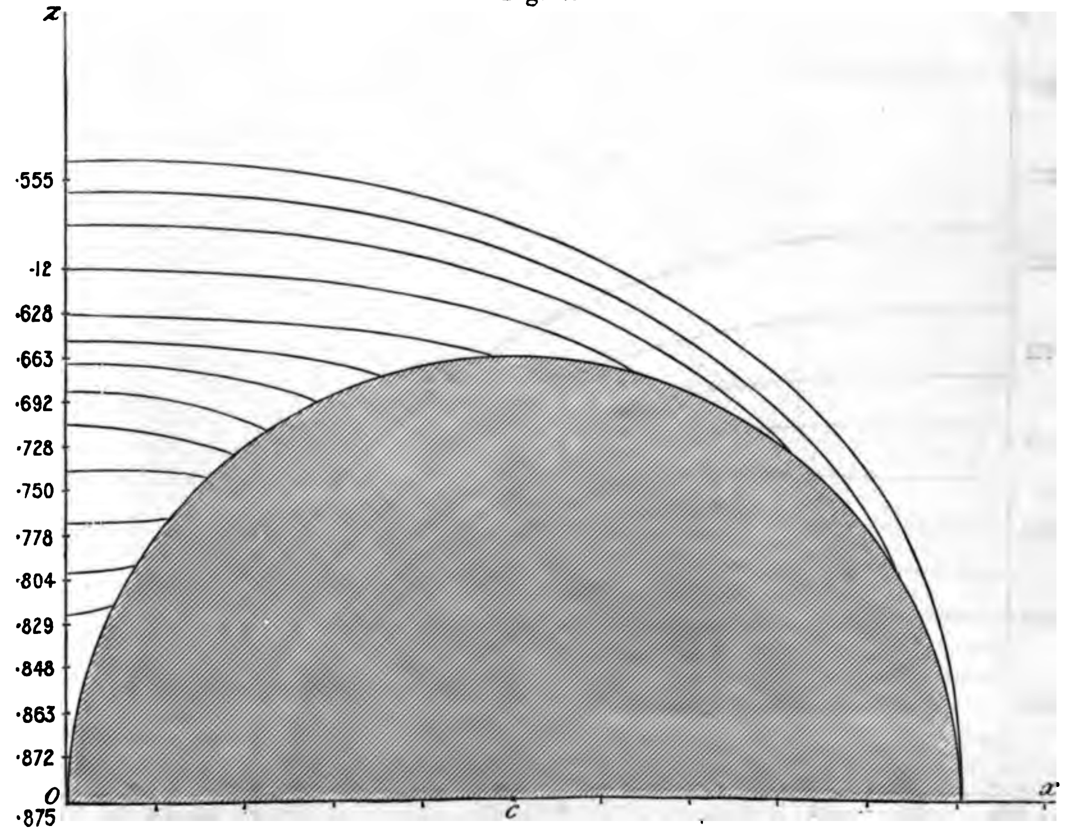

z
·519
·565
·619
·678
·742
·805
·861
·901
·912
O
·928
C
·618
·577
·544
·515
x

Fig. 6.


z
·555
·12
·628
·663
·692
·728
·750
·778
·804
·829
·848
·863
·872
O
·875
C
x

Section III.—*Electric Potential of an Anchor Ring.*

§ 9. The potential at any external point may be taken to be

$$A_1I_1 + a^2A_2I_2 + a^4A_3I_3 + \&c.$$

For this is a solution of LAPLACE's equation, finite at all points outside the ring, and vanishing at infinity.

The constants are to be determined by the condition that at the surface of the ring the potential is the constant V_0.

Using the expansions of § 3 (A.), we have that at any external point

$$\begin{aligned} Vc = {} & A_1\left\{l + 2 + \frac{2l+1}{16}s^2 + \frac{108l+27}{2048}s^4\right\} + A_2\sigma^2\left\{\frac{2l+3}{4} + \frac{12l+5}{128}s^2\right\} + A_3\sigma^4\frac{36l+51}{32} \\ & + \cos\chi\left\{A_1\left(\frac{l+1}{2}s + \frac{9l+3}{64}s^3\right) + A_2\frac{\sigma^2}{s}\left(1 + \frac{12l+11}{32}s^2\right) + A_3\sigma^4\frac{9}{4s}\right\} \\ & + \cos 2\chi\left\{A_1\left(\frac{3l+2}{16}s^2 + \frac{20l+4\frac{1}{3}}{256}s^4\right) + A_2\sigma^2\left(\tfrac{1}{4} + \frac{12l+9}{64}s^2\right)\right. \\ & \qquad\qquad\qquad \left. + A_3\sigma^4\left(\frac{1}{s^2} + \tfrac{3}{4}\right) + A_4\sigma^6\frac{5}{s^2}\right\} \\ & + \cos 3\chi\left\{A_1\frac{5l+2\frac{1}{3}}{64}s^3 + A_2\sigma^2\frac{3s}{32} + A_3\sigma^4\frac{1}{4s} + A_4\sigma^6\frac{2}{s^3}\right\} \\ & + \cos 4\chi\left\{A_1\frac{35l+11\frac{1}{3}}{1024}\sigma^4 + A_2\sigma^2\tfrac{5}{128}s^2 + A_3\sigma^4\tfrac{3}{32} + A_4\sigma^6\frac{1}{2s^2} + A_5\sigma^8\frac{6}{s^4}\right\}. \end{aligned}$$

All terms have been retained which will, when R is put $= a$ or s put $= \sigma$, be of the order σ^4.

$$\frac{dV}{dn} = \frac{dV}{dR} = \frac{dV}{a},$$

s being put $= \sigma$, after differentiation.

Therefore

$$\begin{aligned} ac\frac{dV}{dn} = {} & \left\{A_1\left(-1 + \frac{\lambda}{4}\sigma^2 + \frac{27\lambda}{128}\sigma^4\right) + A_2\left(-\frac{\sigma^2}{2} + \frac{12\lambda-1}{64}\sigma^4\right) - \tfrac{9}{8}A_3\sigma^4\right\} \\ & + \sigma\cos\chi\left\{A_1\left(\frac{\lambda}{2} + \frac{27\lambda}{64}\sigma^2\right) + A_2\left(-1 + \frac{12\lambda-1}{32}\sigma^2\right) - \tfrac{9}{4}A_3\sigma^2\right\} \\ & + \sigma^2\cos 2\chi\left\{A_1\frac{6\lambda+1}{16} + \frac{10\lambda-\frac{1}{3}}{32}\sigma^2) + A_2\frac{12\lambda+3}{32}\sigma^2 - 2A_3 - 10A_4\sigma^2\right\} \\ & + \sigma^3\cos 3\chi\left\{A_1\frac{15\lambda+2}{64} + A_2\tfrac{3}{32} - A_3\tfrac{1}{4} - A_4 6\right\} \\ & + \sigma^4\cos 4\chi\left\{A_1\frac{140\lambda+10\frac{1}{3}}{1024} + A_2\tfrac{5}{64} - A_4 - 24A_5\right\}. \end{aligned}$$

The density at any point is $-\frac{1}{4\pi}\frac{dV}{dn}$.

The charge

$$
\begin{aligned}
&= -\frac{1}{4\pi}\iint \frac{dV}{dn}\,ds \\
&= -\int_0^{2\pi}\int_0^{2\pi} \frac{1}{4\pi}\frac{dV}{dn}(c - a\cos\chi)\,a\,d\chi\,d\phi \\
&= -\frac{ac}{2}\int_0^{2\pi}\frac{dV}{dn}(1 - \sigma\cos\chi)\,d\chi \\
&= \pi A_1.
\end{aligned}
$$

This value is evidently correct, for the potential at a great distance is $A_1 J_1$ or

$$A_1\int_0^{\pi}\frac{d\phi}{\sqrt{r^2 + c^2 - 2cr\sin\theta\cos\phi}} = \frac{\pi A_1}{r}.$$

Putting $s = \sigma$ in the expression for V, we find

$$
\begin{aligned}
A_2 &= -A_1\left(\frac{\lambda+1}{2} - \frac{12\lambda^2 + 23\lambda + 8}{64}\sigma^2\right) \\
A_3 &= -A_1\left(\frac{\lambda}{16} - \frac{12\lambda^2 + 21\lambda - 1}{256}\sigma^2\right) \\
A_4 &= -A_1\frac{\lambda - \frac{2}{3}}{128} \\
A_5 &= -A_1\frac{5\lambda - 6}{6144}.
\end{aligned}
$$

Substituting these values we find that

$$V_0 = \frac{A_1}{c}\left\{\lambda + 2 - \frac{4\lambda^2 + 8\lambda + 5}{16}\sigma^2 + \frac{192\lambda^3 + 416\lambda^2 + 448\lambda + 171}{2048}\sigma^4, \text{ \&c.}\right\}$$

and

$$
\begin{aligned}
-\frac{1}{4\pi}\frac{dV}{dn} = \frac{A_1}{4\pi ac}&\left\{1 - \left(\frac{2\lambda+1}{2} - \frac{24\lambda^2 + 7}{64}\sigma^2\right)\left(\frac{\sigma^2}{2} + \sigma\cos\chi\right)\right. \\
&\left. - \left(\frac{8\lambda - 1}{16} - \frac{36\lambda^2 + \lambda + 13}{128}\sigma^2\right)\sigma^2\cos 2\chi - \frac{16\lambda - 3}{64}\sigma^3\cos 3\chi - \frac{11(8\lambda - 1)}{1024}\sigma^4\cos 4\chi\right\}.
\end{aligned}
$$

[The capacity (q)

$$
\begin{aligned}
&= \pi A_1/V \\
&= \pi c\Big/\left\{\lambda + 2 - \frac{4\lambda^2 + 8\lambda + 5}{16}\sigma^2 + \frac{192\lambda^3 + 416\lambda^2 + 448\lambda + 171}{2048}\sigma^4\right\}.
\end{aligned}
$$

When

$$\sigma = 1 \qquad \lambda = 2{\cdot}3820 \qquad \frac{\pi c}{q} = 4{\cdot}3820 - {\cdot}0286 + {\cdot}0003$$

$$\sigma = 2 \qquad \lambda = 1{\cdot}6889 \qquad \frac{\pi c}{q} = 3{\cdot}6889 - {\cdot}0798 + {\cdot}0022$$

$$\sigma = 3 \qquad \lambda = 1{\cdot}2834 \qquad \frac{\pi c}{q} = 3{\cdot}2834 - {\cdot}1229 + {\cdot}0073$$

$$\sigma = 4 \qquad \lambda = {\cdot}9957 \qquad \frac{\pi c}{q} = 2{\cdot}9957 - {\cdot}1691 + {\cdot}0152$$

$$\sigma = 5 \qquad \lambda = {\cdot}7726 \qquad \frac{\pi c}{q} = 2{\cdot}7726 - {\cdot}1818 + {\cdot}0260$$

The numbers are given which arise from each term, as they serve to indicate roughly the convergency of the series.

They give

$$\begin{aligned} q &= {\cdot}7216c && \text{for } \sigma = 1, \\ q &= {\cdot}8700c && \text{for } \sigma = 2, \\ q &= {\cdot}9917c && \text{for } \sigma = 3, \\ q &= 1{\cdot}106c && \text{for } \sigma = 4, \\ q &= 1{\cdot}200c && \text{for } \sigma = 5, \end{aligned}$$

which agree with some figures given by Mr. HICKS in the 'Phil. Trans.,' 1881. Aug., 1892.]

Section IV.—*Motion of an Anchor Ring in an Infinite Fluid.*

§ 10. All cases of motion of an anchor ring in a fluid may be found by compounding

(i.) Linear motion parallel to the axis of the ring.
(ii.) Linear motion perpendicular to the axis of the ring.
(iii.) Rotation round a diameter of the central circle.
(iv.) Cyclic motion through the ring.

Let Φ be the velocity potential in cases (i.), (ii.), or (iii.), and Ψ be STOKES' stream-line function in cases (i.) or (iv.).

Then (*a*) Φ and Ψ are single-valued functions;

(*b*) They and their differential coefficients are finite and continuous at all points of the fluid, and vanish at infinity;

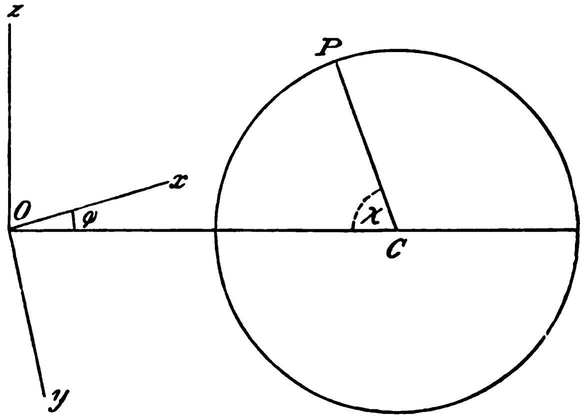
(*c*) Φ satisfies the equation $\nabla^2\Phi = 0$, and Ψ satisfies the equation

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} = 0;$$

(*d*) $d\Phi/dn$ has a definite value at the surface of the ring;
Ψ has a definite value at the surface of the ring.

The different cases must be considered separately.

Let the figure represent a section of the anchor ring through the axis Oz, at an azimuth ϕ.



Let P be a point on the surface and let $\angle$ OCP $= \chi$.

(i.) When the ring moves parallel to Oz, with velocity w;

$$\frac{d\Phi}{dn} = w \sin \chi.$$

(ii.) When it moves parallel to Ox with velocity u,

$$\frac{d\Phi}{dn} = -u \cos \chi \cos \phi.$$

(iii.) When it rotates round Oy with angular velocity ω_2,

$$\frac{d\Phi}{dn} = -\mathrm{C}\omega_2 \sin \chi \cos \phi.$$

The stream function for motion parallel to Oz, with velocity w, satisfies at the surface the equation

$$\text{(i.)}\quad \Psi = \mathrm{C} + \tfrac{1}{2} w (c - a \cos \chi)^2.$$

For the cyclic motion,

$$\text{(iv.)}\quad \Psi = \mathrm{C}' \text{ at the surface of the ring.}$$

The formulæ of Section I. show that we may take

$$\Phi = \left\{\mathrm{A}_1 \frac{d\mathrm{I}_1}{dz} + \mathrm{A}_2 a^2 \frac{d\mathrm{I}_2}{dz} + \mathrm{A}_3 a^4 \frac{d\mathrm{I}_3}{dz} + \&\text{c.}\right\} \text{ in case (i.)}$$

$$\Phi = \{\mathrm{A}_1\mathrm{J}_1 + \mathrm{A}_2 a^2 \mathrm{J}_2 + \mathrm{A}_3 a^4 \mathrm{J}_3{}^2 + \&\text{c.}\} \cos \phi \text{ in case (ii.)}$$

$$\Phi = \left\{\mathrm{A}_1 \frac{d\mathrm{J}_1}{dz} + \mathrm{A}_2 a^2 \frac{d\mathrm{J}_2}{dz} + \mathrm{A}_3 a^4 \frac{d\mathrm{J}_3}{dz} + \&\text{c.}\right\} \cos \phi \text{ in case (iii.),}$$

and that we may also take

$$\Psi = r \sin \theta \{A_1 J_1 + A_2 a^2 J_2 + A_3 a^4 J_3 + \&c.\} \text{ in cases (i.) and (iv.).}$$

A_1, A_2, &c., are constants which must be found separately in each case.

§ 11. *Motion Parallel to the Axis of the Ring.*

Let the ring move with velocity w parallel to its axis.

The value of the normal velocity is $w \sin \chi$ at the surface of the ring.

At the surface of the ring $R = a$.

Let $\sigma = a/c$ = value of s or R/c at the surface of the ring.

Let $\lambda = \log 8c/a - 2$ = value of l or $\log 8c/R - 2$ at the surface of the ring.

Let us assume that

$$\Phi = A_1 \frac{dI_1}{dz} + A_2 a^2 \frac{dI_2}{dz} + A_3 a^4 \frac{dI_3}{dz} + \&c.$$

Then A_1, A_2, &c., are to be chosen, so that

$$\frac{d\phi}{dn} = w \sin \chi,$$

i.e.,

$$\frac{d\phi}{ds} = cw \sin \chi.$$

Differentiating the formulæ (B) of § 3, we find

$$\frac{d}{ds}\left(\frac{dI_1}{dz}\right) = \frac{1}{c^2 s^2}\left\{\sin \chi - \left(\frac{4l+1}{32} \sin \chi + \tfrac{3}{32} \sin 3\chi\right) s^2 - \left(\frac{6l+3}{32} \sin 2\chi + \tfrac{5}{64} \sin 4\chi\right) s^3\right.$$
$$\left. - \left(\frac{18l+3}{256} \sin \chi + \frac{360l+183}{2048} \sin 3\chi + \tfrac{105}{2048} \sin 5\chi\right) s^4 - \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_2}{dz}\right) = \frac{1}{c^4 s^3}\left\{2 \sin 2\chi + \left(\tfrac{3}{4} \sin \chi + \tfrac{1}{4} \sin 3\chi\right) s\right.$$
$$\left. - \left(\frac{12l+1}{64} \sin \chi + \tfrac{27}{128} \sin 3\chi + \tfrac{5}{128} \sin 5\chi\right) s^3 - \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_3}{dz}\right) = \frac{2!}{c^6 s^4}\left\{3 \sin 3\chi + \left(\tfrac{5}{2} \sin 2\chi + \tfrac{1}{2} \sin 4\chi\right) s\right.$$
$$\left. + \left(\tfrac{15}{16} \sin \chi + \tfrac{15}{32} \sin 3\chi + \tfrac{3}{32} \sin 5\chi\right) s^2 + \&c.\right\}$$

$$\frac{d}{ds}\left(\frac{dI_4}{dz}\right) = \frac{3!}{c^8 s^5}\left\{4 \sin 4\chi + \left(\tfrac{21}{4} \sin 3\chi + \tfrac{3}{4} \sin 5\chi\right) s + \ldots\right\}$$

$$\frac{d}{ds}\left(\frac{dI_5}{dz}\right) = \frac{4!}{c^{10} s^6} 5 \sin 5\chi.$$

At the surface of the ring $s = \sigma = a/c$: and we have the following equations to determine the constants :—

$$\left.\begin{aligned}
&A_1\left(1 - \frac{4\lambda + 1}{32}\sigma^2 - \frac{18\lambda + 3}{256}\sigma^4\right) + A_2\left(\tfrac{3}{4}\sigma^2 - \frac{12\lambda + 1}{64}\sigma^4\right) + A_3\tfrac{15}{8}\sigma^4 = a^2cw \\
&A_1\left(-\frac{6\lambda + 3}{32}\sigma^2\right) + 2A_2 + A_3 5\sigma^2 = 0 \\
&A_1\left(-\tfrac{3}{32} - \frac{360\lambda + 183}{2048}\sigma^2\right) + A_2\left(\tfrac{1}{4} - \tfrac{27}{128}\sigma^2\right) + A_3\left(6 + \tfrac{15}{16}\sigma^2\right) + A_4\tfrac{63}{2}\sigma^2 = 0 \\
&\quad -\tfrac{5}{64}A_1 + A_3 + 24A_4 = 0 \\
&\quad -\tfrac{105}{2048}A_1 - \tfrac{5}{128}A_2 + \tfrac{3}{16}A_3 + \tfrac{9}{2}A_4 + 120A_5 = 0
\end{aligned}\right\}$$

These equations give

$$A_1 = \left\{1 + \frac{4\lambda + 1}{32}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{1024}\sigma^4\right\}a^2cw.$$

$$A_2 = \frac{12\lambda + 1}{128}\sigma^2 a^2cw.$$

$$A_3 = \frac{1}{64}\left(1 + \frac{360\lambda + 57\frac{2}{3}}{64}\sigma^2\right)a^2cw.$$

$$A_4 = \frac{1}{384}a^2cw.$$

$$A_5 = \frac{5}{2^{14}}a^2cw.$$

Thus

$$\Phi = a^2cw\left\{\left(1 + \frac{4\lambda + 1}{32}\sigma^2 + \ldots\right)\frac{dI_1}{dz} + \frac{12\lambda + 1}{128}\sigma^2 a^2\frac{dI_2}{dz} + \ldots\right\}.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2T &= -\rho\iint\Phi\frac{d\Phi}{dn}\,dS \\
&= -\rho w\,2\pi\int_0^{2\pi}(c - a\cos\chi)\sin\chi\,\Phi\,a\,d\chi \\
&= -\,2\pi\rho acw\int_0^{2\pi}\left(\sin\chi - \frac{\sigma}{2}\sin 2\chi\right)\Phi\,d\chi \\
&= 2\pi^2\rho\frac{ac}{c^2\sigma}w\left\{A_1\left(1 + \frac{4\lambda + 1}{32}\sigma^2 - \frac{6\lambda + 9}{256}\sigma^2\right) + A_2\sigma^2\left(\tfrac{1}{4} + \frac{12\lambda + 1}{64}\sigma^2\right) + A_3\sigma^4\tfrac{5}{8}\right\} \\
&= Mw^2\left\{1 + \frac{4\lambda + 1}{16}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{512}\sigma^4 + \ldots\right\},
\end{aligned}$$

where M is the mass of the fluid displaced.

§ 12. *Velocity Potential for a Ring moving at Right Angles to its Axis.*—The velocity potential Φ is of the form

$$\{A_1J_1 + A_2a^2J_2 + A_3a^4J_3 + \ldots\} \cos \phi.$$

The constants are to be determined from the condition that the normal velocity at the surface of the ring is $-u \cos \chi \cos \phi$.

Thus

$$\frac{d\Phi}{dR} = -u \cos \chi \cos \phi$$

$$\frac{d\Phi}{ds} = -cu \cos \chi \cos \phi.$$

Now the formulæ (C) of § 3 give on differentiation

$$\begin{aligned}\frac{dJ_1}{ds} = \frac{1}{cs}\Big\{ &-1 + \frac{l-2}{2} s \cos \chi + \left(\frac{3l-1}{4} + \frac{6l-11}{16} \cos 2\chi\right) s^2 \\ &+ \left(\frac{99l-30}{64} \cos \chi + \frac{15l-28}{64} \cos 3\chi\right) s^3 \\ &+ \left(\frac{135l-27}{128} + \frac{240l-83}{192} \cos 2\chi + \frac{420l-809}{3072} \cos 4\chi\right) s^4 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_2}{ds} = \frac{1}{c^3s^2}\Big\{ &-\cos \chi - \frac{s}{2} - \left(\frac{4l+}{32} \cos \chi - \tfrac{3}{32} \cos 3\chi\right) s^2 \\ &+ \left(\frac{12l-1}{64} - \frac{12l+13}{32} \cos 2\chi + \tfrac{5}{64} \cos 4\chi\right) s^3 \\ &+ \left(\frac{54l-15}{256} \cos \chi - \frac{105l+61}{256} \cos 3\chi + \tfrac{105}{2048} \cos 5\chi\right) s^4 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_3}{ds} = \frac{1}{c^5s^3}\Big\{ &-2 \cos 2\chi - \left(\tfrac{9}{4} \cos \chi + \tfrac{1}{4} \cos 3\chi\right) s - \tfrac{13}{8} s^2 \\ &+ \left(-\frac{20l+107}{64} \cos \chi + \tfrac{21}{128} \cos 3\chi + \tfrac{5}{128} \cos 5\chi\right) s^3 + \ldots\Big\}.\end{aligned}$$

$$\begin{aligned}\frac{dJ_4}{ds} = \frac{1}{c^7s^4}\{ &-6 \cos 3\chi - (10 \cos 2\chi + \cos 4\chi) s \\ &- (\tfrac{81}{8} \cos \chi + \tfrac{21}{16} \cos 3\chi + \tfrac{3}{16} \cos 5\chi) + \ldots\}.\end{aligned}$$

$$\frac{dJ_5}{ds} = \frac{1}{c^9s^5}\{-24 \cos 4\chi - (\tfrac{105}{2} \cos 3\chi + \tfrac{9}{2} \cos 5\chi) s - \ldots\}.$$

$$\frac{dJ_6}{ds} = \frac{1}{c^{11}s^6}\{-120 \cos 5\chi - \ldots\}.$$

Therefore, the equations giving A_1, A_2, &c., are

$$\left.\begin{array}{l}
A_1\left(-1+\frac{3\lambda-1}{4}\sigma^2+\frac{135\lambda-27}{128}\sigma^4\right)+A_2\sigma^2\left(-\frac{1}{2}+\frac{12\lambda-1}{64}\sigma^2\right)-A_3\frac{13}{8}\sigma^4=0. \\
A_1\left(\frac{\lambda-2}{2}+\frac{99\lambda-30}{64}\sigma^2\right)+A_2\left(-1-\frac{4\lambda+17}{32}\sigma^2+\frac{54\lambda-15}{256}\sigma^4\right) \\
\qquad\qquad +A_3\sigma^2\left(-\frac{9}{4}-\frac{20\lambda+107}{64}\sigma^2\right)-\frac{81}{8}A_4\sigma^4=-c^2u. \\
A_1\left(\frac{6\lambda-11}{16}+\frac{240\lambda-83}{192}\sigma^2\right)+A_2\sigma^2\left(-\frac{12\lambda+13}{32}\right)-2A_3-10\sigma^2A_4=0. \\
A_1\frac{15\lambda-28}{64}+A_2\left(+\frac{3}{32}-\frac{105\lambda+61}{256}\sigma^2\right)+A_3\left(-\frac{1}{4}+\frac{21}{128}\sigma^2\right) \\
\qquad\qquad +A_4\left(-6-\frac{21}{16}\sigma^2\right)-\frac{105}{2}\sigma^2A_5=0. \\
A_1\frac{420\lambda-809}{3072}+\frac{5}{64}A_2-A_4-24A_5=0, \\
\qquad \frac{105}{2048}A_2+\frac{5}{128}A_3-\frac{3}{16}A_4-\frac{9}{2}A_5-120A_6=0.
\end{array}\right\}$$

These equations give

$$A_1=-\frac{\sigma^2}{2}\left(1-\frac{\sigma^2}{4}\right)c^2u.$$

$$A_2=\left(1-\frac{12\lambda+1}{32}\sigma^2+\frac{48\lambda^2+344\lambda+159}{1024}\sigma^4\right)c^2u.$$

$$A_3=-\frac{18\lambda+7}{64}\sigma^2c^2u.$$

$$A_4=\left(\frac{1}{64}-\frac{24\lambda+25}{384}\sigma^2\right)c^2u.$$

$$A_5=\frac{1}{384}c^2u.$$

$$A_6=\frac{5}{2^{14}}c^2u.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2T &= -\iint\Phi\frac{d\Phi}{dn}dS \\
&= \int_0^{2\pi}\int_0^{2\pi}\{A_1J_1+A_2a^2J+\ldots\}\cos\phi\, u\cos\chi\cos\phi\,(c-a\cos\chi)\,a\,d\chi\, d\phi \\
&= \pi acu\int_0^{2\pi}\{A_1J_1+\ldots\}\left\{\cos\chi-\frac{\sigma}{2}-\frac{\sigma}{2}\cos 2\chi\right\}d\chi.
\end{aligned}$$

This is evaluated by picking out the coefficients of cos χ, &c., in the expansions of J_1, J_2, &c., given in § 3 (C), and

$$= \pi^2 au \left\{ A_1 \left(-\frac{\lambda+1}{2}\sigma + \frac{3\lambda+5}{64}\sigma^3 \right) + A_2 \left(\sigma - \frac{20\lambda+17}{32}\sigma^3 + \frac{18\lambda+29}{256}\sigma^5 \right) \right.$$
$$\left. + A_3 \tfrac{7}{4}\sigma^3 + A_4 \tfrac{61}{8}\sigma^5 \right\}$$
$$= \frac{Mu^2}{2}\left\{ 1 - \frac{12\lambda+5}{16}\sigma^2 + \frac{144\lambda^2+24\lambda+57}{512}\sigma^4 \right\}.$$

§ 13. *Ring rotating about a Line through its Centre in the Plane of its Central Circle.*—The velocity potential Φ may be taken to be of the form

$$\left\{ A_1 \frac{dJ_1}{dz} + A_2 a^2 \frac{dJ_2}{dz} + \ldots \right\} \cos\phi.$$

The constants A_1, A_2, &c., are determined by the boundary condition

$$\frac{d\Phi}{dR} = -c\omega_2 \sin\chi \cos\phi,$$

or

$$\frac{d\Phi}{ds} = -c^2\omega_2 \sin\chi \cos\phi.$$

Differentiating the formulæ (D) of § 3,

$$\frac{d}{ds}\left(\frac{dJ_1}{dz}\right) = \frac{1}{c^2 s^2}\left\{ \sin\chi + \left(\frac{12l-1}{32}\sin\chi - \tfrac{3}{32}\sin 3\chi \right) s^2 \right.$$
$$+ \left(\frac{18l+1}{32}\sin 2\chi - \tfrac{5}{64}\sin 4\chi \right) s^3$$
$$\left. + \left(\frac{90l-3}{256}\sin\chi + \frac{1080l+9}{2048}\sin 3\chi - \tfrac{105}{2048}\sin 5\chi \right) s^4 \right\}$$

$$\frac{d}{ds}\left(\frac{dJ_2}{dz}\right) = \frac{1}{c^3 s^3}\left\{ 2\sin 2\chi + \left(\tfrac{3}{4}\sin\chi + \tfrac{1}{4}\sin 3\chi \right) s \right.$$
$$\left. + \left(\frac{35l-5}{64}\sin\chi - \tfrac{3}{128}\sin 3\chi - \tfrac{5}{128}\sin 5\chi \right) s^3 + \ldots \right\}$$

$$\frac{d}{ds}\left(\frac{dJ_3}{dz}\right) = \frac{1}{c^4 s^4}\left\{ 6\sin 3\chi + (5\sin 2\chi + \sin 4\chi) s \right.$$
$$\left. + \left(\tfrac{17}{8}\sin\chi + \tfrac{11}{16}\sin 3\chi + \tfrac{3}{16}\sin 5\chi \right) s^2 + \ldots \right\}$$

$$\frac{d}{ds}\left(\frac{dJ_4}{dz}\right) = \frac{1}{c^5 s^5}\left\{ 24\sin 4\chi + \left(\tfrac{63}{2}\sin 3\chi + \tfrac{9}{2}\sin 5\chi \right) s + \ldots \right\}$$

$$\frac{d}{ds}\left(\frac{dJ_5}{dz}\right) = \frac{1}{c^6 s^6}\, 120\sin 5\chi + \ldots$$

The equations to determine A_1, A_2, &c., are, therefore,

$$\left.\begin{aligned}
&A_1\left(1+\frac{12\lambda-1}{32}\sigma^2+\frac{90\lambda-3}{256}\sigma^4\right)+A_2\sigma^2\left(\tfrac{3}{4}+\frac{36\lambda-5}{64}\sigma^2\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad+A_3\sigma^4\tfrac{17}{8}=-a^2c^2\omega_2\\
&A_1\frac{18\lambda+1}{32}\sigma^2+2A_2+5A_3\sigma^2=0\\
&A_1\left(-\tfrac{3}{32}+\frac{1080\lambda+9}{2048}\sigma^2\right)+A_2\left(\tfrac{1}{4}-\tfrac{3}{128}\sigma^2\right)+A_3\left(6+\tfrac{11}{16}\sigma^2\right)+A_4\tfrac{63}{2}\sigma^2=0\\
&-\tfrac{5}{64}A_1+A_3+24A_4=0\\
&-\tfrac{105}{2048}A_1-\tfrac{5}{128}A_2+\tfrac{3}{16}A_3+\tfrac{9}{2}A_4+120\,A_5=0
\end{aligned}\right\}$$

These equations give

$$A_1=-\left(1-\frac{12\lambda-1}{32}\sigma^2+\frac{144\lambda^2-168\lambda+21}{1024}\sigma^4\right)a^2c^2\omega_2\,;$$

$$A_2=+\frac{36\lambda+7}{128}\sigma^2.a^2c^2\omega_2\,;$$

$$A_3=-\left(\tfrac{1}{64}-\frac{42\lambda+7\frac{1}{24}}{512}\sigma^2\right)a^2c^2\omega_2\,;$$

$$A_4=-\tfrac{1}{384}a^2c^2\omega_2\,;$$

$$A_5=-\frac{5}{2^{14}}a^2c^2\omega_2.$$

The kinetic energy is given by the equation

$$\begin{aligned}
2T&=-\iint\Phi\frac{d\Phi}{dn}\,dS\\
&=\int_0^{2\pi}\!\!\int_0^{2\pi}c\omega_2\sin\chi\,(c-a\cos\chi)\left(A_1\frac{dJ_1}{dz}+A_2a^2\frac{dJ_2}{dz}+\ldots\right)\cos^2\phi\,a\,d\chi\\
&=\pi ac^2\omega_2\int_0^{2\pi}\left(\sin\chi-\frac{\sigma}{2}\sin\chi\right)\left(A_1\frac{dJ_1}{dz}+\ldots\right)d\chi\\
&=\pi^2a^2c^3\omega_2{}^2\left\{\left(1-\frac{12\lambda-1}{32}\sigma^2+\frac{144\lambda^2-168\lambda+21}{1024}\sigma^4\right)\left(1-\frac{12\lambda+15}{32}\sigma^2+\frac{6\lambda+11}{256}\sigma^4\right)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.-\frac{36\lambda+7}{128}\sigma^2\frac{\sigma^2}{4}+\tfrac{1}{64}\cdot\tfrac{7}{8}\sigma^4\right\}\\
&=M\frac{c^2\omega_2{}^2}{2}\left\{1-\frac{12\lambda+7}{16}\sigma^2+\frac{18\lambda^2-3\lambda+5}{64}\sigma^4-\ldots\right\}.
\end{aligned}$$

§ 14.—*Stream Line Function for a Ring moving parallel to its Axis, and for the Cyclic Motion through the Ring.*—Let the velocity of the ring be w, and the circulation be κ.

Let ψ be the stream-line function.

Then

$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0$$

at all points of the fluid.

At the surface of the ring

$$\psi - \tfrac{1}{2}w\varpi^2 = \text{const.},$$

or

$$\psi = \mathrm{A}\kappa + wc^2\left(\mathrm{B} - \sigma\cos\chi + \frac{\sigma^2}{4}\cos 2\chi\right).$$

We may therefore assume ψ to be of the form

$$\{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\mathrm{J}_2\varpi + \mathrm{A}_3 a^4\mathrm{J}_3\varpi + \ldots\}.$$

Using the values of $\mathrm{J}_1\varpi$, $\mathrm{J}_2\varpi$, &c., given in § 4 (E),

The equations giving the constants A_1, A_2, &c., are

$$\left.\begin{aligned}
&\mathrm{A}_1\left(\lambda + \frac{2\lambda+5}{16}\sigma^2 + \frac{12\lambda+11}{2048}\sigma^4\right) + \mathrm{A}_2\sigma^2\left(\frac{2\lambda-3}{4} + \frac{20\lambda+47}{128}\sigma^2\right) \\
&\qquad\qquad + \mathrm{A}_3\sigma^4\frac{52\lambda-57}{32} = \mathrm{A}\kappa + \mathrm{B}wc^2 \\
&\mathrm{A}_1\left(-\frac{\lambda+1}{2} + \frac{3\lambda+5}{64}\sigma^2\right) + \mathrm{A}_2\left(1 - \frac{20\lambda+17}{32}\sigma^2\right) + \mathrm{A}_3\tfrac{7}{4}\sigma^2 = -wc^2 \\
&\mathrm{A}_1\left(-\frac{\lambda}{16} + \frac{12\lambda+17}{768}\sigma^2\right) + \mathrm{A}_2\left(-\tfrac{1}{4} - \frac{8\lambda+1}{64}\sigma^2\right) + \mathrm{A}_3(1 - \tfrac{3}{4}\sigma^2) + 2\mathrm{A}_4\sigma^2 = \frac{wc^2}{4} \\
&\mathrm{A}_1\frac{3\lambda-1}{192} - \frac{\mathrm{A}_2}{32} - \frac{\mathrm{A}_3}{4} + 2\mathrm{A}_4 = 0 \\
&-\mathrm{A}_1\frac{15\lambda-8}{3072} - \frac{\mathrm{A}_2}{128} - \frac{\mathrm{A}_3}{32} - \frac{\mathrm{A}_4}{2} + 6\mathrm{A}_5 = 0.
\end{aligned}\right\}$$

Another equation is found by using the fact that the circulation is κ.

Integrating round a circle concentric with the crosss-ection (radius $< c$), we have

$$\kappa = \int_0^{2\pi}\left(-\frac{1}{\varpi}\frac{d\psi}{d\mathrm{R}}\right)\mathrm{R}\,d\chi$$

$$= -\int_0^{2\pi}\frac{d}{d\mathrm{R}}\left\{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\mathrm{J}_2\varpi + \ldots\right\}\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi}.$$

Differentiating the equations of § 3 (E),

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_1\varpi) = \frac{1}{\mathrm{R}}\Big\{-1 - \frac{l}{2}s\cos\chi + \left(\frac{l+2}{4} - \frac{2l-1}{16}\cos 2\chi\right)s^2$$

$$+ \left(\frac{9l+12}{64}\cos\chi - \frac{3l-2}{64}\cos 3\chi\right)s^3 + \left(\frac{3l+2}{128} + \frac{6l+7}{96}\cos 2\chi - \frac{60l-47}{3072}\cos 4\chi\right)s^4\Big\}.$$

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi) = \frac{1}{c\mathrm{R}^2}\Big\{-\cos\chi - \frac{s}{2} - \left(\frac{20l-3}{32}\cos\chi + \tfrac{1}{32}\cos 3\chi\right)s^2$$

$$+ \left(\frac{20l+37}{64} - \frac{3l-1}{16}\cos 2\chi - \tfrac{1}{64}\cos 4\chi\right)s^3 + \ldots\Big\}.$$

$$\frac{d}{d\mathrm{R}}(\mathrm{J}_3\varpi) = \frac{1}{c^2\mathrm{R}^3}\{-2\cos 2\chi - (\tfrac{7}{4}\cos\chi - \tfrac{1}{4}\cos 3\chi)s - \tfrac{13}{8}s^2\ldots\}.$$

Also

$$\frac{1}{c - \mathrm{R}\cos\chi} = \frac{1}{c}(1 - s\cos\chi)^{-1}$$

$$= \frac{1}{c}\Big\{1 + \frac{s^2}{2} + \frac{3s^4}{8} + 2\cos\chi\frac{s}{2}\left(1 + \frac{3s^2}{4} + \frac{5s^4}{8}\right)$$

$$+ 2\cos 2\chi\frac{s^2}{4}(1 + s^2) + 2\cos 3\chi\frac{s^3}{8} + \ldots\Big\}.$$

Therefore

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_1\varpi)\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = \frac{2\pi}{c}\Big\{\left(1 + \frac{s^2}{2} + \frac{3s^4}{8} + \ldots\right)\left(1 - \frac{l+2}{4}s^2 - \frac{3l+2}{128}s^4\right)$$

$$+ \frac{s^2}{4}\left(1 + \frac{3s^2}{4}\right)\left(l - \frac{9l+12}{32}s^2\right) + \frac{2l-1}{64}s^4 + \ldots\Big\}$$

$$= \frac{2\pi}{c}$$

Similarly

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi)\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = \frac{2\pi}{c^3}$$

$$-\int_0^{2\pi}\frac{d}{d\mathrm{R}}(\mathrm{J}_2\varpi)\frac{\mathrm{R}\,d\chi}{c - \mathrm{R}\cos\chi} = 3\,\frac{2\pi}{c^5}$$

&c.

Therefore

$$\kappa = \frac{2\pi}{c}\left\{\mathrm{A}_1 + \frac{\mathrm{A}_2 a^2}{c^2} + 3\,\frac{\mathrm{A}_3 a^4}{c^4} + \ldots\right\}$$

$$= \frac{2\pi}{c}\{\mathrm{A}_1 + \mathrm{A}_2\sigma^2 + 3\mathrm{A}_3\sigma^4 + \text{\&c.}\}.$$

Thus we have, in addition to the equations above, for the determinations of A_1, A_2, &c.,

$$A_1 + A_2\sigma^2 + 3A_3\sigma^4 + \text{\&c.} = \frac{\kappa c}{2\pi}\cdot$$

Solving the equations

$$A_1 = wc^2\sigma^2\left(1 + \frac{4\lambda + 1}{32}\sigma^2\right) + \frac{\kappa c}{2\pi}\left(1 - \frac{\lambda + 1}{2}\sigma^2 - \frac{4\lambda^2 + 17\lambda + 6}{64}\sigma^4\right)$$

$$A_2 = -wc^2\left(1 + \frac{4\lambda + 1}{32}\sigma^2\right) + \frac{\kappa c}{2\pi}\left(\frac{\lambda + 1}{2} + \frac{4\lambda^2 - 19\lambda - 18}{64}\sigma^2\right)$$

$$A_3 = -wc^2\sigma^2\frac{12\lambda - 1}{128} + \frac{\kappa c}{2\pi}\left(\frac{3\lambda + 2}{16} + \frac{12\lambda^2 + \lambda + 1}{256}\sigma^2\right)$$

$$A_4 = -wc^2\,\tfrac{1}{64} + \frac{\kappa c}{2\pi}\,\frac{9\lambda + 10}{384}$$

$$A_5 = -wc^2\,\tfrac{1}{256} + \frac{\kappa c}{2\pi}\,\frac{81\lambda + 56}{18432}.$$

Substituting these values we obtain

$$A = \frac{c}{2\pi}\left\{\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4\right\}$$

$$B = \sigma^2\left\{\frac{2\lambda + 3}{4} + \frac{4\lambda^2 + 5\lambda - 2}{64}\sigma^4\right\}.$$

The kinetic energy is given by the equation

$$\begin{aligned}
T &= -\pi\rho\int_0^{2\pi}\left[\frac{\psi}{\omega}\cdot\frac{d\psi}{dR}\right]_{R=a}\cdot\ a\,d\chi \\
&= -\pi\rho\int_0^{2\pi}\frac{A\kappa + wc^2\left(B - \sigma\cos\chi + \frac{\sigma^2}{4}\cos 2\chi\right)}{c - a\cos\chi}\,\frac{d\psi}{dR}\,a\,d\chi \\
&= \pi\rho\int_0^{2\pi}\left\{A\kappa + wc^2\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4}\right)\right\}\left(-\frac{1}{\varpi}\frac{d\psi}{dR}\right)a\,d\chi \\
&\qquad\qquad - \frac{\pi\rho acw}{2}\int_0^{2\pi}(1 - \sigma\cos\chi)\frac{d\psi}{dR}\,d\chi \\
&= \pi\rho A\kappa^2 + \pi\rho c^2\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4}\right)\kappa\,w \\
&\qquad + \pi^2\rho wc\left\{A_1\left(1 - \frac{\lambda + 1}{2}\sigma^2 + \frac{3\lambda + 5}{64}\sigma^4\right) - A_2\frac{20\lambda + 17}{32}\sigma^4 + \tfrac{3}{4}A_3\sigma^4\right. \\
&= \pi\rho\left\{A\kappa^2 + c^2\kappa w\left(B - \tfrac{1}{2} - \frac{\sigma^2}{4} + \tfrac{1}{2} - \frac{\lambda + 1}{2}\sigma^2 - \frac{4\lambda^2 + 5\lambda - 2}{64}\sigma^4\right)\right. \\
&\qquad\qquad \left. + w^2c^3\sigma^2\left(1 + \frac{4\lambda + 1}{16}\sigma^2\right)\right\}.
\end{aligned}$$

The coefficient of κw vanishes, as it necessarily should (BASSET, 'Hydrodynamics,' vol. 1, art. 171), and the kinetic energy of the forward motion and the cyclic motion is given by

$$2\mathrm{T} = \mathrm{M}w^2\left(1 + \frac{4\lambda + 1}{16}\sigma^2 + \ldots\right)$$
$$+ \rho c\,\kappa^2\left(\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4 + \ldots\right).$$

The first part of this result agrees, as far as it goes, with the result of § 11.

§ 15. As the cyclic motion through a ring is of considerable interest, it seems worth while to give other proofs of some of the above results.

To find the circulation we may integrate the velocity along the axis of the ring from $-\infty$ to ∞, and then along a semicircle from ∞ to $-\infty$.

$$\psi = \{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\,\mathrm{J}_2\varpi + \ldots\}.*$$

The velocity of the fluid at any point on the axis is the limit of $\frac{1}{\varpi}\frac{d\psi}{d\varpi}$, where ϖ is indefinitely small.

Consider the part arising from the term $\mathrm{A}_1\mathrm{J}_1\varpi$ or

$$\mathrm{A}_1\int_0^\pi \frac{\varpi\cos\phi\,d\phi}{\sqrt{(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)}}.$$

$$\text{It} = \text{limit of } \frac{\mathrm{A}_1}{\varpi}\frac{d}{d\varpi}\int_0^\pi \frac{\varpi\cos\phi\,d\phi}{\sqrt{(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)}}$$

$$= \text{limit of } \int_0^\pi \frac{\mathrm{A}_1(z^2 + c^2 - \varpi c\cos\phi)}{\varpi(z^2 + \varpi^2 - 2\varpi c\cos\phi + c^2)^{\frac{3}{2}}}\cos\phi\,d\phi$$

$$= \text{limit of } \int_0^\pi \frac{\mathrm{A}_1}{\varpi(z^2 + c^2)^{\frac{1}{2}}}\left(1 - \frac{\varpi c\cos\phi}{z^2 + c^2}\right)\left(1 + \frac{3\varpi c\cos\phi}{z^2 + c^2}\right)\cos\phi\,d\phi$$

$$= \frac{\pi\mathrm{A}_1 c}{(z^2 + c^2)^{\frac{3}{2}}}.$$

We may notice that at the centre of the ring this

$$= \frac{\pi\mathrm{A}_1}{c^2}.$$

---

* As we are only considering the cyclic motion, only the parts of A_1, A_2, &c., which involve κ, are denoted by these letters now.

Again, the part of the velocity of the fluid arising from the term $A_1J_1\varpi$ in the stream function along an arc of a very great circle whose centre is at the origin, is the limit of $-\frac{1}{\varpi}\frac{d}{dr}(A_1J_1\varpi)$

$$= \text{limit of } -\frac{A_1}{r\sin\theta}\int_0^\pi \frac{\sin\theta\,(c^2 - cr\sin\theta\cos\phi)}{(r^2 + c^2 - 2cr\sin\theta\cos\phi)^{\frac{3}{2}}}\cos\phi\,d\phi$$

$$= \frac{\pi}{2}\frac{A_1 c\sin\theta}{r^3}.$$

Therefore, the part of the circulation rising from the term $A_1J_1\varpi$ of the stream function is

$$\int_{-\infty}^{\infty}\frac{\pi A_1 c}{(z^2 + c^2)^{\frac{3}{2}}}\,dz + \int_0^\pi \frac{\pi}{2}\frac{A_1 c\sin\theta}{r^3}\,r\,d\theta$$

$$= \frac{2\pi}{c}A_1.$$

Now the part arising from $A_2a^2J_2\varpi_2$ or $-A_2a^2\frac{d}{c\,dc}(J_1\varpi)$, is obtained by differentiation with respect to c, and

$$= A_2a^2\frac{2\pi}{c^3}.$$

Therefore, the circulation is given by

$$\kappa = \frac{2\pi}{c}\left\{A_1 + A_2\sigma^2 + 1.3\,A_3\sigma^4 + 1.3.5.A_4\sigma^6 + \ldots\right\},$$

agreeing with the result already obtained.

We also see that the velocity of the fluid at the centre of the ring is

$$\frac{\pi A_1}{c^2} - A_2a^2\frac{d}{c\,dc}\left(\frac{\pi}{c^2}\right) + \ldots = \frac{\pi}{c^2}\{A_1 + 2\sigma^2A_2 + 8\sigma^4A_3 + \ldots\}.$$

The kinetic energy of the cyclic motion may be obtained simply in the following way—

$$2T = \rho\iint\kappa\frac{d\Phi}{dn}dS,$$

the integral being taken over a barrier.

This

$$= \rho\iint\kappa\frac{1}{\varpi}\frac{d\psi}{d\varpi}\varpi\,d\varpi\,d\phi$$

$$= 2\pi\rho\kappa\int\frac{d\psi}{d\varpi}d\varpi$$

$$= 2\pi\rho\kappa\,(\psi_B - \psi_A),$$

where ψ_{R} denotes the value of ψ at the surface of the ring and ψ_{A} at the axis. This result applies to the cyclic motion through a circular ring of any cross-section.

We have taken ψ_{A} to be zero, and called ψ_{R}, A: so that we have $\text{T} = \pi\rho\kappa\,\text{A}$.

The Linear Momentum of the Cyclic Motion is parallel to the axis of the ring and

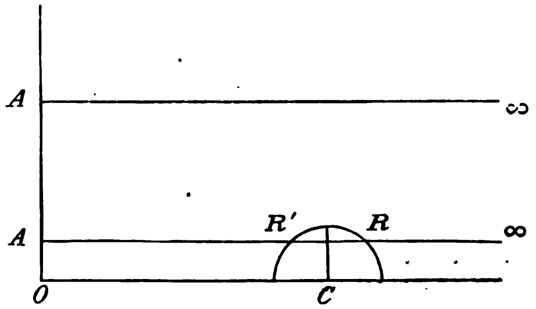
$$= \rho \iiint \frac{1}{\varpi} \frac{d\psi}{d\varpi} \varpi \, d\varpi \, dz \, d\phi :$$

the integral being taken all over the fluid.

This integral

$$= 4\pi\rho \int_0^\infty dz \int \frac{d\psi}{d\varpi} d\varpi$$

$$= 4\pi\rho \int_0^a (\psi_\infty - \psi_{\text{R}} + \psi_{\text{R}'} - \psi_{\text{A}}) \, dz + 4\pi\rho \int_a^\infty (\psi_\infty - \psi_{\text{A}}) \, dz.$$



We have taken $\psi_{\text{A}} = 0$.

ψ is constant on the ring, so that

$$\psi_{\text{R}'} - \psi_{\text{R}} = 0.$$

Therefore, the resultant linear momentum

$$= 4\pi\rho \int_0^\infty \psi_\infty \, dz,$$

where ψ_∞ means the value of ψ at an infinite distance from the axis.

Consider first the part of ψ_∞ arising from the term $\text{A}_1 \text{J}_1 \varpi$.

This is the limit of

$$\text{A}_1 \int_0^\pi \frac{\varpi \cos\phi \, d\phi}{\sqrt{(z^2 + c^2 - 2\varpi c \cos\phi + \varpi^2)}}$$

where ϖ is infinite.

This

$$= \text{A}_1 \int_0^\pi \frac{\varpi}{\sqrt{(\varpi^2 + z^2)}} \left\{ 1 + \frac{\varpi c \cos\phi}{\sqrt{(z^2 + c^2)}} \right\} \cos\phi \, d\phi$$

$$= \text{A}_1 \frac{\pi}{2} \frac{c\varpi^2}{(\varpi^2 + z^2)^{\frac{3}{2}}} \cdot$$

Therefore the momentum of the cyclic motion is

$$
\begin{aligned}
&4\pi\rho\,\frac{\pi}{2}\left(\mathrm{A}_1 c - \mathrm{A}_2\frac{a^2}{c} - \mathrm{A}_3\frac{a^4}{c^3} - 1.3.\mathrm{A}_4.\frac{a^6}{c^5} + \dots\right)\\
&\quad = 2\pi^2\rho c\,\{\mathrm{A}_1 - \mathrm{A}_2\,\sigma^2 - \mathrm{A}_3\,\sigma^4 - 1.3.\mathrm{A}_4\,\sigma^6 - 1.3.5.\mathrm{A}_5\,\sigma^8 + \dots\}\\
&\quad = \pi^2\kappa\rho c\left\{1 - (\lambda + 1)\,\sigma^2 - \frac{4\lambda^2 + 5\lambda - 2}{32}\,\sigma^4 - \&\mathrm{c.}\right\}.
\end{aligned}
$$

§ 16. The cyclic motion might have been started by applying an impulsive pressure $\kappa\rho$ at all points of the aperture of the ring, this would produce an impulsive pressure at all points of the ring. To determine this, the velocity potential of the cyclic must be found at these points. It is easy to find an expression for the velocity potential at points *not far from the surface of the ring.*

For

$$
\begin{aligned}
\frac{d\Phi}{\mathrm{R}\,d\chi} &= -\frac{1}{\varpi}\,\frac{d\psi}{d\mathrm{R}}\\
&= -\frac{1}{\varpi}\,\frac{d}{d\mathrm{R}}\,\{\mathrm{A}_1\mathrm{J}_1\varpi + \mathrm{A}_2 a^2\,\mathrm{J}_2\varpi + \mathrm{A}_3 a^4\,\mathrm{J}_3\varpi + \dots\}\\
&= \frac{\mathrm{A}_1}{c\mathrm{R}\,(1 - s\cos\chi)}\left\{1 + \frac{l}{2}\,s\cos\chi - \left(\frac{l+2}{4} - \frac{2l-1}{16}\cos 2\chi\right)s^2\right.\\
&\qquad - \left(\frac{9l+12}{64}\cos\chi - \frac{3l-2}{64}\cos 3\chi\right)s^3\\
&\qquad \left. - \left(\frac{3l+2}{128} + \frac{6l+7}{96}\cos 2\chi - \frac{60l-47}{3072}\cos 4\chi\right)s^4 \dots\right\}\\
&\quad + \frac{\mathrm{A}_2 a^2}{c^2\mathrm{R}^2\,(1 - s\cos\chi)}\left\{\cos\chi + \frac{s}{2} + \left(\frac{20l-3}{32}\cos\chi + \tfrac{1}{32}\cos 3\chi\right)s^2\right.\\
&\qquad \left. - \left(\frac{20l+37}{64} - \frac{3l-1}{16}\cos 2\chi - \tfrac{1}{64}\cos 4\chi\right)s^3,\ \&\mathrm{c.}\right\}\\
&\quad + \frac{\mathrm{A}_3 a^4}{c^3\mathrm{R}^3\,(1 - s\cos\chi)}\,\{2\cos 2\chi + (\tfrac{7}{4}\cos\chi - \tfrac{1}{4}\cos 3\chi)\,s - \tfrac{13}{8}\,s^2\}\\
&\quad + \frac{\mathrm{A}_4 a^6}{c^4\mathrm{R}^4\,(1 - s\cos\chi)}\,\{6\cos 3\chi + (8\cos 2\chi - \cos 4\chi)\,s\} + \&\mathrm{c}
\end{aligned}
$$

Expanding $\frac{1}{1 - \sigma\cos\chi}$ in ascending powers of σ, and multiplying, $d\Phi/d\chi$ is easily expressed in cosines of multiples of χ

The term independent of χ is $\kappa/2\pi$

On integration, we obtain

$$
\begin{aligned}
\Phi = \text{const} &+ \frac{\kappa}{2\pi}\chi \\
&+ \frac{\mathrm{A}_1}{c}\left\{\left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi + \left(\frac{6l+7}{32}s^2 + \frac{15l+7}{384}s^4\right)\sin 2\chi\right. \\
&\qquad\qquad \left. + \frac{5l+4}{32}s^3\sin 3\chi + \frac{35l+20\frac{1}{12}}{1024}s^4\sin 4\chi\right\} \\
&+ \frac{\mathrm{A}_2\sigma^2}{cs}\left\{\left(1 + \frac{20l+37}{32}s^2\right)\sin\chi + \left(\frac{s}{4} + \frac{16l+21}{64}s^3\right)\sin 2\chi\right. \\
&\qquad\qquad \left. + \tfrac{3}{32}s^2\sin 3\chi + \tfrac{5}{128}s^3\sin 4\chi\right\} \\
&+ \frac{\mathrm{A}_3\sigma^4}{c^2s^2}\left\{\tfrac{11}{4}s\sin\chi + (1 + \tfrac{7}{8}s^2)\sin 2\chi + \tfrac{1}{4}s\sin 3\chi + \tfrac{3}{32}s^3\sin 4\chi\right\} \\
&+ \frac{\mathrm{A}^4\sigma^6}{c^3s^3}\left\{\tfrac{11}{2}s\sin 2\chi + 2\sin 3\chi + \frac{s}{2}\sin 4\chi\right\} \\
&+ \frac{\mathrm{A}_5\sigma^8}{c^4s^3}\,4\sin 4\chi.
\end{aligned}
$$

This holds only at points not far from the surface of the ring.

At the surface of the ring $s = \sigma$. Substituting the values of the constants given in § 14, we find that here

$$
\begin{aligned}
\Phi = \frac{\kappa}{2\pi}\left\{\chi + \left(\frac{2\lambda+3}{2}\sigma + \frac{8\lambda^2+26\lambda+11}{64}\sigma^3\right)\sin\chi + \frac{16\lambda+15}{32}\sigma^2\sin 2\chi\right. \\
\left. + \frac{57\lambda+49}{192}\sigma^3\sin 3\chi + \&\text{c.}\right\}.
\end{aligned}
$$

This gives the impulsive pressure at any point on the surface of the ring.

The momentum of the cyclic motion may be deduced.

It

$$
\begin{aligned}
&= \pi\,(c-a)^2\,\kappa\rho - \iint \Phi\sin\chi\,d\mathrm{S} \\
&= \pi\kappa\rho\,(c-a)^2 - 2\pi a\int_0^{2\pi}\Phi\sin\chi\,(c - a\cos\chi)\,d\chi \\
&= \pi\kappa\rho\,(c-a)^2 - \frac{\kappa\rho}{2\pi}2\pi ac\int_0^{2\pi}(\sin\chi - \frac{\sigma}{2}\sin 2\chi) \\
&\quad \left(\chi + \left\{\frac{2\lambda+3}{2}\sigma + \frac{8\lambda^2+26\lambda+11}{64}\sigma^3\right\}\sin\chi + \frac{16\lambda+15}{32}\sigma^2\sin 2\chi\right)d\chi \\
&= \pi\kappa\rho c^2\left\{1 - 2\sigma + \sigma^2 + \left(2\sigma - \frac{\sigma^3}{2}\right) - \sigma^2\left(\frac{2\lambda+3}{2} + \frac{8\lambda^2+26\lambda+11}{64}\sigma^2\right) + \frac{16\lambda+15}{64}\sigma^4\right\} \\
&= \pi\kappa\rho c^2\left\{1 - (\lambda+1)\,\sigma^2 - \frac{4\lambda^2+5\lambda-2}{32}\sigma^4\right\},
\end{aligned}
$$

which agrees with the result already obtained.

If the cross-section of the ring is very small

$$A_1 = \frac{\kappa c}{2\pi}$$

and

$$\Phi = \text{const} + \frac{\kappa}{2\pi}\left\{\chi + \left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi + \left(\frac{6l+7}{32}s^2 + \frac{15+7}{384}s^4\right)\sin 2\chi \right.$$
$$\left. + \frac{5l+4}{32}s^3 \sin 3\chi + \frac{35l+20\frac{1}{12}}{1024}s^4 \sin 4\chi + \ldots\right\}.$$

But in this case Φ is proportional to Ω the solid angle subtended by the ring at any point.

Hence the solid angle subtended by a circular ring at a point near it is

$$2\left\{\pi - \chi - \left(\frac{l+2}{2}s + \frac{3l+2}{64}s^3\right)\sin\chi - \left(\frac{6l+7}{32}s^2 + \frac{15l+7}{384}s^4\right)\sin 2\chi - \&\text{c.}\right\}$$

§ 17. The kinetic energy of the fluid has been calculated for the different motions of the ring. To find the kinetic energy of the solid ring, its moments of inertia round the axis, and round a perpendicular to the axis through its centre must be found

Let ρ' be the density of the ring, and M' be the mass of the ring.

The moment of inertia round the axis is

$$\int_0^a\int_0^{2\pi} 2\pi\rho'(c - R\cos\chi)^3 R\,dR\,d\chi$$
$$= 4\pi^2\rho' \int_0^a (c^3 + \tfrac{3}{2}cR^2) R\,dR$$
$$= M'(c^2 + \tfrac{3}{4}a^2).$$

The moment of inertia round a perpendicular axis is

$$\int_0^a\int_0^{2\pi} 2\pi\rho'(c - R\cos\chi)\left\{\frac{(c - R\cos\chi)^2}{2} + R^2\sin^2\chi\right\}R\,dR\,d\chi$$
$$= M'\left(\frac{c^2}{2} + \tfrac{3}{8}a^2\right) + M'.\frac{a^2}{4}$$
$$= M'\left(\frac{c^2}{2} + \tfrac{5}{8}a^2\right).$$

§ 18. Therefore, ρ being the density of the fluid, M and M' the masses of the fluid displaced by the ring, and of the ring, respectively, the kinetic energy is given by

$$2T = P(u^2 + v^2) + Rw^2 + A(\omega_1^2 + \omega_2^2) + C\omega_3^2 + K\kappa^2,$$

where

$$P = \frac{M}{2}\left(1 - \frac{12\lambda + 1}{16}\sigma^2 + \frac{144\lambda^2 + 24\lambda + 57}{512}\sigma^4\right) + M',$$

$$R = M\left(1 + \frac{4\lambda + 1}{16}\sigma^2 + \frac{16\lambda^2 + 8\lambda - 23}{512}\sigma^4\right) + M',$$

$$A = M\frac{c^2}{2}\left(1 - \frac{12\lambda + 7}{16}\sigma^2 + \frac{18\lambda^2 - 3\lambda + 5}{64}\sigma^4\right) + M'\frac{c^2}{2}(1 + \tfrac{5}{4}\sigma^2),$$

$$C = M'c^2(1 + \tfrac{3}{4}\sigma^2),$$

$$K = \rho c\left(\lambda - \frac{4\lambda^2 + 8\lambda + 1}{16}\sigma^2 - \frac{2\lambda^3 + 9\lambda^2 + 6\lambda - \frac{43}{32}}{64}\sigma^4\right),$$

Also the linear momentum of the cyclic motion is given by

$$Z = \pi c^2 \kappa \rho \left\{1 - (\lambda + 1)\sigma^2 - \frac{4\lambda^2 + 5\lambda - 2}{32}\sigma^4\right\}.$$

§ 19. The motion of a ring in an infinite fluid is discussed in BASSET'S 'Hydrodynamics,' vol. 1, pp. 196–208.

It is there shown that if a ring be set rotating with angular velocity ω about a diameter of its circular axis, it will make complete revolutions if

$$\omega > Z\sqrt{\frac{R}{AP(R - P)}},$$

but will oscillate if

$$\omega < Z\sqrt{\frac{R}{AP(R - P)}}.$$

Putting in the values of the constants, a very fine ring will make complete revolutions if

$$\omega > \frac{\kappa}{\pi a^2}\sqrt{\frac{2\rho}{\rho + 2\rho'}},$$

and will oscillate if

$$\omega < \frac{\kappa}{\pi a^2}\sqrt{\frac{2\rho}{\rho + 2\rho'}}.$$

A possible steady motion of the ring occurs when it moves with velocity w parallel to the axis, and has also an angular velocity Ω round its axis. It is shown that this motion will be stable if

$$(Rw + Z)\,[(R - P)\,w + Z] + \frac{\Omega^2 c^2}{4A}P$$

is positive.

Substituting, we see that the motion will be stable for any value of Ω if

$$\left(w + \frac{\kappa c}{\pi a^2}\right)\left(w + \frac{\kappa c \rho}{2\pi a^2 (\rho + \rho')}\right) > 0.$$

Therefore the motion is stable for all positive values of w, and for all negative values numerically less than

$$\frac{\kappa c}{\pi a^2} \frac{\rho}{2(\rho + \rho')},$$

or numerically greater than

$$\frac{\kappa c}{\pi a^2}.$$

The motion is stable for values of w between these values if

$$\Omega^2 > -\frac{2\rho(\rho + \rho')^2}{\rho'^2(\rho + 2\rho')}\left(w + \frac{\kappa c}{\pi a^2}\right)\left(w + \frac{\kappa c}{\pi a^2} \frac{\rho}{2(\rho + \rho')}\right).$$

The greatest value the right-hand side can have is

$$\frac{\kappa^2 c^2}{\pi^2 a^4} \frac{\rho(3\rho + 2\rho')}{\rho'^2}.$$

Therefore the motion is always stable if

$$\Omega > \frac{\kappa c}{\pi a^2} \sqrt{\frac{\rho(3\rho + 2\rho')}{\rho'^2}}.$$

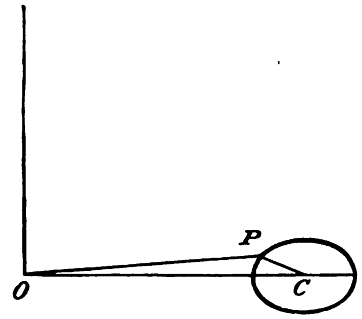
Another possible steady motion of the ring is for it to move round in a circle, as if it were a rigid body attached to an axis in the plane of the ring. Mr. BASSET shows that if T be the time of a complete oscillation, and r the radius of the circle described by the centre of the ring, then $r = \mathrm{T}/r\pi \,.\, \mathrm{Z}/\mathrm{R}$. When the ring is very fine this becomes

$$r = \frac{\mathrm{T}}{16\pi^2} \frac{c\kappa}{a^2} \frac{\rho}{\rho + \rho'}.$$

Section V.—*Annular Form of Rotating Fluid.*

§ 20. In order that a surface may be a possible figure of equilibrium of rotating fluid, it is necessary that $\mathrm{V} + \frac{1}{2}\omega^2 r^2 \sin^2\theta$ should be constant over the boundary.

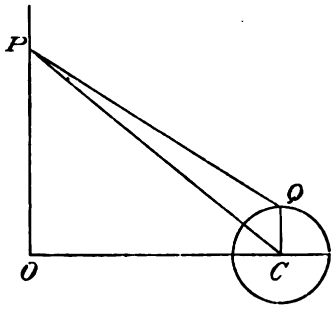
Let us assume that $\rho = a(1 + \beta_1 \cos\chi + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \ldots)$ is the equation to the cross-section of the annulus, and that β_1, β_2, &c., are small quantities.



Then, by making C the centre of gravity of the cross-section, we obtain the equation $\beta_1 + \beta_1\beta_2 + \beta_2\beta_3 + \ldots = 0$; so that β_1 vanishes compared with β_2.

We shall show that β_2 is of order σ^2, β_3 of order σ^3, &c., where σ denotes a/c, and is taken fairly small.

To the first order of the small quantities β_2, β_3, β_4, the potential of the ring is the potential of the solid ring $r = a$, together with the potential of a distribution on the ring of surface density $a\ (\beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)$, it will be most convenient to find the potential in this way, and take account of the terms arising from $\beta_2{}^2$ separately.



§ 21.—*To find the Potential of a Surface Distribution* $a\beta_2 \cos 2\chi$ *on the Ring at a Point on the Axis.*

Let

$$\angle \mathrm{OCQ} = \chi : \angle \mathrm{OCP} = \alpha : \angle \mathrm{PCQ} = \phi.$$
$$\mathrm{OC} = c : \mathrm{CQ} = a : \mathrm{CP} = \mathrm{R}.$$

Then

$$\mathrm{V} = 2\pi a^2 \int_0^{2\pi} \frac{(c - a\cos\chi)\,\beta_2 \cos 2\chi}{\sqrt{(\mathrm{R}^2 - 2a\mathrm{R}\cos\phi + a^2)}}\,d\phi$$

$$= \frac{2\pi a^2 c\,\beta_2}{\mathrm{R}} \int_0^{2\pi} \left\{\cos 2(\phi + \alpha) - \frac{a}{2c}\cos(\phi + \alpha) - \frac{a}{2c}\cos 3(\phi + \alpha)\right\}$$
$$\left\{1 + \frac{a}{\mathrm{R}}\mathrm{P}_1 + \frac{a}{\mathrm{R}^2}\mathrm{P}_2 + \&\text{c.}\right\} d\phi$$

$$= \frac{2\pi a^2 c\beta_2}{\mathrm{R}} \int_0^{2\pi} \left\{\cos 2\phi \cos 2\alpha - \frac{\sigma}{2}\cos\phi\cos\alpha - \frac{\sigma}{2}\cos 3\phi \cos 3\alpha\right\}$$
$$\left\{1 + \frac{a}{\mathrm{R}}\mathrm{P}_1 + \frac{a^2}{\mathrm{R}^2}\mathrm{P}_2 + \ldots\right\} d\phi,$$

since such integrals as

$$\int_0^{2\pi} \sin 2\phi \, \mathrm{P}_n (\cos\phi) \, d\phi$$

vanish. Therefore

$$\begin{aligned} \mathrm{V} = {} & \frac{2\pi a^2 c}{\mathrm{R}} \beta_2 \cos 2\alpha \int_0^{2\pi} \left\{ \frac{a^2}{\mathrm{R}^2} \cos 2\phi \, \mathrm{P}_2 + \frac{a^4}{\mathrm{R}^4} \cos 2\phi \, \mathrm{P}_4 + \frac{a^6}{\mathrm{R}^6} \cos 2\phi \, \mathrm{P}_6 + \ldots \right\} d\phi \\ & + \frac{2\pi a^2 c}{\mathrm{R}} \beta_2 \cos \alpha \int_0^{2\pi} \left\{ -\frac{\sigma}{2} \frac{a}{\mathrm{R}} \cos\phi \, \mathrm{P}_1 - \frac{\sigma}{2} \frac{a^3}{\mathrm{R}^3} \cos\phi \, \mathrm{P}_1 - \ldots \right\} d\phi \\ & + \frac{2\pi a^2 c}{\mathrm{R}} \beta_2 \cos 3\alpha \int_0^{2\pi} \left\{ -\frac{\sigma}{2} \frac{a^3}{\mathrm{R}^3} \cos 3\phi \, \mathrm{P}_3 - \frac{\sigma}{2} \frac{a^5}{\mathrm{R}^5} \cos 3\phi \, \mathrm{P}_5 \right\} d\phi \end{aligned}$$

$$\begin{aligned} = {} & \beta_2 \frac{2\pi^2 a^2 c}{\mathrm{R}} \cos 2\alpha \left\{ \tfrac{3}{4} \frac{a^2}{\mathrm{R}^2} + \tfrac{5}{16} \frac{a^4}{\mathrm{R}^4} + \tfrac{105}{512} \frac{a^6}{\mathrm{R}^6} + \ldots \right\} \\ & - \beta_2 \frac{2\pi^2 a^2 c}{\mathrm{R}} \frac{\sigma}{2} \cos \alpha \left\{ \frac{a}{\mathrm{R}} + \tfrac{3}{8} \frac{a^3}{\mathrm{R}^3} + \ldots \right\} \\ & - \beta_2 \frac{2\pi^2 a^2 c}{\mathrm{R}} \frac{\sigma}{2} \cos 3\alpha \left\{ \tfrac{5}{8} \frac{a^3}{\mathrm{R}^3} + \tfrac{35}{128} \frac{a^5}{\mathrm{R}^5} + \ldots \right\}, \end{aligned}$$

the integrals being found at once by using the expansions of P_1, P_2, &c., in cosines of multiples of ϕ.

Substituting now $\frac{c}{\mathrm{R}}$ for $\cos\alpha$, $\frac{2c^2}{\mathrm{R}^2} - 1$ for $\cos 2\alpha$, $\frac{4c^3}{\mathrm{R}^2} - \frac{3c}{\mathrm{R}}$ for $\cos 3\alpha$: we find for the value of the potential

$$\mathrm{V} = 2\pi a^2 c \beta_2 \left\{ -\tfrac{5}{4} \frac{a^2}{\mathrm{R}^3} + \left(\tfrac{3}{2} a^2 c^2 + \tfrac{7}{16} a^4\right) \frac{1}{\mathrm{R}^5} - \tfrac{5}{8} a^4 c^2 \frac{1}{\mathrm{R}^7} - \tfrac{35}{256} \frac{a^6 c^2}{\mathrm{R}^9} + \text{\&c.} \right\}.$$

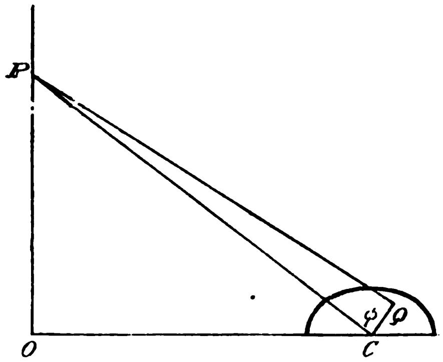
Similarly, the surface density $\alpha\beta_3 \cos 3\chi$ gives

$$\mathrm{V} = 2\pi a^2 c \beta_3 \left\{ -\tfrac{21}{8} \frac{a^3 c}{\mathrm{R}^5} + \tfrac{5}{2} \frac{a^3 c^3}{\mathrm{R}^7} - \tfrac{35}{32} \frac{a^5 c^3}{\mathrm{R}^9} + \ldots \right\},$$

and the surface density $\alpha\beta_4 \cos 4\chi$ gives

$$\mathrm{V} = 2\pi a^2 c \beta_4 \left\{ \tfrac{35}{8} \frac{a^4 c^4}{\mathrm{R}^9} \right\}.$$

To find the part of the potential involving β_2^2.



The potential of the ring at P is given by

$$V = \int_0^{2\pi}\int_0^{\rho} 2\pi \frac{(c - r\cos\chi)\, r\, dr\, d\chi}{\sqrt{R^2 - 2Rr\cos\phi + r^2}}$$

$$= 2\pi \int_0^{2\pi} \left\{ \left(\frac{c\rho^2}{2} - \frac{\rho^3}{3}\cos\chi\right)\frac{1}{R} + \left(\frac{c\rho^3}{3} - \frac{\rho^4}{4}\cos\chi\right)\frac{\rho_1(\cos\phi)}{R^2} + \ldots \right\} a\chi.$$

Now

$$\rho = a(1 + \beta_2\cos 2\chi + \ldots),$$

therefore

$$\rho^2 = a^2\left\{1 + 2\beta_2\cos 2\chi + \ldots + \frac{\beta_2^2}{4} + \frac{\beta_2^2}{4}\cos 4\chi + \ldots\right\},$$

$$\rho^3 = a^3\left\{1 + 3\beta_2\cos 2\chi + \ldots + \frac{3\beta_2^2}{2} + \frac{3\beta_2^2}{2}\cos 4\chi + \ldots\right\},$$

&c.

Therefore, the terms which will give rise to terms of highest order in β_2^2 at surface of ring are

$$2\pi a^2 c\beta_2^2 \int_0^{2\pi} \left\{\frac{1}{4R} + \tfrac{5}{4}\frac{a^4}{R^5}\cos 4\chi\, P_4(\cos 4\phi)\right\} d\phi$$

$$= 2\pi^2 a^2 c\beta_2^2 \left\{\frac{1}{2R} + \tfrac{5}{4}\cdot\tfrac{35}{64}\cos 4\alpha\,\frac{a^4}{R^5}\right\}$$

$$= 2\pi^2 a^2 c\beta_2^2 \left\{\frac{1}{2R} + \tfrac{175}{32}\frac{a^4c^4}{R^9}\right\}.$$

Collecting these results, the potential of the ring at P is given by

$$\frac{V}{2\pi a^2 c} = \frac{1}{R}\left(1 + \frac{\beta_2^2}{2}\right) + \frac{ac}{R^3}\left(-\frac{\sigma}{8} - \tfrac{5}{4}\beta_2\sigma\right) + \frac{a^2c^2}{R^5}\left\{-\frac{\sigma^2}{64} + \beta_2\left(\tfrac{3}{2} + \tfrac{7}{16}\sigma^2\right) - \tfrac{21}{8}\sigma\beta_3\right\}$$

$$+ \frac{a^3c^3}{R^7}\left\{-\frac{5\sigma^3}{1024} - \tfrac{5}{8}\sigma\beta_2 + \tfrac{5}{2}\beta_3\right\} + \frac{a^4c^4}{R^9}\left\{-\tfrac{35}{2^{14}}\sigma^4 - \tfrac{35}{256}\sigma^2\beta_2 - \tfrac{35}{32}\sigma\beta_3 + \tfrac{35}{8}\beta_4 + \tfrac{175}{32}\beta_2^2\right\}$$

$$+ \text{\&c.}$$

§ 22. The potential at any point external to the ring whose polar coordinates are r, θ, is found by writing

$$\frac{1}{\pi}\int_0^\pi \frac{d\phi}{\sqrt{(r^2 + c^2 - cr\sin\theta\cos\phi)}}, \text{ \&c.,}$$

instead of $1/\text{R}$, $1/\text{R}^3$, &c., as in Section II., § 5.

Instead of $1/\text{R}$, $1/\text{R}^3$, $1/\text{R}^5$, &c., we write

$$\frac{\text{I}_1}{\pi},\quad \frac{\text{I}_2}{\pi},\quad \frac{\text{I}_3}{3\pi},\quad \frac{\text{I}_4}{3.5\pi},\quad \frac{\text{I}_5}{3.5.7\pi}.$$

Therefore, the potential at any point outside the ring is given by

$$\frac{\text{V}}{2\pi a^2 c} = \text{A}_1\text{I}_1 + \text{A}_2 ac\,\text{I}_2 + \text{A}_3 a^2c^2\,\text{I}_3 + \text{\&c.,}$$

where

$$\text{A}_1 = 1 + \frac{\beta_2^2}{2}$$

$$\text{A}_2 = -\frac{\sigma}{8} - \tfrac{5}{4}\sigma\beta_2$$

$$\text{A}_3 = -\frac{\sigma^2}{192} + \beta_2\left(\tfrac{1}{2} + \tfrac{7}{48}\sigma^2\right) - \tfrac{7}{8}\sigma\beta_3$$

$$\text{A}_4 = \tfrac{1}{3}\left(-\frac{\sigma^3}{1024} - \tfrac{1}{8}\sigma\beta_2 + \tfrac{1}{2}\beta_3\right)$$

$$\text{A}_5 = \tfrac{1}{3}\left(-\frac{\sigma^4}{2^{14}} - \frac{\sigma^2\beta_2}{256} - \frac{\sigma\beta_3}{32} + \frac{\beta_4}{8} + \frac{5\beta_2^2}{32}\right).$$

Using the expansions given in § 3 (A), the potential at a point R, χ near the surface of the ring is given by

$$\begin{aligned}
\frac{\text{V}}{2\pi a^2} = {} & \text{A}_1\left(l + 2 + \frac{2l+1}{16}s^2 + \frac{108l+27}{2048}s^4\right) + \text{A}_2\sigma\left(\frac{2l+3}{4} + \frac{12l+5}{128}s^2\right) + \text{A}_3\sigma^2\frac{36l+51}{32} \\
& + \cos\chi\left\{\text{A}_1\left(\frac{l+1}{2}s + \frac{9l+3}{64}s^3\right) + \text{A}_2\sigma\left(\frac{1}{s} + \frac{12l+11}{32}s\right) + \text{A}_3\sigma^2\frac{9}{4s}\right\} \\
& + \cos 2\chi\left\{\text{A}_1\left(\frac{3l+2}{16}s^2 + \frac{20l+4\frac{1}{3}}{256}s^4\right) + \text{A}_2\sigma\left(\tfrac{1}{4} + \frac{12l+9}{64}s^2\right)\right. \\
& \qquad\qquad \left. + \text{A}_3\sigma^2\left(\frac{1}{s^2} + \tfrac{3}{4}\right) + \text{A}_4\sigma^3\frac{5}{s^2}\right\} \\
& + \cos 3\chi\left\{\text{A}_1\frac{5l+2\frac{1}{3}}{64}s^3 + \text{A}_2\sigma\tfrac{3}{32}s + \text{A}_3\sigma^2\frac{1}{4s} + \text{A}_4\sigma^3\frac{2}{s^3}\right\} \\
& + \cos 4\chi\left\{\text{A}_1\frac{35l+11\frac{1}{3}}{1208}s^4 + \text{A}_2\sigma\tfrac{5}{128}s^2 + \text{A}_3\sigma^2\tfrac{3}{32} + \text{A}_4\sigma^3\frac{1}{2s^2} + \text{A}_5\sigma^4\frac{6}{s^4}\right\}
\end{aligned}$$

where as before, l stands for $\log(8c/\text{R}) - 2$, s for R/c, and σ for a/c.

§ 23. At the surface of the ring

$$V + \frac{\omega^2 \varpi^2}{2}$$

is constant. Therefore

$$V + \frac{\omega^2}{2}(c - R\cos\chi)^2$$

is constant at the surface of the ring, or

$$\frac{V}{2\pi a^2} + \frac{\omega^2 c^2}{4\pi a^2}(1 - s\cos\chi)^2$$

is constant, when

$$s = \sigma(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi).$$

Let

$$\frac{V}{2\pi a^2} + \frac{\omega^2 c^2}{4\pi a^2}(1 - s\cos\chi)^2$$

be called $f(s)$.

Then

$$f(s) = f(\sigma) + f'(\sigma)\,\sigma\,(\beta_2 \cos 2\chi + \ldots) + \tfrac{1}{2} f''(\sigma)\,\sigma^2 \beta_2{}^2 \frac{1 + \cos 4\chi}{2}.$$

Now

$$\begin{aligned} f'(s) = {} & A_1\left(-\frac{1}{s} + \frac{l}{4}s\right) - A_2\sigma\frac{1}{2s} + \frac{\omega^2 c^2}{4\pi a^2}s \\ & + \cos\chi\left\{A_1\frac{l}{2} - \frac{A_2\sigma}{s^2} - \frac{\omega^2 c^2}{2\pi a^2}\right\}\cos\chi \\ & + \cos 2\chi\left\{A_1\frac{6l+1}{16}s - \frac{2A_3\sigma^2}{s^3} + \frac{\omega^2 c^2}{4\pi a^2}s\right\} \end{aligned}$$

and

$$f''(s) = \left(\frac{A_1}{s^2} + \frac{\omega^2 c^2}{4\pi a^2}\right) + \frac{\omega^2 c^2}{4\pi a^2}\cos 2\chi.$$

Therefore, $\sigma f'(\sigma)(\beta_2 \cos 2\chi + \&c.)$

$$\begin{aligned} = {} & \left(A_1\frac{6\lambda+1}{32}\sigma^2 - A_3 + \frac{\omega^2 c^2}{4\pi a^2}\cdot\frac{\sigma^2}{2}\right)\beta_2 \\ & + \left(A_1\frac{\lambda}{4}\sigma - \frac{A_2}{2} - \frac{\omega^2 c^2}{4\pi a^2}\sigma\right)\beta_2\cos\chi \\ & + \left\{\left[A_1\left(-1 + \frac{\lambda}{4}\sigma^2\right) - \frac{A_2}{2}\sigma + \frac{\omega^2 c^2}{4\pi a^2}\right]\beta_2 + A_1\left[\frac{\lambda}{4}\sigma - \frac{A_2}{2} - \frac{\omega^2 c^2}{4\pi a^2}\sigma\right]\beta_3\right\}\cos 2\chi \\ & + \left\{\left(A_1\frac{\lambda}{4}\sigma - \frac{A^2}{2} - \frac{\omega^2 c^2}{4\pi a^2}\sigma\right)\beta_2 - A_1\beta_3\right\}\cos 3\chi \\ & + \left\{\left(A_1\frac{6\lambda+1}{32}\sigma^2 - A_3 + \frac{\omega^2 c^2}{4\pi a^2}\frac{\sigma^2}{2}\right)\beta_2 + \left(A_1\frac{\lambda}{4}\sigma - \frac{A_2}{2} - \frac{\omega^2 c^2}{4\pi a^2}\sigma\right)\beta_3 - A_1\beta_4\right\}\cos 4\chi, \end{aligned}$$

and

$$\frac{\sigma^2}{2} f''(\sigma)\,\beta_2^2\,\frac{1+\cos 4\chi}{2} = \frac{A_1}{4}\beta_2^2 + \frac{A_1}{4}\beta_2^2\cos 4\chi.$$

All terms which are of order higher than σ^4 have been neglected; and throughout, it has been assumed that $\omega^2 c^2/4\pi a^2$ is of zero order in σ, β_2 of the second order, β_3 of the third, and so on—an assumption which will be justified by the result.

The value of

$$\frac{V}{2\pi a^2} + \frac{\omega^2 c^2}{4\pi a^2}(1 - s\cos\chi)^2$$

at the surface of the ring is

$$\begin{aligned}
&A_1\left(\lambda + 2 + \frac{2\lambda+1}{16}\sigma^2 + \frac{108\lambda+27}{2048}\sigma^4 + \frac{6\lambda+1}{32}\sigma^2\beta_2 + \tfrac{1}{4}\beta_2^2\right)\\
&\quad + A_2\left(\frac{2\lambda+3}{4}\sigma + \frac{12\lambda+5}{128}\sigma^3\right) + A_3\left(\frac{36\lambda+51}{32}\sigma^2 - \beta_2\right) + \frac{\omega^2c^2}{4\pi a^2}\left(1 + \frac{\sigma^2}{2} + \beta_2\frac{\sigma^2}{2}\right)\\
&+ \cos\chi\left\{A_1\left(\frac{\lambda+1}{2}\sigma + \frac{9\lambda+3}{64}\sigma^3 + \frac{\lambda}{4}\sigma\beta_2\right) + A_2\left(1 + \frac{12\lambda+11}{32}\sigma^2 - \frac{\beta_2}{2}\right)\right.\\
&\qquad\qquad\left. + A_3\tfrac{9}{4}\sigma + \frac{\omega^2c^2}{4\pi a^2}(-2\sigma - \sigma\beta_2)\right\}\\
&+ \cos 2\chi\left\{A_1\left(\frac{3\lambda+2}{16}\sigma^2 + \frac{20\lambda+4\frac{1}{3}}{256}\sigma^4 - \beta_2 + \frac{\lambda}{4}\sigma^2\beta_2 + \frac{\lambda}{4}\sigma\beta_3\right)\right.\\
&\qquad + A_2\left(\frac{\sigma}{4} + \frac{12\lambda+9}{64}\sigma^3 - \frac{\sigma}{2}\beta_2 - \tfrac{1}{2}\beta_3\right) + A_3\left(1 + \tfrac{3}{4}\sigma^2\right)\\
&\qquad\qquad\left. + A_4\,5\sigma + \frac{\omega^2c^2}{4\pi a^2}\left(\frac{\sigma^2}{2} + \beta_2 - \sigma\beta_3\right)\right\}\\
&+ \cos 3\chi\left\{A_1\left(\frac{5\lambda+2\frac{1}{3}}{64}\sigma^3 + \frac{\lambda}{4}\sigma\beta_2 - \beta_3\right) + A_2\left(\tfrac{3}{32}\sigma^2 - \tfrac{1}{2}\beta_2\right)\right.\\
&\qquad\qquad\left. + A_3\frac{\sigma}{4} + 2A_4 - \frac{\omega^2c^2}{4\pi a^2}\sigma\beta_2\right\}\\
&+ \cos 4\chi\left\{A_1\left(\frac{35\lambda+11\frac{1}{3}}{1024}\sigma^4 + \frac{6\lambda+1}{32}\sigma^2\beta_2 + \frac{\lambda}{3}\sigma\beta_3 - \beta_4 + \frac{\beta_2^2}{4}\right)\right.\\
&\qquad + A_2\left(\tfrac{5}{128}\sigma^3 - \tfrac{1}{2}\beta_3\right) + A_3\left(\tfrac{3}{32}\sigma^2 - \beta_2\right)\\
&\qquad\qquad\left. + A_4\frac{\sigma}{2} + 6A_5 + \frac{\omega^2c^2}{4\pi a^2}\left(\frac{\sigma^2}{2}\beta_2 - \sigma\beta_3\right)\right\}.
\end{aligned}$$

Equating the coefficients of $\cos\chi$, $\cos 2\chi$, $\cos 3\chi$, $\cos 4\chi$ to zero, we obtain four equations to find

$$\frac{\omega^2c^2}{4\pi a^2},\ \beta_2,\ \beta_3,\ \text{and}\ \beta_4.$$

Substituting the values of A_1, A_2, &c., these equations are

$$\left.\begin{aligned}
&-\frac{\omega^2c^2}{2\pi a^2}\left(1+\frac{\beta_2}{2}\right)+\frac{4\lambda+3}{8}+\frac{12\lambda-1}{128}\sigma^2+\frac{4\lambda-1}{16}\beta_2=0,\\
&\frac{3\lambda+\frac{17}{12}}{16}\sigma^2+\frac{7\lambda-\frac{19}{24}}{128}\sigma^4+\frac{\omega^2c^2}{\pi a^2}\frac{\sigma^2}{8}+\beta_2\left\{-\tfrac{1}{2}+\frac{4\lambda+1}{16}\sigma^2+\frac{\omega^2c^2}{\pi a^2}\frac{\sigma^2}{4}\right\}\\
&\qquad\qquad+\beta_3\left\{\left(\lambda-\tfrac{1}{12}\right)\frac{\sigma}{4}-\frac{\omega^2c^2}{\pi a^2}\frac{\sigma}{4}\right\}=0,\\
&\frac{\lambda+\frac{35}{24}}{64}\sigma^3+\left(\lambda+\tfrac{5}{12}-\frac{\omega^2c^2}{\pi a^2}\right)\frac{\sigma}{4}\beta_2-\tfrac{2}{3}\beta_3=0,\\
&\frac{35\left(\lambda+\frac{19}{120}\right)}{1024}\sigma^4+\frac{3\lambda+\frac{7}{8}}{16}\sigma^2\beta_2+\frac{\lambda+\frac{1}{3}}{4}\sigma\beta_3-\tfrac{3}{4}\beta_4+\frac{\beta_2^2}{16}+\frac{\omega^2c^2}{\pi a^2}\left(\frac{\sigma^2}{8}\beta_2-\frac{\sigma}{4}\beta_3\right)=0.
\end{aligned}\right\}$$

These equations give

$$\left.\begin{aligned}
\frac{\omega}{\pi}&=\left(\lambda+\tfrac{3}{4}\right)\sigma^2-\tfrac{1}{8}\left(\lambda+\tfrac{19}{12}\right)\sigma^4\\
\beta_2&=\frac{\frac{5}{8}\left(\lambda+\frac{7}{12}\right)\sigma^2+\frac{3}{64}\left(\lambda-\frac{11}{12}\right)\sigma^4}{1-\left(\lambda+\frac{1}{2}\right)\sigma^2}\\
\beta_3&=\tfrac{5}{128}\left(\lambda-\tfrac{7}{24}\right)\sigma^3\\
\beta_4&=\frac{75\lambda^2+80\lambda+21}{256}\sigma^4,
\end{aligned}\right\}$$

where $\sigma = a/c$ and $\lambda = \log_e(8c/a) - 2$.

The result $\omega^2/\pi = (\lambda + \frac{3}{4})\sigma^2$ is given by **Mme. Kowalewski** in a paper on Saturn's rings in the 'Astronomische Nachrichten' for 1885. She finds $\beta_2 = \frac{1}{2}(\lambda + \frac{3}{4})\sigma^2$. M. **Poincaré** also gives $\omega^2/\pi = (\lambda + \frac{3}{4})\sigma^2$, and finds $\beta_2 = \frac{3}{4}(\lambda + \frac{3}{4})\sigma_2$. Both papers are given in **Tisserand's** 'Mécanique Céleste,' vol. 2. The value found above has been kindly verified for me by Mr. **Herman**, Fellow of Trinity College, Cambridge.

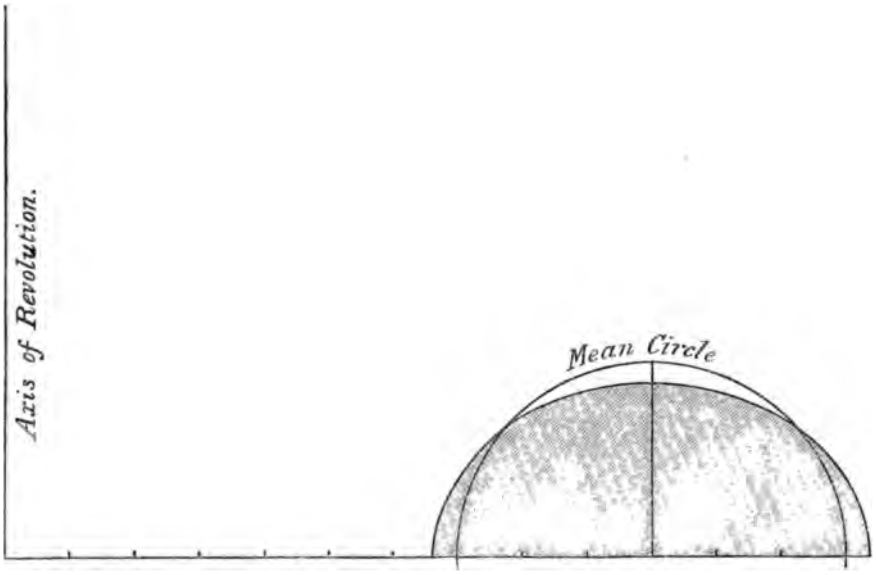
The numerical values of ω^2/π, β_2, &c., are given below when $\sigma = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$.

$\sigma = \cdot 1$	$\lambda = 2\cdot 3820$	$\beta_2 = \cdot 0189$	$\beta_3 = \cdot 0001$	$\beta_4 = \cdot 0003$	$\frac{\omega^2}{\pi\rho} = \cdot 0313$
$\sigma = \cdot 2$	$\lambda = 1\cdot 6889$	$\beta_2 = \cdot 0570$	$\beta_3 = \cdot 0004$	$\beta_4 = \cdot 0023$	$\frac{\omega^2}{\pi\rho} = \cdot 0970$
$\sigma = \cdot 3$	$\lambda = 1\cdot 2834$	$\beta_2 = \cdot 1268$	$\beta_3 = \cdot 0010$	$\beta_4 = \cdot 0078$	$\frac{\omega^2}{\pi\rho} = \cdot 1836.$

A figure is given for the case of $\sigma = \cdot 3$.

The cross-section is roughly an ellipse whose major axis is perpendicular to the axis of the ring. The eccentricity of this ellipse increases with the angular velocity.

[When the paper was read, numerical results were given for larger values of σ. The above method cannot, however, be employed in such cases. For, as the eccentricity of the cross-section increases, the expression given for the potential at external points will become divergent at the extremity of the minor axis of the cross-section. A similar result will occur if the potential of an elliptic cylinder be found at external points, considering it as an approximation to a circular cylinder. In this case it is easy to show that the eccentricity must be $< 1/\sqrt{2}$. Assuming the same result for the case of a ring, it will be seen that $\beta_2 = \cdot 2076$, the value the above method gives for $\sigma = \cdot 4$, gives the cross-section of too great eccentricity for the series to be convergent. That the series were divergent for large values of β_2 was also suggested by one of the Referees. Aug., 1892.]



III. BAKERIAN LECTURE.—*The Rate of Explosion in Gases.*

By HAROLD B. DIXON, *M.A., F.R.S., late Fellow of Balliol College, Oxford; Professor of Chemistry in Owens College, Manchester.*

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[PLATE 1.]

CAP. I.—INTRODUCTION.

THE experiments described in this memoir were undertaken with two objects : in the first place, to obtain information concerning the course of chemical change pursued by reacting gases ; and, secondly, to examine the nature of the "explosion-wave" in gaseous mixtures discovered by M. BERTHELOT.

The idea of using the rate of explosion as a means of determining the course of a chemical reaction occurred to me in 1877, when investigating the influence of steam on the union of carbonic oxide and oxygen. If steam acts as a carrier of oxygen to the carbonic oxide by a series of alternate reductions and oxidations, an increase in the amount of steam present, beyond that required to initiate the reaction, should be accompanied by an increase in the rate of combination up to a certain limit. Attempts were therefore made to detect such an increase by measuring the velocity of the flame in a tube.* But while the difference in the rate of explosion between the nearly dry and the moist gases was well marked, the attempts to directly measure the rate of the explosion of the moist gases failed, owing to the great rapidity of the flame. In the spring of 1881 I attempted to measure the rate of explosion of carbonic oxide and oxygen with varying quantities of steam by photographing on a moving plate the flashes at the beginning and end of a closed tube 20 feet long. The two flashes appeared to be *simultaneous* to the eye, but no record of the rate was obtained, for the apparatus was broken to pieces by the violence of the explosion. Shortly after this attempt was made the first of the brilliant series of papers by MM. BERTHELOT and VIEILLE, and by MM. MALLARD and LE CHATELIER, was read before the French Academy of Sciences. The work of these French chemists has opened a new era in the theory of explosions.

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* 'Phil. Trans.,' 1884, Pt. II., p. 635.

Thirty-five years ago BUNSEN* described a method of measuring the rapidity of the flame in gas explosions. Passing a mixture of explosive gases through an orifice at the end of a tube and igniting the gases as they issued into the air, he determined the rate at which the gases must be driven through the tube to prevent the flame passing back through the opening. By this method he found that the rate of propagation of the ignition of hydrogen and oxygen was 34 metres per second, while the rate of ignition of carbonic oxide and oxygen was less than 1 metre per second. BUNSEN *applied these results to the rate of explosion of gases in closed vessels:* his results were accepted without cavil for four and twenty years.

By 1880 facts began to accumulate which seemed inconsistent with BUNSEN'S conclusions. For instance, between 1876–80 I had several times observed that the flame produced by igniting a mixture of moist carbonic oxide and oxygen travelled in a long eudiometer too quickly to be followed by the eye. Again, Mr. A. V. HARCOURT,† in his investigation of an explosion of coal-gas and air, which happened in a large gas main near the Tottenham Court Road in 1880, was led to the conclusion that the flame travelled at a rate exceeding 100 yards per second. In the winter of 1880–1 I was startled by the rapid increase of velocity and violence as a flame of carbon bisulphide with nitric oxide travelled down a long glass vessel.

In July, 1881, two papers appeared in the 'Comptes Rendus,' one by M. BERTHELOT, the other by MM. MALLARD and LE CHATELIER. Both papers announced the discovery of the enormous velocity of explosion of gaseous mixtures. Other papers quickly followed by the same authors. M. BERTHELOT made the important discovery that the rate of explosion rapidly increases from its point of origin until it reaches a maximum which remains constant, however long the column of gases may be. Each mixture of gases has a definite maximum velocity of explosion. The rate of explosion thus forms a new physico-chemical constant, having important theoretical and practical bearings. The name "L'Onde Explosive" is given by BERTHELOT to the flame when propagated through an explosive mixture of gases at the maximum velocity.

While BERTHELOT, associated with VIEILLE, was measuring the rate of the "explosion-wave" for various mixtures of gases, MALLARD and LE CHATELIER continued the study of the preliminary phenomena of explosion which precede the formation of the "wave." They showed by photographing on a revolving cylinder:—(1) that when a mixture such as nitric oxide and carbon bisulphide is ignited at the open end of a tube, the flame travels a certain distance (depending on the diameter and length of the tube) at a uniform velocity; (2) that at a certain point in the tube vibrations are set up, which alter the character of the flame, and that these vibrations become more intense, the flame swinging backwards and forwards with oscillations of increasing amplitude; and (3) that the flame either goes out altogether, or the rest of the gas detonates with extreme velocity. Again, when a mixture of gases was fired near the closed end of the tube, they found the velocity of the flame regularly increased, as far as their instruments were able to record the rapidly increasing pace.

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* 'Gasometrische Methoden,' 1857.

† Report to the Board of Trade.

Mixtures of coal-gas with air, and of fire-damp with air, show phenomena of the first and second kind. Ignited at the open end of a tube these mixtures burn at a uniform rate for a certain distance, and then the flame begins to vibrate. The vibrations acquire greater or less velocity according to the nature of the mixture and the conditions of the experiment; but the third *régime* of uniform maximum velocity is not set up. In narrow tubes the explosion soon dies out.

CAP. II.—BERTHELOT'S EXPERIMENTS.

BERTHELOT'S experiments were made partly with a tube of lead, and partly with a tube of thick caoutchouc, usually 5 mm. in internal diameter. The rate of explosion was determined by making the flame break two strips of thin tin stretched across the tube, each carrying a current. The interval between the interruption of the two circuits was measured by a Le Boulengé chronograph. The gases were fired by an electric spark near one of the interrupters. To ensure the fracture of the strips of tin a grain of fulminate was placed in a fold of the metal. The tube was 40 metres long, and was supported on a wooden screen in horizontal layers.

The following are the chief conclusions reached by BERTHELOT concerning the propagation of the explosion-wave in gases :—

(1) The propagation is uniform. Measurements made in tubes of 20, 30, and 40 metres length show the same velocity for the same mixture.

(2) The velocity is independent of the material of the tube: the explosion travels at the same rate in tubes of lead as in tubes of caoutchouc.

(3) The velocity is independent of the diameter of the tube above a small limit: the explosion travels at the same rate in a tube of 5 mm. diameter as in a tube of 15 mm. diameter.

(4) The velocity is independent of the pressure : the explosion travels at the same rate when the gas is under a pressure of 560 mm. as it does under a pressure of 1580 mm.

(5) The velocity of the explosion equals, or approximates closely to, the mean velocity of translation of the molecules at the moment of combination, on the supposition that they retain all the heat developed in the reaction. The rates of explosion of some twenty different mixtures agree with the theoretical rate calculated by the formula of CLAUSIUS :—

$$\theta = 29{\cdot}354 \sqrt{\frac{T}{d}},$$

where T is the absolute temperature reached in the explosion, and *d* the density of the products of combustion referred to air.

In the following tables the rates of explosion of simple mixtures as measured by BERTHELOT are compared with the theoretical velocity of the products of combustion calculated from CLAUSIUS' formula. The temperature T is calculated on the supposition (1) that the gases are heated *at constant pressure*, and (2) that the specific heat of a compound gas is the sum of the specific heats of its constituents.

TABLE I.—BERTHELOT'S Experiments. Combustible Gases with Oxygen.

Gases.	Velocity in metres per second.	
	Calculated.	Found.
Hydrogen	2831	2810
Carbonic oxide	1940	1090
Marsh gas	2427	2287
Ethane	2483	2363
Ethylene	2517	2210
Acetylene	2660	2482
Cyanogen	2490	2195

TABLE II.—Combustible Gases with Nitrous Oxide.

Gases.	Velocity in metres per second.	
	Calculated.	Found.
Hydrogen	2250	2284
Carbonic oxide	1897	1106
Cyanogen	2198	2036

BERTHELOT observes that the formula does not apply to mixtures of carbonic oxide either with oxygen or with nitrous oxide. *It is to be noticed that* BERTHELOT *worked with dry gases.*

TABLE III.—Mixtures of Combustible Gases with Oxygen.

Gases.	Velocity in metres per second.	
	Calculated.	Found.
Hydrogen and carbonic oxide.		
(1) $H_2 + CO + O_2$	2236	2008
(2) $3H_2 + 2CO + O_5$ . . .	2321	2170
Ethylene and hydrogen.		
(1) $C_2H_4 + H_2 + O_7$ . . .	2551	2417
(2) $C_2H_4 + 2H_2 + O_8$ . . .	2588	2579
Ethane and hydrogen.		
$C_2H_6 + H_2 + O_8$	2522	2250

With mixtures of hydrogen and carbonic oxide the formula is found to hold good. BERTHELOT explains this by saying that "the hydrogen communicates to the carbonic oxide a law of detonation analogous to its own."

The general concordance between the observed velocities and the calculated rate of translation of the molecules shows, according to BERTHELOT, that dissociation plays but a small part in these phenomena, perhaps because of the high pressure developed: a result which is confirmed by the fact that the velocities are found to be independent of the pressure.

A comparison is next made between the rates of explosion of "isomeric mixtures," *i.e.*, mixtures of different gases which yield the same products of combustion. For instance, the rate of explosion of marsh gas and oxygen is compared with the rate of explosion of a mixture of ethylene, hydrogen, and oxygen yielding identical products; the object being to eliminate the influence of the individual gases burnt. The same conclusions are drawn.

TABLE IV.—Combustible Gases with Oxygen and Nitrogen.

Gases.	Velocity in metres per second.	
	Calculated.	Found.
$H_2 + O + N_2$	1935	2121
$H_2 + O + N_4$ . . . . (Air)	1820	1439
$CO + O + N_2$.	1661	1000 ?
$CO + O + N_4$.	1236	Detonation not propagated
$CH_4 + O_4 + N_4$	2002	1858
$CH_4 + O_4 + N_8$	1744	1151
$CH_4 + O_4 + N_{15}$ . . . . (Air)	1450	Detonation not propagated
$C_2N_2 + O_4 + N_2$	2334	2044
$C_2N_2 + O_4 + N_4$	2152	1203
$C_2N_2 + O_4 + N_8$	1920	Detonation not propagated

When the explosive gases are mixed with an inert gas, nitrogen, which takes no part in the reaction, the same law holds good—except when the nitrogen is added in excess. Before the gases are diluted sufficiently to stop the explosion, there is found a marked falling off in the velocity. The formula gives the theoretically highest rate the explosion can attain—a maximum reached in few cases only, but approached in a large number.

Berthelot's Conclusions.

These results show, according to BERTHELOT, that the velocity of the explosion-wave constitutes, for each inflammable mixture, a true specific constant. The wave is propagated by the impact of the products of combustion of one layer upon the unburnt gases in the next layer, and so on to the end of the tube at the rate of movement of the products of combustion themselves. In a word, the mean velocity of translation of the gaseous molecules retaining the total *vis viva* which corresponds to the heat developed in the reaction *may be regarded as a limit representing the maximum rate of propagation of the explosion-wave* ('Sur la Force des Matières Explosives,' 1, p. 159).

If this theory is true, it accounts not only for the extreme rapidity of explosion of gaseous mixtures, and gives us the means of calculating the maximum velocity obtainable with any mixture of gases, but it also affords us information on the specific heats of gases at very high temperatures, and it explains the phenomena of *detonation* whether of gases or of solid or liquid explosives.

CAP. III.—REPETITION OF BERTHELOT'S EXPERIMENTS.

§ 1. *Objects in View.*

A consideration of BERTHELOT'S results, published in the 'Annales de Chimie'* in 1883, led me to think it would be useful to repeat and extend these experiments. The close coincidence between the rates of explosion of hydrogen, both with oxygen and with nitrous oxide, and the calculated velocities of the products of combustion showed that the formula held good for gases which could readily be prepared in a pure state; and, again, the great discordance between the found and calculated rates for carbonic oxide, both with oxygen and nitrous oxide, was what I should have expected from my own experiments on the part taken by steam in the oxidation of carbonic oxide. On the other hand, BERTHELOT'S contention that the gases are heated at constant pressure appeared improbable, and his results obtained with the addition of an inert gas seemed to vary capriciously. The chief objects I had in view in continuing these experiments were:—

(1) To determine as accurately as possible the rate of the explosion-wave for some simple mixtures under varying conditions, *e.g.*, diameter of tube, initial pressure, initial temperature.

(2) To measure the rate of the explosion-wave in carbonic oxide and oxygen with different quantities of steam.

(3) To compare the effect of inert gases and of excess of one or other of the reacting gases on the rate of explosion.

§ 2. *Methods and Apparatus used.*

The mixtures of hydrogen, carbonic oxide, and marsh gas with oxygen were prepared in a graduated 5-feet iron gas-holder over water; the mixtures containing ethylene, acetylene, cyanogen, and nitrous oxide were prepared in a 1-foot iron holder over mercury; the mixtures of hydrogen and chlorine were passed directly from the generating and purifying apparatus into the explosion tube.

The gases were driven from the holders through drying vessels into the explosion tube by placing weights upon the holders. When all air-traps were avoided in the drying tubes and connections, very little diffusion was found to occur in driving out the air by the explosive mixture. One "Drechsel" washing bottle and three towers packed with pumice, all containing boiled oil of vitriol, were usually employed as drying vessels. When the explosive mixture contained either ethylene or acetylene the gas was driven from the mercury holder through one washing bottle containing strong aqueous potash and through two towers packed with lumps of caustic potash. The cyanogen mixtures were passed direct from the mercury holder into the explosion tube, the gases having been dried in preparation.

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* 'Ann. Chim. et Phys.,' [V.] vol. 28; also 'Sur la force des Matières Explosives,' vol. 1., chap. 7.

The explosion tube was, in most of the experiments, a leaden pipe, 100 metres long and 9 mm. in diameter. With some of the more violent explosives—*e.g.*, cyanogen and acetylene, a leaden tube of 6·5 mm. was used; and in the chlorine experiments straight wrought-iron pipes lined with glass were employed.

The leaden tube was, in most cases, coiled on a drum, 2 feet in diameter, which was immersed in an iron vessel containing water. By heating the water and pumping dry air through the tube, the apparatus could be quickly dried after an explosion in which water was formed. For determinations of the rate of explosion of gases at 100° C. this arrangement was also suitable. Two inconveniences arose from this disposition of the pipe: (1) the difficulty of determining its exact length, and (2) the uncertainty whether the explosion-wave was propagated along the axis of the coiled pipe, or whether it followed a shorter path nearer to the inside wall.

It was found impossible to coil a small leaden pipe without lengthening it appreciably; the outside of the tube was therefore measured after each coil was wound on the drum, and the length of the axis of the pipe calculated. After a series of experiments the coils were then unwound, by rolling the drum along a corridor, and the length was measured directly. The length so obtained did not vary more than an inch, or at most two, from that calculated on winding.

To determine whether the flame was propagated centrally, or whether it took a shorter cut, measurements of the rate of explosion of samples of oxygen and hydrogen from the same mixture were made alternately, in a leaden pipe 9 mm. in diameter (1) coiled on the drum, and (2) lying straight on the floor. The experiments were thrice repeated, and no appreciable difference in the rate could be detected; from which it may be concluded that in a pipe of this bore the wave-front travels quickest along the axis. Possibly in the experiments with tubes of a larger diameter the curvature may have slightly affected the results.

In the experiments made under reduced pressure, the tube was filled from the holder in the ordinary way. The steel stopcock* near the second bridge was closed, and the gas was then sucked from the tube by a powerful pump, maintaining a vacuum of about 26 inches of mercury. A gauge showed when the desired reduction of pressure was reached. The other stopcock was then closed, and the spark immediately passed. In the experiments made under increased pressure, the gases were driven into the tube from a strong reservoir, either by water or by mercury, according to the nature of the gaseous mixture. In all cases the gases were fired by an electric spark near one end of the tube; about four feet from the firing point, the flame broke a piece of silver foil stretched across the tube, the moment of rupture being marked on a moving plate by an electro-magnetic style. On reaching the further end of the tube, the flame broke a second strip of silver, the moment of rupture being recorded by a second style. To eliminate errors due to retardation of the electro-magnets, a blank experiment was made immediately before firing the gases. The plate of the chronograph passed over two breaks, which could be adjusted so as to be broken by it at the same instant. In the preliminary experiment, the circuits of the two magnets were completed through these two breaks, and were thus broken simultaneously, the two styles registering their marks on the moving plate. The magnets, without their position being altered, were then connected with the silver bridges across the explosion tube, and the gases were fired by the moving plate striking one or other of the two breaks. The flame travelling down the tube broke the two silver bridges in turn, releasing the two styles, which again registered their marks on the moving plate. The two marks made by the first style gave the interval of time between the spark and the breaking of the first bridge, independently of the error of the electro-magnet; for both marks were equally affected by any error of retardation. In the same way, the two marks made by the second style gave the interval of time between the spark and the breaking of the second bridge, independently of the error of the electro-magnet. The difference between the intervals was the time taken by the flame to travel between the two bridges.

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* These steel stopcocks, made for me by Mr. J. J. HICKS, of Hatton Garden, have stood many hundreds of explosions without leaking. The gun-metal stopcocks employed at first were indented by the explosions and soon leaked.

This method of cutting out the errors of the chronograph is only valid when the current flowing through the electro-magnets is constant. A small resistance coil was interposed in the preliminary experiment in each chronograph circuit, to match the resistance of the silver-bridge and connections used in the actual determination. The current was supplied by a storage cell (Elwell and Parker) to each electro-magnet, and the time between the preliminary and the final experiment was never more than one minute. In all the later portion of the work a confirmatory experiment was made after the gases were fired, the connections being made as in the preliminary. In most cases the styles passed absolutely over the preliminary marks—merely deepening the lines, thus showing that nothing had been displaced. In some cases the preliminary and confirmatory marks were just visibly separated; in these cases the mean position of the two was used to calculate the rate. In the few instances in which the confirmatory did not closely agree with the preliminary marks the experiment was rejected. In order further to eliminate errors of the instrument a second experiment was always made as soon as possible after the first, the connections of the chronograph being all reversed—the first style being connected with the second bridge, and *vice versâ*. The mean result of the two experiments is taken as one determination.

The chief differences between my apparatus and that employed by Berthelot were the following:—

a. The explosion tube was longer and wider.

b. The interrupters or "bridges" were of silver foil, and no fulminate was used.

c. A longer space was allowed between the firing spark and the first "bridge" where the record of the rate was begun. This was found essential in several cases, especially when the mixture was fired under reduced pressure, or in presence of inert gases, in which cases the explosion does not reach its maximum rate for some feet.

§ 3. *On the Constant Velocity of the Explosion Wave.*

Experiments on the rate of explosion of hydrogen and oxygen in a leaden tube, 55 metres long and 8 mm. in internal diameter, gave a mean rate of 2817 metres per second. In a tube of the same diameter and 100 metres long, the mean rate was 2821 metres; and in a tube 100 metres long and 13 mm. diameter the mean rate was 2819.

The general mean of these experiments—viz., 2819, is in close agreement with the mean result obtained by BERTHELOT in a shorter tube—viz., 2810.

The constancy of the rate of the explosion-wave, under ordinary conditions of temperature and pressure, was, therefore, fully confirmed.

In the course of this investigation I have made many measurements of the rate of explosion of hydrogen and oxygen under ordinary conditions. Some of these were made with more precautions than others, so that in arriving at the most probable value of the rate, I have given greater weight to some sets than to others. The sets, each of which consists of from eight to ten distinct measurements, are lettered in the following table A, B, C, etc., in the order in which they were made:—

TABLE V.—General Mean of $H_2 + O$ Rates.

Date.	Sets.	Mean.
1884, 1885	A + B + C	2819
1888	D	2822
1889	E + F + G	2824
1890	H	2818
		2821

The details of these experiments will be found in the Appendix.

§ 4. *The Influence of Pressure on the Velocity of the Explosion Wave.*

Preliminary experiments on the rate of explosion of hydrogen and oxygen under 500 mm. and 1000 mm. pressure having shown an appreciable difference, careful measurements of the rate were made at pressures varying from 200 mm. to 1500 mm. at a temperature close to 10° C. In the experiments at low pressures it was, of course, essential that the firing tube should be absolutely gas-tight; the tube was therefore tested before each experiment. An important fact concerning the development of the explosion-wave was detected in these experiments. As the pressure was reduced the flame was found to travel a greater distance before its rate became constant; so that, although four feet start was allowed before the first measurement was taken, this distance was found insufficient. Accordingly a tube 20 feet long was interposed between the firing point and the first bridge, after it had been shown that the flame acquired its maximum velocity in this distance under the lowest pressure used. Electrolytic gas can be exploded under lower pressures than 200 mm., but the flame travels irregularly. At 100 mm. pressure the flame went out in the tube, and at 150 mm. the rates found were not constant.

TABLE VI.—Pressure Experiments. $H_2 + O$.

Pressure.	200 mm.	300 mm.	500 mm.	760 mm.	1100 mm.	1500 mm.
Mean rate . .	2627	2705	2775	2821	2856	2872

These figures show that the rate of explosion increases rapidly with increase of pressure from 200 mm.; that the rate of increase diminishes, and that the velocity becomes nearly constant at two atmospheres pressure. This effect of pressure on the rate is plainly seen when the results are expressed graphically (see Plate 1).

BERTHELOT'S conclusion that the explosion-wave is independent of the initial pressure of the gases is, therefore, not strictly accurate. At lower pressures the rate falls off, but above a certain *crucial pressure*, which, in the case of hydrogen and oxygen, seems to be about two atmospheres, the velocity is independent of the pressure.

It will be observed that the mean rate of explosion of hydrogen and oxygen at 1500 mm. pressure is appreciably higher than BERTHELOT'S θ—the calculated rate of translation of the steam molecules. But BERTHELOT, in his calculation, has not taken into account the fact that the gases are not at absolute zero to start with. Assuming that he worked at about 13° C., the theoretical rate would be 2900 metres per second.

Analogous results were found in experiments with other mixtures under varying pressures. With hydrogen and nitrous oxide the rate is slower at 500 mm. than at 760 mm., but no increase is found on raising the pressure to 1000 mm. The crucial pressure is in this instance below the atmospheric pressure. The same was found to be the case with marsh gas, ethylene, acetylene, and cyanogen exploded with oxygen.

In the following table some of these results are shown :—

TABLE VII.—Pressure Experiments.

$H_2 + N_2O$.

Pressure.	500 mm.	760 mm.	1000 mm.
Mean rate . .	2094	2307	2302

$CH_4 + O_4$.

Pressure.	500 mm.	760 mm.	1000 mm.
Mean rate . .	2280	2322	2319

$CH_4 + O_3$.

Pressure.	500 mm.	760 mm.	1000 mm.
Mean rate . .	2418	2470	2488

$C_2N_2 + O_2$ (in 5 mm. tube).

Pressure.	500 mm.	760 mm.	1000 mm.
Mean rate . .	2536	2677	2671

§ 5. *Influence of Temperature on the Velocity of the Explosion-Wave.*

BERTHELOT has published no experiments on the influence of initial temperature on the explosion-wave; but, if the wave is propagated in a manner analogous to a sound wave, it might be anticipated that an increase in the initial temperature of the gases would increase the velocity. The reverse effect is found in the explosion of hydrogen and oxygen. Experiments made at 10° C. and 100° C. showed a small, but distinct, lowering of the rate at the higher temperature. The experiments were made alternately at the high and low temperature, in order to eliminate the effect which might be produced by any mistake in the mixture or error in the chronograph. The comparative experiments made at 10° agreed with those previously obtained under the same conditions, so that the results found at 100° must be accepted.* A similar lowering of the rate at 100° was found to occur at low and high pressures.

TABLE VIII.

TEMPERATURE Experiments. $H_2 + O$.

Temperature.	10°.	100°.
Mean rate . .	2821	2790

$H_2 + O$ at 100° C. at different pressures.

Pressure.	390 mm.	500 mm.	760 mm.	1000 mm.	1450 mm.
Mean rate . .	2697	2738	2790	2828	2842

When the rates at 100° and varying pressures are compared graphically with the rates found at 10° and varying pressures, it is seen that the two curves run parallel one with the other (see fig. 1 on Plate 1).

The explosion-rates of ethylene and oxygen at 10° C. and 100° C. were also compared. Samples of the same mixture were exploded alternately at the low and at the high temperature. Five concordant determinations at 10° C. gave a mean velocity of 2581 metres per second; four concordant determinations at 100° C. gave a mean velocity of 2538 metres per second. Similar experiments were made with cyanogen.

TABLE IX.

RATE of Explosion of Ethylene and Oxygen. $C_2H_4 + 2O_2$.

Temperature.	10° C.	100° C.
Mean rate . .	2581	2538

RATE of Explosion of Cyanogen and Oxygen. $C_2N_2 + O_2$.

Temperature.	10° C.	100° C.
Mean rate . .	2728	2711

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* The expansion of the metal tube is taken into account in calculating the rates at 100° C.

The small differences in the rate of explosion of hydrogen and oxygen, and of ethylene and oxygen at 10° and at 100° show that the effect of ordinary atmospheric changes of temperature may be neglected.

§ 6. *Comparison of the Rate of Explosion of different Gaseous Mixtures with* BERTHELOT'S *results.*

The results obtained, under ordinary conditions of pressure and temperature, with hydrogen and oxygen, with hydrogen and nitrous oxide, and with marsh gas and oxygen, in exact proportions for complete combustion, were in close accordance with the mean results of BERTHELOT; for ethylene, acetylene, and cyanogen my numbers differed appreciably, but in no case differed by more than 7 per cent. from the rates observed by BERTHELOT. In the following table our measurements for the same mixtures are compared :—

TABLE X.—Mean Velocity of Explosion in Metres per Second.

		BERTHELOT.	DIXON.
Hydrogen and oxygen	H_2+O . .	2810	2821
Hydrogen and nitrous oxide	H_2+N_2O .	2284	2305
Marsh gas and oxygen	CH_4+O_4 .	2287	2322
Ethylene and oxygen	$C_2H_4+O_6$.	2210	2364
Acetylene and oxygen	$C_2H_2+O_5$.	2482	2391
Cyanogen and oxygen	$C_2N_2+O_4$.	2195	2321

The general agreement between these measurements leaves, I think, no room for doubt about the substantial accuracy of the results.

CAP. IV.—THE INFLUENCE OF STEAM ON THE RATE OF EXPLOSION OF CARBONIC OXIDE AND OXYGEN.

The results already described show that the formula proposed by BERTHELOT expresses with a close degree of approximation the rates of explosion of several gaseous mixtures. The formula fails for the explosion of carbonic oxide with oxygen or with nitrous oxide. This was to be expected if, in the *detonation* of carbonic oxide in a long tube, the oxidation is effected indirectly by means of steam—as it is in the ordinary combustion of the gas. Measurements of the rate of explosion of carbonic oxide and oxygen in a long tube showed that the rate increased as steam was added to the dry mixture, until a maximum velocity was attained when between 5 and 6 per cent. of steam was present.

The experiments were conducted in the following way:—In the "dry" experiments the coil of pipe was heated and a current of dry air drawn through it. The mixture of carbonic oxide and oxygen was passed into the dry coil through drying apparatus, consisting of two wash-bottles and three pumice towers containing boiled oil of vitriol, and then two long tubes packed with anhydrous phosphoric acid. The mixture under these conditions is called "well-dried"; it readily transmits the explosion. In the next experiments the phosphoric acid tubes were omitted; the mixture is called "dried." For saturating the gases with steam the mixture was led through water in three small wash-bottles before entering the coil. The first wash-bottle was kept about 5° C., the second about 3° C., and the third exactly 1° above the temperature at which the mixture was to be exploded. The last bottle and the connections between it and the coil were kept entirely surrounded by water. The gas, on its passage through the wash-bottles, carried over steam, which was partly condensed in the cooler wash-bottles and in the coil. While the gas was being admitted to the coil the water jacket was kept about ·5° above the final temperature required. When the tube was full the source of heat was removed, and the water in the jacket well stirred. When the exact temperature was reached the stop-cocks were closed and the gases fired. This arrangement ensured that the gases were *saturated* with moisture, and prevented any accumulation of liquid in the coil which might have retarded the explosion.

In the following table the *mean* results obtained with the dried and with the moistened gases are given:—

TABLE XI.—Rate of Explosion of Carbonic Oxide and Oxygen saturated with Steam at Different Temperatures.

Condition.	Per cent. of steam present.	Mean rate in metres per second.
Well dried	. .	1264
Dried	. .	1305
Saturated at 10° C.	1·2	1676
" 20° C.	2·3	1703
" 28° C.	3·7	1713
" 35° C.	5·6	1738
" 45° C.	9·5	1693
" 55° C.	15·6	1666
" 65° C.	24·9	1526
" 75° C.	38·4	1266

The maximum rate was obtained when the mixture was saturated at 35° C.—*i.e.*, contained 5·6 per cent. of steam. Excess of steam slowly retards the rate. A few experiments were made above and below the atmospheric pressure, which showed that the maximum rate always occurred when the same percentage of steam was present. Under 1100 mm. pressure the fastest rate was found when the mixture was saturated at 43°—*i.e.*, contained 5·6 per cent. of steam; and under 400 mm. pressure the fastest rate was found when the mixture was saturated at 25°—*i.e.*, contained 5·6 per cent. of steam. Increase of pressure increased, and diminution of pressure diminished, the rate of explosion of moist carbonic oxide and oxygen. Under 300 mm. pressure the explosion-wave was not propagated in the mixture of carbonic oxide and oxygen saturated at 10° C. In one experiment the flame reached the end of the tube, taking about 30 seconds to travel 100 metres. Under 400 mm. pressure the explosion-wave was propagated.

TABLE XII.—Rate of Explosion of Carbonic Oxide and Oxygen.

1. UNDER 1100 mm. pressure.

Condition.	Per cent. of steam present.	Mean rate in metres per second.
Saturated at 26° C.	2·3	1737
„ 43° C.	5·6	1782
„ 53° C.	9·5	1742

2. UNDER 400 mm. pressure.

Condition.	Per cent. of steam present.	Mean rate in metres per second.
Saturated at 10° C.	2·3	1576
„ 25° C.	5·6	1616
„ 33° C.	9·5	1570

These results are shown graphically by curves on fig. 2, Plate 1, in which the ordinates are the rates of explosion, and the abscissæ the percentages of steam present in the mixture.

These curves show the very marked increase of the rate of explosion on adding 1 or 2 per cent. of steam to the dried gases. Now the addition of steam to other dry gaseous mixtures, such as hydrogen and oxygen, ethylene and oxygen, cyanogen and oxygen, has been found to *lower* the rate of explosion. I think these results, therefore, are strong evidence that *at the extreme temperatures of the explosion-wave, as well as in ordinary combustion, carbonic oxide is oxidised by the steam, and not directly by the oxygen.*

Cap. V.—The Mode of Burning of Gaseous Carbon.

§ 1. The fact that carbonic oxide is not acted on directly by oxygen, even in the explosion-wave, points to the possibility that the carbon in gaseous hydrocarbons, and in other volatile carbon compounds, is not oxidised directly to carbonic acid when these gases are burnt with an excess of oxygen. When cyanogen, for instance, is exploded with oxygen, the carbon may burn in one or in two stages. Each atom may either combine directly with two atoms of oxygen to form carbonic acid, or each atom may combine first with a single atom of oxygen to form carbonic oxide, which afterwards combines with a second atom of oxygen to form carbonic acid. If the oxidation is effected in *two* stages, the chemical change first occurring, viz., the oxidation of the carbon to carbonic oxide, must take place in the presence of an excess of oxygen. Will this excess of oxygen influence the velocity with which the *initiation* of the chemical change is propagated, *i.e.*, the rate of explosion? The interest of tracing the *course* of any chemical change, and the scientific importance of this particular reaction, led me to study the influence of the presence of an excess of oxygen, and of an inert gas, on the rate of explosion of gaseous mixtures—with a view to obtain evidence on their mode of burning.

§ 2. *The Retardation of the Wave caused by Inert Gases.*

When electrolytic gas is mixed with oxygen, nitrogen, or steam, the explosion-wave is retarded, the oxygen having most effect, the steam least. For instance, the addition of one volume of steam to three volumes of electrolytic hydrogen and oxygen reduced the rate of explosion to 2494 metres per second in experiments made at 65° C. At the same temperature the addition of one volume of nitrogen to three of electrolytic gas reduced the rate to 2402 metres per second. According to Berthelot's formula a retardation of this order was to be expected—the greater specific heat of the steam being more than counterbalanced by the greater density of the nitrogen. Experiments made with hydrogen and oxygen at ordinary temperatures, with successive additions of oxygen and of nitrogen respectively, showed that, so long as the explosion-wave was propagated in both cases, the retarding influence of nitrogen was less than that of an equal volume of oxygen. Three volumes of electrolytic gas can be exploded with five volumes of nitrogen, but with seven volumes of nitrogen the explosion is not propagated through a long tube; whereas three volumes of electrolytic gas can still be exploded when mixed with eight of oxygen. But so long as the volume of nitrogen added is not sufficient to stop the explosion, the addition of this inert gas, incapable of taking part in the chemical change, produces less effect than the addition of oxygen, one of the reacting substances.

Some experiments published by BERTHELOT in 1885* seem to be at variance with this statement. When electrolytic gas, largely diluted with oxygen, was exploded in a bomb, the flame traversed the bomb more quickly than when an equal volume of nitrogen was used as a diluent. But the rate measured in this case was not that of the explosion-wave; it was the mean rate of the flame from the point of inflammation to a point about 250 mm. distant. On the other hand, with a smaller volume of diluent gas, BERTHELOT found the rate of explosion to be less retarded by nitrogen than by oxygen:—

	Time taken by flame to travel 250 mm.
$H_2 + O$	2·14 thousandths of a second.
$H_2 + O + 3O_2$. . . .	16·04 ,, ,,
$H_2 + O + 3N_2$ . . .	24·45 ,, ,,
$H_2 + O + O_2$	8·16 ,, ,,
$H_2 + O + N_2$	6·87 ,, ,,

I have found that the explosion-wave is sooner initiated when electrolytic gas is largely diluted with oxygen than when it is diluted with the same volume of nitrogen; a fact which explains the apparent anomaly in BERTHELOT'S experiments.

On repeating my measurements of the rate of explosion of electrolytic gas with large volumes of diluent gases, some discrepancies were found, which could not at first be explained. A tube of larger diameter (13 mm.) was then substituted for the smaller tube (9 mm.). The results were more regular. It was then discovered that the explosion frequently died out near the end of the smaller tube, whilst transmitting a sound-wave sufficiently powerful to break the silver foil composing the "second bridge." A record was thus obtained on the chronograph plate, indicating a smaller velocity than the true rate of explosion. By inserting a glass tube just before the second bridge, and watching for the flame, the observers made sure that the explosion-wave reached the end of the pipe. A second precaution, which must also be observed in experiments on largely diluted gases, is to make sure that the explosion-wave has been set up before the flame reaches the first bridge. M. BERTHELOT does not state the free run he allowed his flame before beginning the measurements; but, from an inspection of the figure of his apparatus ('Sur la Force des Matierès Explosives,' vol. 1, p. 138) it would appear that a few inches only were allowed between the firing spark and his first "interrupter." No doubt, the use of a grain of fulminate in the interrupter hastened the establishment of the explosion-wave in his experiments, but I think it possible that in some of his measurements he began to record the rate before the wave was started, and so obtained a rate below the true one. In all my later experiments on diluted gases, some pure electrolytic gas was passed into the long firing piece, just before the explosion, so as to fill it for a length of two or three feet.

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* 'Ann. Chim. Phys.' [VI.], vol. 4, p. 44.

The explosion-wave, rapidly started in this mixture, was communicated to the dilute mixture beyond, without break of continuity, before the first bridge was reached.

A comparison between the few experiments made by **Berthelot** with diluted mixtures and the similar ones made by me will, I think, show the need of these precautions. To three volumes of electrolytic gas **Berthelot** added two volumes of nitrogen : the rate of explosion was found to be 2121 metres per second. To three volumes of electrolytic gas he added four volumes of nitrogen : the rate of explosion was only 1439 metres per second. By interpolation* I find from my own experiments with one, three, and five volumes of nitrogen (added to three volumes of electrolytic gas) the rates 2200 and 1930, for the explosion of electrolytic gas with two volumes and four volumes of nitrogen respectivly. The first agrees fairly with **Berthelot**'s number, the second is far higher than his. I imagine in the second case the interrupter was broken before the explosion-wave was established in his experiment. Similarly, in two experiments with cyanogen, **Berthelot** found a rapid falling off of the rate which I did not observe. On firing a mixture of cyanogen with twice its volume of oxygen and one volume of nitrogen, he found the rate to be 2044 metres; on adding another volume of nitrogen the rate fell to 1203 metres. By interpolation I find from my experiments 2125 and 1960 as the rates of these two mixtures. The first agrees fairly well with, the second is far higher than, **Berthelot**'s number.

The *mean* results obtained in a number of experiments on the retarding effect of oxygen and nitrogen are given in the following table :—

Table XIII.—Electrolytic Gas with excess (1) of Oxygen and (2) of Nitrogen.

(1.)

Mixture.	2 vols. hydrogen 1 vol. oxygen. $H_2 + O$	With addition of 1 vol. oxygen. $H_2 + O + O$	With addition of 3 vols. oxygen. $H_2 + O + O_3$	With addition of 5 vols. oxygen. $H_2 + O + O_5$	With addition of 7 vols. oxygen. $H_2 + O + O_7$
Mean rate . .	2821	2328	1927	1707	1281

(2.)

Mixture.	2 vols. hydrogen 1 vol. oxygen. $H_2 + O$	With addition of 1 vol. nitrogen. $H_2' + O + N$	With addition of 3 vols. nitrogen. $H_2 + O + N_3$	With addition of 5 vols. nitrogen. $H_2 + O + N_5$	With addition of 7 vols. nitrogen. $H_2 + O + N_7$
Mean rate . .	2821	2426	2055	1822	—

These results are shown graphically in fig. 3, Plate 1, where the ordinates are volumes of nitrogen (or oxygen) added to three volumes of electrolytic gas, and the abscissæ are the rates of explosion of the mixtures in metres per second.

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* See fig. 3, Plate 1.

These curves plainly show that the retardation of the explosion-wave depends upon the density of the gas added as well as upon its volume—a result in accordance with BERTHELOT's theory.

I think it a fair inference from these facts to conclude, when the addition of a gas to an explosive mixture retards the rate of explosion by an amount which depends upon its volume and density, that such added gas is inert as far as the propagation of the wave is concerned, and that any change which it may undergo takes place after the wave-front has passed by—in other words, is a *secondary* change.

§ 3. *Oxygen Inert to Carbonic Oxide in Explosion of Carbon Compounds.*

This principle can be applied to determine whether, in the combustion of gaseous carbon, the oxidation to carbonic acid is effected in one or two stages. If, for instance, the carbon is first burnt to carbonic oxide, which subsequently is burnt to carbonic acid, the rate of the explosion-wave should correspond with the carbonic oxide reaction retarded by an excess of oxygen; whereas, if the carbon burns to carbonic acid directly, in one stage, then the rate of the explosion-wave should correspond with the complete and unretarded reaction.

Now, if we adopt BERTHELOT's theory as a working hypothesis, we can calculate the theoretical rates of explosion of marsh gas, ethylene, or cyanogen: (1) on the supposition that, with sufficient oxygen, the carbon burns directly to carbonic acid, and (2) on the supposition that the carbon burns first to carbonic oxide, and the further oxidation is a subsequent or secondary reaction. On the first supposition, if 100 represents the rate of explosion of these three gases with oxygen only sufficient to burn them to carbonic oxide, the addition of the oxygen required to burn the gases to carbonic acid should *increase* the rate of explosion :—

	Marsh gas.	Ethylene.	Cyanogen.
Calculated rate of explosion when burnt to CO_2 .	104	103	107

Whereas, if these gases really burn first to carbonic oxide, and the extra oxygen is inert in propagating the explosion-wave, then the addition of this inert oxygen would *diminish* the rate of explosion :—

	Marsh gas.	Ethylene.	Cyanogen.
Calculated rate of explosion when burnt to CO with inert oxygen present	92	88	87

The experiments show that if 100 be taken as the rate of explosion when the oxygen is only sufficient to burn the carbon to carbonic oxide, the following are the rates found when oxygen is added sufficient to burn the carbon to carbonic acid :—

	Marsh gas.	Ethylene.	Cyanogen.
Rates found	94	92	84

The results are, therefore, in favour of the view that, in the explosion of these gases, the carbon is first burnt to carbonic oxide.*

But stronger evidence on this point is obtained by comparing the explosion rates of these gases (1) when fired with oxygen sufficient to burn the carbon in them to carbonic acid, and (2) when nitrogen is substituted for the oxygen in excess of that required to burn the carbon to carbonic oxide. We have seen that oxygen, added to electrolytic gas, hinders the explosion more than nitrogen. In precisely the same way oxygen, added to a mixture of equal volumes of cyanogen and oxygen, hinders the explosion more than the same volume of nitrogen. The conclusion we must come to is, that the oxygen, added to the mixture expressed by the formula $C_2N_2 + O_2$, is as inert (so far as the propagation of the explosion-wave is concerned) as oxygen added to the mixture expressed by the formula $H_2 + O$. The same phenomena occur in the explosion of marsh gas, ethylene, and acetylene.

The following tables contain the results of my last experiments on this subject. The rates come out on the whole somewhat higher than in my earlier experiments. This is mainly due to the later explosions having been made in a tube of larger diameter—9 mm. instead of 5 mm.

TABLE XIV.—The Rate of Explosion of Cyanogen (1) with Oxygen, and (2) with Oxygen and Nitrogen.

(1.)

Mixture.	1 vol. cyanogen 1 vol. oxygen $C_2N_2+O_2$	With addition of 1 vol. oxygen. $C_2N_2+2O_2$	With addition of 2 vols. oxygen. $C_2N_2+3O_2$
Mean rate . .	2728	2321	2110

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* [It might be urged that, with defect of oxygen, the reactions really are—

1. $CH_4 + O_3 = CO_2 + H_2O + H_2$
2. $C_2H_4 + O_4 = 2CO_2 + 2H_2$
3. $C_2N_2 + O_2 = CO_2 + C + N_2$

If this were true, and 100 represented the observed rates of explosion in each case, then on adding oxygen sufficient for complete combustion, the rates should be 101, 99, and 107 respectively.—Jan., 1893.]

(2.)

Mixture.	1 vol. cyanogen 1 vol. oxygen. $C_2N_2+O_2$	With addition of 1 vol. nitrogen. $C_2N_2+O_2+N_2$	With addition of 2 vols. nitrogen. $C_2N_2+O_2+2N_2$
Mean rate . .	2728	2398	2165

Cyanogen will explode when mixed with four times its volume of oxygen. When three volumes of nitrogen are added to one volume of cyanogen and one of oxygen, the mixture will not explode, but burns at the mouth of the tube. So long, however, as the nitrogen does not prevent the explosion, its addition to the mixture has less retarding effect than the addition of an equal volume of oxygen.

TABLE XV.—The Rate of Explosion of Marsh Gas (1) with Oxygen and (2) with Oxygen and Nitrogen.

(1.)

Mixture.	2 vols. methane 3 vols. oxygen CH_4+O_3	With addition of 1 vol. oxygen. CH_4+O_4	With addition of 3 vols. oxygen. CH_4+O_6	With addition of 5 vols. oxygen. CH_4+O_8
Mean rate . .	2470	2322	2146	1963

(2.)

Mixture.	2 vols. methane 3 vols. oxygen CH_4+O_3	With addition of 1 vol. nitrogen. CH_4+O_3+N	With addition of 3 vols. nitrogen. $CH_4+O_3+N_3$	With addition of 5 vols. nitrogen. $CH_4+O_3+N_5$
Mean rate . .	2470	2349	2154	1880 (goes out sometimes)

The addition of one and of three volumes of nitrogen to the mixture of marsh gas and oxygen ($CH_4 + O_3$) is found to retard the rate less than the addition of the same volumes of oxygen. When, however, five volumes of nitrogen are added, the retardation is greater than with oxygen: this mixture is near the limit of explosion, and once or twice the flame died out before reaching the end of the tube.*

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* Many attempts were made to explode mixtures of marsh gas and air in a tube. A vigorous explosion, initiated by firing electrolytic gas down the first few feet of the tube, soon dies out in the mixtures of marsh gas and air. In a metal tube 1 inch in diameter the flame travelled a considerable distance, but in a slow and irregular way. Similar results were obtained with mixtures of coal-gas and air. On the other hand, mixtures of ethylene and air explode readily down a narrow tube.

TABLE XVI.—The Rate of Explosion of Ethylene (1) with Oxygen, and (2) with Oxygen and Nitrogen.

(1.)

Mixture.	1 vol. ethylene 2 vols. oxygen $C_2H_4+2O_2$	With addition of 1 vol. oxygen. $C_2H_4+3O_2$	With addition of 2 vols. oxygen. $C_2H_4+4O_2$	With addition of 4 vols. oxygen. $C_2H_4+6O_2$	With addition of 6 vols. oxygen. $C_2H_4+8O_2$	With addition of 8 vols. oxygen. $C_2H_4+10O_2$
Mean rate.	2581	2368	2247	2118	1980	1856

(2.)

Mixture.	1 vol. ethylene 2 vols. oxygen $C_2H_4+2O_2$	With addition of 1 vol. nitrogen. $C_2H_4+2O_2+N_2$	With addition of 2 vols. nitrogen. $C_2H_4+2O_2+2N_2$	With addition of 4 vols. nitrogen. $C_2H_4+2O_2+4N_2$	With addition of 6 vols. nitrogen. $C_2H_4+2O_2+6N_2$	With addition of 8 vols. nitrogen. $C_2H_4+2O_2+8N_2$
Mean rate.	2581	2413	2211	2024	1878	1734

In the case of ethylene, the retarding action of oxygen, which is greater than that of nitrogen for the first volume added to the mixture ($C_2H_4+2O_2$), becomes less and less as a large excess of oxygen is added; so that the rate of explosion of ethylene and air would be considerably increased by the substitution of oxygen for the nitrogen of the air. With a large excess of oxygen the *secondary* reaction, whereby carbonic acid is formed, influences the result, for the conditions are eminently favourable for the interaction of the carbonic oxide, steam, and oxygen.

TABLE XVII.—The Rate of Explosion of Acetylene (1) with Oxygen, and (2) with Oxygen and Nitrogen.*

(1.)

Mixture.	2 vols. acetylene 3 vols. oxygen $C_2H_2+O_3$	With addition of 2 vols. oxygen. $C_2H_2+O_5$
Mean rate . .	2716	2391

(2.)

Mixture.	2 vols. acetylene 3 vols. oxygen $C_2H_2+O_3$	With addition of 2 vols. nitrogen. $C_2H_2+O_3+N_2$
Mean rate . .	2716	2414

These experiments show that in all the cases examined, viz., the combustion of cyanogen, marsh gas, ethylene, and acetylene, the substitution of nitrogen for the oxygen required to burn the carbon from carbonic oxide to carbonic acid *increases* the velocity of the explosion. *These facts seem only consistent with the view that the carbon burns directly to carbonic oxide, and the formation of carbonic acid is an after-occurrence.*

§ 4. BERTHELOT'S *Experiments bearing on the mode of Combustion of Carbon.*

M. BERTHELOT found that the pressures produced in the explosion of cyanogen were greater for the *incomplete* than for the *complete* combustions.†

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* No doubt with a large excess of oxygen the explosion of acetylene would be less retarded than with a large excess of nitrogen, as in the case of ethylene. Owing to the expense of preparing the acetylene, experiments with a large excess of oxygen were not made.

† 'Ann. Chim. et Phys.' [VI.], vol. 4, p. 38.

Mixtures.	Pressures.
$C_2N_2 + O_2$	25·11 atmospheres.
$C_2N_2 + 2O_2$	20·96 ,,
$C_2N_2 + 2N_2O$	26·02 ,,
$C_2N_2 + 4N_2O$	22·66 ,,
$C_2N_2 + 2NO$	23·34 ,,
$C_2N_2 + 4NO$	16·92 ,,

It is difficult to reconcile these numbers with the received view as to the burning of carbon; they are readily explained on the hypothesis that carbon burns in two stages.

BERTHELOT has also made some observations of the initial rate of explosion of cyanogen when burnt to carbonic oxide and when burnt to carbonic acid. The time taken by the flame in travelling a distance of about 130 mm. from the point of inflammation was *less* for the incomplete than for the complete combustion; a result which agrees with my measurements of the rate of the explosion-wave in the two cases. When a mixture of carbonic oxide, oxygen and nitrogen, having a composition corresponding to the first stage of the combustion of cyanogen, was fired in the same apparatus, the rate of the flame was found by BERTHELOT to be far slower than in the complete combustion of cyanogen:—

	Time taken by flame to travel 130 mm.
$C_2N_2 + O_2$	1·05 thousandths of a sec.
$C_2N_2 + 2O_2$	1·55 ,, ,,
$2CO + O_2 + N_2$	17·78 ,, ,,

M. BERTHELOT adds:—"D'après ces nombres, il ne paraît pas que la combustion totale du cyanogène s'effectue en deux temps, en formant d'abord en totalité de l'oxyde de carbone qui brûlerait ensuite; car la combustion totale est beaucoup plus rapide que la somme de ces deux effets séparés."*

On first reading this passage I understood M. BERTHELOT to mean that in the complete combustion of cyanogen there was no intermediate formation of carbonic oxide. But I am led to think that this interpretation cannot be correct for two reasons. First, because the conditions of temperature and pressure are so entirely different in the two cases compared, that no argument drawn from one can be applied to the other. In one case, a reaction is initiated in a mixture of carbonic oxide, oxygen, and nitrogen at the atmospheric temperature and pressure; in the other, the same gases are at an enormously high temperature and pressure when the reaction begins. Secondly, because M. BERTHELOT, in the same paper, has drawn from analogous facts an argument in an exactly opposite sense. Having determined the initial rate of explosion of hydrogen and oxygen, and of carbonic oxide and oxygen in a bomb, he found the rate of explosion of the mixture of the two to be less than the mean of the two separately :—

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* 'Ann. Chim. et Phys.,' [VI.] vol. 4, p. 43.

	Time.
$H_2 + O$	1·04
$CO + O$	12·86
$H_2 + CO + O_2$	3·88

"La vitesse de combustion n'est, dans aucun cas, la moyenne de celle des composants mélangés. Mais les deux gaz paraissent tendre à brûler séparément, chacun avec sa vitesse propre."

The meaning of the passage quoted, concerning the combustion of cyanogen, would therefore appear to be: "In the complete combustion of cyanogen there are not two flames propagated from the point of inflammation with different velocities—the quicker due to the burning of cyanogen to carbonic oxide, the slower to the burning of carbonic oxide to carbonic acid." With this literal interpretation of the passage I entirely agree: there are not two flames or two explosions, but the burning of each molecule takes places in two stages, the second stage prolonging the duration of the flame due to the first.

§ 5. *Other Experiments on the burning of Carbon.*

It is usually assumed that carbon, in all its states, burns to carbonic acid, and then the excess of carbon, if it is present, reduces the carbonic acid, first formed, to carbonic oxide. This view, no doubt, originated from the observation that solid carbon burns in oxygen, without apparent flame, to form carbonic acid; and that carbonic acid is readily reduced to carbonic oxide when passed over heated carbon. The experiments, however, of Mr. BRERETON BAKER ('Phil. Trans.,' 1888) show that purified charcoal may be heated to redness in a current of dry oxygen without visible combustion taking place, but with the formation of carbonic oxide—although the oxygen is in large excess. It would appear, from Mr. BAKER's observations, that carbonic oxide is the first product of the action of oxygen on solid carbon at a high temperature; the formation of carbonic acid being a secondary change due to the interaction of carbonic oxide, steam and oxygen. Both solid and gaseous carbon, therefore, appear to undergo oxidation in two stages; the second stage being facilitated by the presence of steam.

The evidence, above described, led me, two years ago, to hazard an explanation of the appearance of a cyanogen flame burning in air. The crimson inner zone might be due to the primary reaction—the burning of the cyanogen to carbonic oxide and nitrogen, with the limited air supply that reaches it; the blue outer zone might be due to the burning of the carbonic oxide (formed in the crimson zone), as it meets with a plentiful supply of air outside. A few experiments I made showed the presence of carbonic oxide in abundance on the edge of the crimson zone; but I have left the further investigation of the structure of the cyanogen flame to Professor A. SMITHELLS, who has devised a most ingenious method of separating the outer and inner cones of flame, formed when a mixture of a combustible gas and air is burnt in a Bunsen burner. His experiments show that the cyanogen is completely destroyed in the inner red flame, and carbonic oxide is the main product of its oxidation there. This carbonic oxide burns above, on reaching the air, with its well-known blue flame* ('Chem. Soc. Journ.,' 1892, vol. 1, p. 215).

CAP. VI.—THE REACTIONS OF CARBONIC OXIDE IN THE FLAME.

If it be true that in a flame gaseous carbon burns to carbonic oxide, and that carbonic oxide reacts with steam (if present) to a limited extent, and is completely oxidised in presence of steam and oxygen, two questions arise: (1) What happens to the carbonic oxide first formed in the combustion of hydrocarbons or cyanogen with excess of oxygen? and (2) How do these secondary reactions affect the rate of explosion? In the case of cyanogen we should expect, if the gases are fairly dry, that the formation of carbonic acid would be comparatively slow, since no steam is formed in the reaction. We find, in accordance with this expectation, that so long as the mixtures will explode, successive additions of oxygen retard the rate more than the additions of nitrogen—just as in the case of electrolytic gas. This means that in the cyanogen explosion the excess of oxygen and carbonic oxide are as inert towards each other (as far as the propagation of the wave is concerned) as the excess of oxygen and steam are in the explosion of electrolytic gas. On the other hand, the carbonic oxide formed in the explosion of hydrocarbons, with excess of oxygen, meets with the steam also formed in the flame. This is the condition necessary for its most rapid oxidation. We should expect, therefore, to find in the explosions of hydrocarbons that the retarding action of large quantities of oxygen is partly counteracted by this secondary reaction, for the oxygen in presence of the steam is not inert towards the carbonic oxide. As a matter of fact, we find that the first additions of oxygen to the hydrocarbon mixtures retard more than nitrogen, but as the additions are continued the oxygen has less and less retarding influence compared with the nitrogen.

When carbonic oxide, which has the same density as nitrogen, is added to electrolytic gas, the retarding effect on the explosion is very nearly the same as when an equal volume of nitrogen is added:—

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* The greenish-grey tint, observable in the outer flame, may be due to some oxide of nitrogen formed in the inner flame. A little peroxide of nitrogen gives a similar tint to an ordinary Bunsen gas flame; and peroxide of nitrogen is formed when cyanogen is burnt under certain conditions (*vide* 'Chem. Soc. Journ.,' 1886, vol. 1, p. 390).

TABLE XVIII.—Rate of Explosion of Electrolytic Gas (1) with Nitrogen, and (2) with Carbonic Oxide. (K.)

(1.)

Mixture.	2 vols. hydrogen and 1 vol. oxygen. H_2+O	With addition of 1 vol. nitrogen. H_2+O+N	With addition of 3 vols. nitrogen. H_2+O+N_3
Mean rate .	2821	2426	2055

(2.)

Mixture.	2 vols. hydrogen and 1 vol. oxygen. H_2+O	With addition of 1 vol. carbonic oxide. H_2+O+K	With addition of 3 vols. carbonic oxide. H_2+O+K_3
Mean rate .	2821	2455	2080

It is true that carbonic acid and free hydrogen are found in the tube after the explosion; but these are due to a *secondary* reaction, which need not necessarily affect the rate; in other words, the carbonic acid may be formed in the heated gases after the wave-front has passed by.

This similarity in the retarding action of carbonic oxide and nitrogen can be explained in two ways. First, the carbonic oxide may be for the most part unaffected in the initial change occurring in the wave-front, and be, consequently, as inert as an equal volume of nitrogen. Secondly, the carbonic oxide—inert towards the hydrogen and oxygen—may react with the steam produced both in the wave-front and behind it. The reaction between the steam and carbonic oxide *in the wave-front* would slightly quicken the explosion by developing heat. The experiments show that nitrogen retards the wave to an extent just appreciably greater than carbonic oxide.

To test the correctness of these views concerning the *rôle* played by carbonic oxide in the flame, an experiment was tried with electrolytic gas diluted (1) with oxygen and nitrogen, and (2) with oxygen and carbonic oxide. We have seen that electrolytic gas, when diluted with carbonic oxide, explodes at nearly the same rate as when diluted with nitrogen. Now, if the views expressed above are correct, we may replace some of the diluent nitrogen by oxygen, and thereby make the explosion slower; but if the diluent carbonic oxide is partly replaced by oxygen, the explosion should be considerably faster. The rates found in the two cases agree with this prediction:—

Table XIX.

(1.)

Mixture.	2 vols. hydrogen 1 vol. oxygen H_2+O	With addition of 3 vols. nitrogen. H_2+O+N_3	With addition of 2 vols. nitrogen and 1 vol. oxygen. $H_2+O+O+N_2$
Mean rate.	2821	2055	2003

(2.)

Mixture.	2 vols. hydrogen 1 vol. oxygen H_2+O	With addition of 3 vols. carbonic oxide. H_2+O+K_3	With addition of 2 vols. carb. ox. and 1 vol. oxygen. $H_2+O+O+K_2$
Mean rate .	2821	2080	2143

A study of these several tables brings us to the conclusion that secondary reactions, whereby carbonic oxide is oxidised in the flame, may affect the velocity of the wave to a smaller extent when the explosion is rapid, to a greater extent when the explosion is slow.

Cap. VII.—Examination of Berthelot's Theory.

The measurements of the rate of explosion of the various gaseous mixtures described in Cap. V. permitted an extended comparison between the theoretical velocities calculated by Berthelot's formula (θ) and the actual rates (ν). This comparison is made in the following tables for electrolytic gas, diluted with oxygen and with nitrogen; for cyanogen burning to carbonic oxide, and when this mixture is diluted with nitrogen; for marsh gas, ethylene, and acetylene, all burning to carbonic oxide and steam, and for the same gases diluted with nitrogen. In calculating θ, I made a correction for the gases being at the ordinary temperature (13° C.) before explosion, and not at absolute zero; the theoretical velocities are, therefore, rather higher than those given by Berthelot. This correction raises the theoretical velocity of explosion of electrolytic gas from 2831 metres per second (Berthelot) to 2900.

TABLE XX.—Electrolytic Gas with excess of Oxygen added.

Mixture.	2 vols. hydrogen } 1 vol. oxygen. } H_2+O	With addition of 1 vol. oxygen. H_2+O+O	With addition of 3 vols. oxygen. H_2+O+O_3	With addition of 5 vols. oxygen. H_2+O+O_5
Theory, θ . .	2900	2252	1730	1476
Found, ν . .	2821	2328	1927	1707

TABLE XXI.—Electrolytic Gas with excess of Nitrogen added.

Mixture.	H_2+O	H_2+O+N	H_2+O+N_3	H_2+O+N_5
Theory, θ . .	2900	2321	1814	1558
Found, ν . .	2821	2426	2055	1822

These measurements show that BERTHELOT'S formula, which gives a calculated rate 3 per cent. too high for pure electrolytic gas, also gives a calculated rate which is found to be 16 and 17 per cent. *too low* for the same gas when largely diluted with oxygen and with nitrogen respectively.

Again, in the explosion of cyanogen to carbonic oxide with its own volume of oxygen, and also when the same mixture is diluted with nitrogen, the theoretical rates calculated by BERTHELOT'S formula are far too low :—

TABLE XXII.—Cyanogen and Oxygen with excess of Nitrogen added.

Mixture.	1 vol. cyanogen } 1 vol. oxygen. } $C_2N_2+O_2$	With addition of 1 vol. nitrogen. $C_2N_2+O_2+N_2$	With addition of 2 vols. nitrogen. $C_2N_2+O_2+2N_2$
Theory, θ . .	2361	2083	1877
Found, ν . .	2728	2397	2166

The observed rates of explosion are 15 per cent. higher than the calculated rates. Similar differences are found between the observed and the calculated rates when the cyanogen mixture is diluted with oxygen.

When marsh gas and ethylene are exploded with oxygen sufficient to burn them to carbonic oxide and steam, the observed rates are higher than those given by the formula. Again, when these mixtures are diluted with nitrogen the rates come out higher than the formula predicts.

TABLE XXIII.—Marsh Gas and Oxygen with excess of Nitrogen added.

Mixture.	2 vols. methane } 3 vols. oxygen. CH_4+O_3	With addition of 1 vol. nitrogen. CH_4+O_3+N	With addition of 3 vols. nitrogen. $CH_4+O_3+N_3$
Theory, θ . .	2387	2211	1958
Found, ν . .	2470	2349	2154

For pure marsh gas and oxygen, the calculated velocity is 3½ per cent. too low; when 3 volumes of nitrogen are added, the calculated velocity is 10 per cent. too low. The mixture cannot be detonated with a large excess of nitrogen.

TABLE XXIV.—Ethylene and Oxygen with excess of Nitrogen added.

Mixture	$C_2H_4+2O_2$	$C_2H_4+2O_2+N_2$	$C_2H_4+2O_2+2N_2$	$C_2H_4+2O_2+4N_2$	$C_2H_4+2O_2+6N_2$	$C_2H_4+2O_2+8N_2$
Theory, θ .	2487	2224	2050	1796	1624	1498
Found, ν	2581	2418	2211	2024	1878	1734

The divergence between the calculated and observed rates varies with the dilution from 4 per cent. to 16 per cent. The calculated velocities are all too low. In the following table are given the rates of explosion of acetylene burning to carbonic oxide and steam, alone, and with successive additions of nitrogen.

TABLE XXV.—Acetylene and Oxygen with excess of Nitrogen added.

Mixture.	2 vols. acetylene 2 vols. oxygen. $C_2H_2+O_2$	$C_2H_2+O_2+N_2$	$C_2H_2+O_2+3N_2$	$C_2H_2+O_2+4N_2$	$C_2H_2+O_2+5N_2$	$C_2H_2+O_2+6N_2$
Theory, θ .	2749	2397	1983	1848	1739	1650
Found, ν .	2716	2414	2209	2116	2019	1908

Here, as in the case of electrolytic gas, the rate for the pure mixture is in close agreement with the calculated velocity. Taken by itself, this experiment (not tried by BERTHELOT) would have offered strong confirmation of the correctness of the theory. But as successive additions of nitrogen are made to the mixture, the calculated velocities fall *below* the observed rates, until, with a large excess of nitrogen, the calculated rate is 16 per cent. too low.

The gradual divergence between the observed and the calculated rates of explosion for different gaseous mixtures, as successive quantities of nitrogen were added to the pure explosive gas, led me to try the effect of diluting electrolytic gas with hydrogen. According to BERTHELOT'S formula a slight increase in the rate should occur; actually a very marked increase in the rate was observed.

TABLE XXVI.—Rate of Explosion of Electrolytic Gas with excess of Hydrogen added.

Mixture.	2 vols. hydrogen 1 vol. oxygen. H_2+O	With addition of 2 vols. hydrogen. H_2+O+H_2	With addition of 4 vols. hydrogen. H_2+O+H_4	With addition of 6 vols. hydrogen. H_2+O+H_6
Theory, θ . .	2900	3055	3061	3028
Found, ν . .	2821	3268	3527	3532

To compare with this series of experiments, and with those given in Table XXI., a mixture of hydrogen and nitrous oxide was exploded (1) by itself, (2) with excess of hydrogen, and (3) with excess of nitrogen. A similar divergence between the calculated and observed rates was found both when the mixture was diluted with hydrogen, and when it was diluted with nitrogen.

TABLE XXVII.—Rate of Explosion of Hydrogen and Nitrous Oxide, with excess of (1) Hydrogen, (2) Nitrogen.

(1.)

Mixture.	2 vols. hydrogen 2 vols. nitrous oxide. H_2+N_2O	With addition of 2 vols. hydrogen. $H_2+N_2O+H_2$	With addition of 4 vols. hydrogen. $H_2+N_2O+H_4$	With addition of 5 vols. hydrogen. $H_2+N_2O+H_5$
Theory, θ . .	2319	2374	2396	2400
Found, ν . .	2305	2545	2705	2732

(2.)

Mixture.	$H_2 + N_2O$	With addition of 2 vols. nitrogen. $H_2 + N_2O + N_2$	With addition of 3 vols. nitrogen. $H_2 + N_2O + N_3$
Theory, θ . .	2319	1912	1782
Found, ν . .	2305	1991	1880

Now M. BERTHELOT especially insists that the formula gives a *maximum* velocity which may be reached, but not surpassed, by the actual explosion-wave. If, then, the rates of explosion of the diluted mixtures had fallen below the calculated rates, such divergence might be explained on the supposition that the diluent gases interfere with the normal propagation of the wave, an explanation which is indeed advanced by BERTHELOT to account for the falling off in the rate of certain diluted mixtures observed by him. But since the theory assumes that the explosion-wave travels at the rate of the gaseous molecules themselves while they still retain all the heat developed by the reaction, the presence of inert molecules could not possibly *increase* this rate of motion.*

Since the amount and regularity of the divergence between the found and calculated rates precluded the idea of experimental error being its sole cause, I was driven to conclude either that the hypothesis was incorrect, or that the formulas used failed to express the hypothesis with exactness.

M. BERTHELOT's theory is that the explosion does not travel with the velocity of *sound* in the heated gases, but with a velocity equal to the mean rate of translation of the molecules produced in the explosion.

Let us examine the mode in which BERTHELOT calculates the theoretical velocity, *i.e.*, the mean rate of translation of the products of combustion at the temperature of the explosion. In CLAUSIUS' formula

$$\nu = 29{\cdot}354 \sqrt{\frac{T}{d}}$$

M. BERTHELOT calculates the absolute temperature of the explosion by dividing the quantity of heat developed in the complete reaction by the specific heat of the products of combustion *taken at constant pressure.* He argues that each layer of gas, in transmitting the explosion, is heated under constant pressure. I cannot follow his reasoning.

"The combustion," he says, "in propagating itself from layer to layer, is preceded by the compression of the gaseous layer which it is about to transform.

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* It will be shown in the sequel that BERTHELOT's formula fails to express the rates of explosion of some undiluted explosive mixtures, viz., hydrogen and chlorine, ammonia and nitrous oxide.

. . . The combustion of each layer produces both heat and at the same time the work necessary to compress the following layer—that is to say, the layer loses on this score just as much heat as it gained by its own compression. The whole proceeds, as far as the elevation of temperature is concerned, precisely as if we had operated under constant pressure." On the facts of the case there is no dispute. The gas is exploded in a closed vessel. Each layer is compressed before being fired; after firing it compresses the layer beyond it. Now, as regards this preliminary compression, each layer, in turn, expends the same energy as was previously communicated to it, and, therefore, it does no work of its own. But a gas heated under conditions where it does no work is raised to the same temperature as it would be had its volume remained constant. M. BERTHELOT admits that it would appear at first sight as if the gases were heated at constant volume; he adds that the concordance of the calculated with the observed numbers supports his explanation of the phenomena.

Again, the fact that each layer is fired by compression involves the preliminary heating of that layer, and this heating must be added to the heat developed by its burning. The temperature of the burning layer must, therefore, be greater than that obtained by dividing the heat developed in the chemical change by the specific heat of the products of combustion.

When a sound-wave alone is transmitted through a gas its velocity shows that each layer of gas forming the wave-front is heated by compression, and there is no reason why this should not happen when the compression is accompanied by a chemical change.

To what extent is each layer heated before combustion? MM. MALLARD and LE CHATELIER[*] state that the explosion-wave will be propagated when each layer is brought by compression to its own temperature of inflammation. For hydrogen and oxygen, they find this temperature to be about 550° C., and calculate that a pressure of 30 atmospheres must be exerted upon the gas to raise it to this point. This temperature may be regarded as the *lowest limit* of the preliminary heating of each layer before combustion. But having regard to the fact that the individual molecules of hydrogen and oxygen in a yet unburnt layer may, in the interval of time between their meetings, be heated above the point at which they combine with each other, we cannot affirm that this limit of temperature may not be greatly surpassed in the explosion.

If we regard the transmission of the explosion to be due to the collisions of the molecules, and assume that molecules which are chemically inactive towards each other act as elastic bodies when they come into collision, and that molecules which combine chemically lose energy of translation and gain energy of vibration, then it must happen that unburnt molecules come into collision with burnt molecules and take up their energy in the form of motion. For instance, in the explosion of hydrogen and chlorine, the energy of the hydrogen chloride formed will be communicated by collision to molecules of hydrogen and chlorine;* these *heated* molecules, moving forwards, will meet *unheated* molecules moving backwards, when combination will occur between those of opposite kind. Heated hydrogen will thus combine with cool chlorine, and heated chlorine with cool hydrogen. The heat of combustion is thus communicated to a molecule of hydrogen or chlorine which *shares it* with a molecule of the opposite kind; each hydrogen chloride molecule formed in turn will therefore have, on the average, a temperature corresponding to the heat of chemical combination *plus* half the heat of a molecule previously formed. According to this view the temperature reached by each successive layer would increase until it was *double* that due to the chemical change alone. The temperature of the explosion would then remain constant, and the wave would advance at a uniform rate as long as it met the same mixture of gases.

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[*] 'Recherches sur la combustion des Mélanges Gazeux Explosifs,' p. 88-91.

My conception may perhaps be illustrated by an analogy, although an imperfect one. Imagine a kilogram of water at 0° falling *in vacuo* through 425 metres into a vessel filled with an equal mass of water at 0°: its motion is stopped, but an equal mass of water is displaced. Imagine that the heat developed (which would raise the fallen water to 1°) be divided equally between the water which fell and that which was displaced. The displaced kilogram of water, now at ·5°, falls into a similar vessel 425 metres below, where it divides its heat with, and displaces, a third kilogram of water, and so on. By successive divisions of heat, the temperature of the *falling* water will approach 1°, beyond which it cannot rise. But if the heat developed were not divided, and remained in the fallen water, the *falling* water would be always at 0°, and the *fallen* water at 1°. The latter case is the analogue of the condition imagined by M. BERTHELOT, where the heat of the burnt layer is not imparted to the unburnt layer next it; the former case is the analogue of the condition which I imagine holds in the propagation of the explosion-wave.

Again, M. BERTHELOT considers the velocity of the wave to be solely governed by the mean rate of translation of the *products* of combustion. If, however, we accept the hypothesis that the explosion is propagated by molecular collisions, we have just seen that the movement of the products of combustion is communicated to the unburnt molecules in front. It will, therefore, follow that the rate of the advancing explosion will depend not only on the rate of translation of the products of combustion, but also on the rate of translation of the heated *but yet uncombined molecules.* If the burnt molecules communicate their rise of temperature without loss to the molecules in front, no difference in the mean rate of motion will be caused by this transference so long as the burnt and the unburnt gases are of the same average density. But if a change in density is produced by the combustion, the average velocity of the molecules in the burnt layer will differ from the average velocity of the molecules in the heated but yet unburnt layer, and this difference must be taken into account. Lastly, there is a correction to be applied for changes of density in the explosion which in part counteracts the foregoing correction. In the oxidation of hydrogen and of carbonic oxide there is a contraction from three volumes to two; in the burning of cyanogen to carbonic oxide there is an expansion of two volumes to three.* If each layer of the reacting gases is heated at constant volume, a change in the number of molecules must affect the temperature reached.

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* [I make the assumption that a very thin layer of gas (comparable in thickness with the mean free path of the molecules) may be raised by collisions to *nearly* the same temperature as the highly heated layer next it.—Jan., 1893.]

If the number of molecules is increased by the chemical change the temperature from which the velocity of the wave is calculated will be higher than that directly deduced from calorimetric observations; and *vice versâ*. In allowing for this change we may suppose that the new formed molecules have been suddenly compressed or expanded, without loss of heat, into the volume previously occupied by the initial molecules, in which case the temperature would be altered according to RANKINE'S formula

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma - 1}$$

where T_1 and T_2 are the absolute temperatures, V_1 and V_2 the volumes before and after the reaction, and γ the ratio of the specific heats at constant pressure and at constant volume.

CAP. VIII.—COMPARISON OF THE RATE OF EXPLOSION TO THAT OF A SOUND-WAVE.

The criticisms which I have ventured to make on BERTHELOT'S method of calculating the mean rate of translation of the gaseous molecules concerned in the propagation of the explosion-wave, tend to show that the rate so calculated must be too small. But in attempting to apply corrections suggested by these criticisms, one is beset with difficulties. As an approximation I assume (1) that the explosion-wave is carried forward by the movements of molecules of density intermediate between that of the products of combustion and that of the unburnt gas; (2) that the temperature of the gas propagating the wave is double that due to the chemical reaction alone; (3) that the temperature is increased when the chemical volume of the products is larger, and is diminished when the chemical volume of the products is smaller than that of the initial gases; (4) that the gases are heated at constant volume, and their specific heats remain constant at high temperatures. On calculating the mean rate of translation of the molecules on these assumptions one arrives at numbers greatly in excess of any of the observed rates of explosion; but some of the observed rates agree with *the velocity of sound* in a gas of the temperature and density so calculated. For instance, when one volume of cyanogen is exploded with an equal volume of oxygen, two volumes of carbonic oxide are formed, and one volume of nitrogen :—

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* M. BERTHELOT rightly takes this change into account in interpreting BUNSEN'S experiments on the temperature produced in explosions. 'Ann. Chim. et Phys.,' [V.], vol. 12, p. 302.

$$C_2N_2 + O_2 = 2CO + N_2.$$

Taking the quantity of heat evolved as 126,100 calories, and the specific heat at constant volume of the products of combustion as $4{\cdot}8 \times 3 = 14{\cdot}4$, the temperature produced by the chemical change is 8694° C. If the gases were initially at 13°, or 286° degrees above absolute zero, the chemical reaction will raise the temperature to 8980°. But, since three molecules are formed where two previously existed, the temperature is further raised by the heat developed in compressing three volumes to two. This will raise the temperature to 10,595°. At *double* this temperature the mean rate of translation of a molecule of the mean density of the burnt and unburnt gases would be 3892 metres per second. If the formula for the velocity of sound under ordinary conditions held good in the explosion, the velocity of the sound wave would be 2670 metres per second—a rate which is about 2 per cent. less than the observed velocity of the explosion-wave. Now, the theoretical velocity of sound is calculated on the assumption that the disturbance is very small; if the displacements are large the velocity of sound should be higher. Direct measurements of the velocity of sound-waves of great intensity have confirmed this anticipation. Under ordinary conditions the rate of the sound-wave is to the mean rate of the molecules as ·688 : 1. If we take the ratio in an explosion as ·7 : 1 the velocity of the sound-wave agrees with the observed rate of explosion in this particular case.

We may now compare the rate of the sound-wave so calculated with the velocity of explosion of cyanogen with oxygen (1) in presence of excess of oxygen, (2) in presence of excess of nitrogen; (3) of cyanogen with nitrous oxide alone, and (4) in presence of excess of nitrogen; and lastly (5) of cyanogen with nitric oxide.

Taking the data furnished by BERTHELOT's experiments, we have, for the quantities of heat evolved, the expansion on explosion, the specific heat of the products, and for the mean density of the burnt and unburnt gases :—

TABLE XXVIII.—Explosion of Cyanogen with Oxygen.

Explosive gases.	Inert gases.	Heat developed.	Expansion.	Specific heat of products.	Mean density.
$C_2N_2 + O_2$ . .	..	126,100 Cals.	2 : 3	4·8 × 3	1·213
,, . .	O_2	,,	3 : 4	4·8 × 4	1·172
,, . .	$2O_2$	,,	4 : 5	4·8 × 5	1·154
,, . .	$3O_2$	,,	5 : 6	4·8 × 6	1·144
EXPLOSION of Cyanogen with Oxygen and Nitrogen.					
$C_2N_2 + O_2$ . .	N_2	126,100 Cals.	3 : 4	4·8 × 4	1·132
,, . .	$2N_2$	,,	4 : 5	4·8 × 5	1·092
EXPLOSION of Cyanogen with Nitrous Oxide and Nitrogen.					
$C_2N_2 + 2N_2O$.	..	167,300 Cals.	3 : 5	4·8 × 5	1·294
,, .	N_2	,,	4 : 6	4·8 × 6	1·213
,, .	$2N_2$	,,	5 : 7	4·8 × 7	1·165
EXPLOSION of Cyanogen with Nitric Oxide.					
$C_2N_2 + 2NO$.	..	169,300 Cals.	3 : 4	4·8 × 4	1·132

TABLE XXIX.—The Rate of Explosion of Cyanogen Mixtures compared with the calculated Velocity of Sound (Σ).

1. CYANOGEN with Oxygen.

Mixture . . .	$C_2N_2 + O_2$.	$C_2N_2 + 2O_2$.	$C_2N_2 + 3O_2$.
Mean rate . .	2728	2321	2110
Σ	2725	2310	2066

2. CYANOGEN with Oxygen and Nitrogen.

Mixture . . .	$C_2N_2 + O_2$.	$C_2N_2 + O_2 + N_2$.	$C_2N_2 + O_2 + 2N_2$.
Mean rate . .	2728	2397	2163
Σ	2725	2397	2166

3. CYANOGEN with Nitrous Oxide and Nitrogen.

Mixture.	$C_2N_2 + 2N_2O$	$C_2N_2 + 2N_2O + N_2$	$C_2N_2 + 2N_2O + 2N_2$
Mean rate .	2454	2283	2098
Σ	2416	2237	2093

4. CYANOGEN with Nitric Oxide.

Mixture.	$C_2N_2 + 2NO$
Mean rate . . .	2760
Σ	2763

This comparison shows that the formula, which makes the velocity of sound in cyanogen burning to carbonic oxide equal to the observed rate of explosion, also gives for the velocity of sound in the same gases diluted with oxygen or nitrogen numbers closely concordant with the observed rates of explosion. The same formula holds for the explosion of cyanogen with nitrous oxide and when this mixture is diluted with nitrogen. Lastly, by firing a cartridge of fulminate in a steel cylinder filled with a mixture of cyanogen and nitric oxide I succeeded in propagating the explosion from the cylinder through a long leaden tube filled with the gases. The rate of explosion so obtained agreed exactly with the calculated sound-wave. I believe this was the first time that a mixture of cyanogen and nitric oxide had been exploded in a tube. BERTHELOT, who fired this mixture in a bomb, states that the *wave* is not propagated in these gases. I found that neither a strong spark, nor an initial explosion of hydrogen and nitric oxide, would set up the explosion-wave in cyanogen and nitric oxide. The details of these experiments are given in the Appendix.

The formula that I have given was thus found to agree with all the cyanogen explosions. It is therefore, at all events, an empirical expression which can be applied to calculate the rates of explosion of cyanogen burning to carbonic oxide and nitrogen under a fairly wide range of conditions. It was accordingly a matter of considerable interest to apply the formula to the rates found when electrolytic gas was exploded by itself and when diluted with hydrogen, oxygen, and nitrogen. In the following table the rates for electrolytic gas with excess of hydrogen and of oxygen are compared with the numbers calculated from BERTHELOT'S formula (θ), and with the sound wave (Σ) calculated as before. In calculating Σ the *contraction* which occurs on the union of hydrogen and oxygen is taken into account, but not the existence (which is possible) of free *atoms* of oxygen :—

TABLE XXX.

Explosive gases.	Inert gases.	Heat developed.	Contraction.	Specific heat of products.	Mean density.
$H_2 + O$	—	59,000 Cals.	3 : 2	4·8 × 1·5	·520
$H_2 + O$	H_6	„	9 : 8	4·8 × 4·5	·1964
$H_2 + O$	O_5	„	8 : 7	4·8 × 4	·910
$H_2 + O$	N_5	„	8 : 7	4·8 × 4	·8302

THE Rate of Explosion of Electrolytic Gas with excess of Hydrogen and with excess of Oxygen compared with Calculated Velocities.

Mixture.	$H_8 + O$	$H_6 + O$	$H_4 + O$	$H_2 + O$	$H_2 + O_2$	$H_2 + O_4$	$H_2 + O_6$
BERTHELOT'S θ.	3028	3061	3055	2900	2252	1730	1476
Rate of explosion . .	3532	3527	3268	2821	2328	1927	1707
Σ	3516	3571	3585	3416	2650	2024	1718

A glance at this table reveals the fact that the sound-wave calculated by my formula does not agree in velocity with the explosion-wave of pure electrolytic gas; but as the electrolytic gas is more and more diluted with hydrogen or with oxygen, the observed and calculated velocities come nearer together, until, not far from the two limits of regular explosion, they are in close agreement. Conversely we have seen that BERTHELOT'S formula gives the correct rate for pure electrolytic gas, but diverges more and more from the observed rates as the gas is diluted. Where BERTHELOT'S "θ" fails to represent the facts, the "Σ" I have calculated does so; and *vice versâ*.

I advance the following hypothesis to account for this divergence. *At the high temperature of the explosion-wave the combination of hydrogen and oxygen is not complete; or, in other words, steam is partly dissociated under these conditions.* The temperature of the wave front is therefore less than that calculated. As more and more inert gases are added to electrolytic gas, the temperature of the explosion is lowered by the division of the heat between the diluent gases; more and more of the explosive gases are thus able to combine. As the gases are diluted, therefore, the temperature of the wave front comes nearer and nearer to that calculated.

A comparison of the sound-wave, Σ, with the rates of explosion of electrolytic gas diluted with nitrogen, and of hydrogen with nitrous oxide and nitrogen, brings out similar divergences which admit of the same explanation. It is to be noted in the case of hydrogen and nitrous oxide that no alteration of volume occurs in the chemical change; no correction in temperature need, therefore, be made in calculating the sound-wave, and the question of the density of the molecules concerned does not arise.

TABLE XXXI.—The Rate of Explosion of Electrolytic Gas with excess of Nitrogen compared with Calculated Velocities.

Mixture.	H_2+O	H_2+O+N	H_2+O+N_3	H_2+O+N_5
BERTHELOT'S θ . . .	2900	2321	1814	1558
Rate of explosion . .	2821	2426	2055	1822
Σ	3416	2731	2122	1813

TABLE XXXII.—The Rate of Explosion of Hydrogen and Nitrous Oxide, with excess of Hydrogen, and of Nitrogen, compared with Calculated Velocities.

Mixture.	H_7+N_2O	H_6+N_2O	H_4+N_2O	H_2+N_2O	$H_2+N_2O+N_2$	$H_2+N_2O+N_3$
BERTHELOT'S θ . . .	2400	2396	2374	2319	1912	1782
Rate of explosion . .	2732	2705	2545	2305	1991	1880
Σ	2776	2781	2765	2706	2227	2067

When electrolytic gas is diluted with excess of nitrogen, the observed rate of explosion closely agrees with the calculated sound-wave. With hydrogen and nitrous oxide the temperature cannot be diminished to the same extent by dilution. When three volumes of hydrogen, or two volumes of nitrogen, are added to the mixture consisting of one volume of hydrogen and one of nitrous oxide, the explosion-wave is not propagated with regularity; consequently the temperature cannot be brought down by dilution below the dissociation-point of steam. As the mixture is diluted the rate of explosion approaches Σ, but agreement is not reached at the limits of dilution at which the explosion-wave will still travel.

The rates of explosion of mixtures of ammonia with oxygen and with nitrous oxide were also determined. The gases were mixed in an iron holder standing over mercury. The rate of explosion of ammonia and oxygen is faster than the velocity calculated by

BERTHELOT's formula, but falls far below the velocity of sound "Σ." On the other hand, the rate of explosion of ammonia and nitrous oxide is close to the calculated sound-wave and is far above "θ."

TABLE XXXIII.

	Explosive gases.	Heat developed.	Expansion.	Specific heat of products.	Mean density.
1.	$4NH_3+3O_2$	305,200	7 : 8	4·8 × 11	·761
2.	$2NH_3+3N_2O$	193,800	5 : 7	4·8 × 8·5	·987

(1.)

Mixture.	$4NH_3+3O_2$.
BERTHELOT's θ . . .	2300
Rate of explosion . .	2390
Σ	2665

(2.)

Mixture.	$2NH_3+3N_2O$.
BERTHELOT's θ . . .	1953
Rate of explosion . .	2200
Σ	2223

Mixtures of oxygen with the hydrocarbons ethylene and acetylene will explode when largely diluted with nitrogen. For instance, the explosion-wave is propagated in a mixture of one volume of ethylene with two of oxygen and eight of nitrogen. The temperature reached in the explosion is comparable with that produced when three volumes of electrolytic gas are fired with five volumes of nitrogen. It is found both with ethylene and acetylene that the rate of explosion exactly agrees with the calculated velocity of the sound-wave when the mixtures are largely diluted with nitrogen, but not when the gases are exploded without dilution :—

TABLE XXXIV.

Explosive gases.	Inert gases.	Heat developed.	Expansion.	Specific heat of products.	Mean density.
$C_2H_4+O_4$	——	185,000	3 : 4	4·8 × 5	·930
$C_2H_4+O_4$	$8N_2$	185,000	11 : 12	4·8 × 13	·954

Rate of Explosion of Ethylene and Oxygen with excess of Nitrogen compared with Calculated Velocities.

Mixture.	$C_2H_4+2O_2$	$C_2H_4+2O_2+N_2$	$C_2H_4+2O_2+2N_2$	$C_2H_4+2O_2+4N_4$	$C_2H_4+2O_2+6N_2$	$C_2H_4+2O_2+8N_2$
BERTHELOT'S θ . .	2487	2234	2050	1796	1624	1498
Rate of explosion .	2581	2413	2211	2024	1878	1734
Σ	2856	2575	2367	2073	1873	1727

TABLE XXXV.

Explosive gases.	Inert gases.	Heat developed.	Expansion.	Specific heat of products.	Mean density.
$C_2H_2+O_3$	—	171,700	5 : 6	4·8 x 3·5	·9406
$C_2H_2+O_3$	$6N_2$	,,	17 : 18	4·8 x 9·5	·9594

Rate of Explosion of Acetylene and Oxygen with excess of Nitrogen compared with Calculated Velocities.

Mixture.	$C_2H_2+O_3$	$C_2H_2+O_3+N_2$	$C_2H_2+O_3+N_6$	$C_2H_2+O_3+N_8$	$C_2H_2+O_3+N_{10}$	$C_2H_2+O_3+N_{12}$
BERTHELOT'S θ . . .	2749	2397	1983	1848	1739	1650
Rate of Explosion .	2716	2414	2209	2116	2019	1908
Σ	3187	2784	2303	2144	2016	1910

Since steam is known to be partially dissociated under atmospheric pressure at the temperature of the oxy-hydrogen flame, it does not seem improbable that dissociation would also occur at the higher pressures and higher temperatures of the explosion-wave. The researches of MM. BERTHELOT and VIEILLE, and those of MM. MALLARD and LE CHATELIER, on the pressures registered in an explosion of gases, have led these investigators to the conclusion that the specific heat of steam rapidly rises with the temperature. The deficiency of "available" pressure, which BUNSEN first observed in the explosion of gases and attributed to incomplete combustion, they consider to be due to an increase of specific heat. Such an increase in the specific heat of steam with rise of temperature would explain the divergence between the observed and calculated rates of explosion of hydrogen with oxygen, and with nitrous oxide. And conversely, it appears to me that the results of the French experimenters would be equally well explained by the temporary dissociation of steam in their explosions.

One fact appears, at first sight, to be opposed to the hypothesis of incomplete combustion. The addition of oxygen, one of the products of the dissociation of steam, retards the explosion of electrolytic gas more than the addition of an equal volume of nitrogen—an inert gas. WURTZ found, in the case of phosphoric chloride, that the amount of dissociation was greater when the chloride was volatilised in air —an inert gas, than when it was volatilised in the presence of the lower chloride— one of the products of its decomposition. Arguing from this observation, one might have expected that excess of oxygen would tend to hinder dissociation, and therefore, if the combustion were really incomplete, that the rate should be faster when oxygen was used as a diluent instead of nitrogen. But the condition of equilibrium under which WURTZ determined the vapour density of phosphoric chloride is not the condition of the wave-front of an explosion. If we imagine in the wave-front a molecule A dashing against a molecule B with such extreme velocity that the compound AB is not formed, the two molecules may be supposed to rebound, exchanging energies, and the motion of B may be communicated to the molecules in front continuing the explosion. This is a case where dissociation of AB occurs in the wave-front. Now if a number of B molecules are present in excess, A will shortly combine with one of them, behind the wave-front, and the dissociation will not be permanent. On the other hand, if there are a number of inert molecules C present, dissociation may occur as before in the wave-front, but the uncombined molecules may be prevented from combining behind the wave by the inert molecules present. The propagation of the wave does not depend on what occurs after it has passed; the rate of the wave will therefore be independent of the permanency of the dissociation occurring in it.

That the combustion of pure electrolytic gas is not *wholly* complete in the explosion-wave has been proved by collecting the residue and exploding it.* In the propagation of the wave the cooling due to expansion is so rapid that some molecules of hydrogen and oxygen, which are unburnt in the wave-front, have not time to combine before they are cooled below the temperature of combination. In a leaden tube, 9 mm. in diameter and 100 metres long, about 1 per cent. of electrolytic gas was found uncombined after the explosion. That this incompleteness of combustion was not due to the cooling effect of the walls was shown by making comparative experiments in tubes 4 mm. and 19 mm. in diameter. Nearly the same percentage of unburnt residue was found in all the tubes, and also when the gases were detonated in an iron bomb 100 mm. in diameter. We have, therefore, positive evidence that in the explosion-wave the combustion of electrolytic gas is incomplete; it seems, therefore, not unreasonable to assume that in the wave-front—*i.e.*, at the highest temperature—a considerable proportion of hydrogen and oxygen is uncombined, and the propagation of the wave is retarded accordingly.

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* "Incompleteness of Combustion in Gaseous Explosions," by H. B. DIXON and H. W. SMITH. 'Manchester Memoirs' [IV.], vol. 2. 1888.

The residual gases, left after the explosion of electrolytic gas, were examined for peroxide of hydrogen; the water condensed on the tube was also washed out and examined. In neither case could peroxide of hydrogen be detected. If the view were correct that hydrogen and oxygen first form peroxide on explosion, it would seem likely that, in the rapid cooling, some of the peroxide would remain undecomposed, and be found in the residue.

In the explosion of carbonic oxide and oxygen a similar residue of unburnt gas is found—indicating the dissociation of carbonic acid at the temperature of the wave. No trace of peroxide of hydrogen could be detected in the tube. These observations give no support to the view put forward by MORITZ TRAUBE[*] that steam promotes the oxidation of carbonic oxide by dividing with it the molecules of oxygen:—

$$CO + OO + OH_2 = CO_2 + O_2H_2$$

and that the peroxide so formed is again reduced to steam by more carbonic oxide:—

$$CO + O_2H_2 = CO_2 + OH_2.$$

CAP. IX.—THE EXPLOSION OF HYDROGEN AND CHLORINE.

The coincidences and the divergences found between the calculated and observed rates of explosion, as described above, made it important to examine a reaction in which no steam should be formed, and, if possible, one in which no alteration in volume should be produced. The simplest case fulfilling these conditions is the explosion of hydrogen and chlorine; a reaction which has the further advantage of forming, as the product of explosion, a gas which approximates to a perfect gas. So far as I know the rate of explosion of hydrogen and chlorine had never been measured.

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[*] 'Ber. Deut. Chem. Ges.,' vol. 18, p. 1890. (See 'Chem. Soc. Journ.,' 1886, vol. 1, p. 94.)

Before devising apparatus to carry out the experiments, it was necessary to determine whether the action between hydrogen and chlorine was a direct one, or whether it was influenced by the presence of other gases, *e.g.*, steam.[*] Electrolytic hydrogen and chlorine were accordingly sealed up in glass vessels with anhydrous phosphoric acid, and were kept in the dark for several weeks. The dried gases were found to be far less sensitive to the action of *light* than when a trace of aqueous vapour was present, a result in accordance with PRINGSHEIM's[†] observations. But on the passage of an induction spark by means of platinum-iridium wires the dried gases exploded violently.

Since these experiments left it doubtful whether the presence of steam influenced the rate of explosion of hydrogen and chlorine, a preliminary series of measurements were made with the dry and moist gases.[‡] The explosion-vessel consisted of two long glass tubes[§] (34 and 30 feet) cemented together. The ends were provided with steel flanges so that they could be rapidly joined to two smaller tubes which carried the silver bridges. The explosion tube was hung to the wall of a long, darkened corridor.

The gas passed directly from the electrolytic cell through washing tubes into the long explosion tube—driving out the air before it: at the further end it passed through an "analyser," which served to determine the composition of the mixture issuing from the tube. When the long tube was completely filled with hydrogen and chlorine, the two smaller tubes carrying the bridges were fastened to its two ends by the steel flanges. These smaller tubes had previously been filled with hydrogen and oxygen. The electric connections were then made to the chronograph, and the gases fired by the passage of a spark through the hydrogen and oxygen in one of the smaller tubes. The flame travelling down this tube, which was about 4 feet long, broke the silver bridge at the end of it, and communicated the explosion to the hydrogen and chlorine. On reaching the end of the glass tube, the flame broke the silver bridge carried by the second smaller tube. In the few seconds that elapsed between the joining on of the small tubes and the firing of the mixture but little diffusion of the gases was possible; and the silver bridges, being coated with paraffin, were hardly acted on by the chlorine. By this arrangement the electrolytic hydrogen and chlorine were kept from contact with metals or india-rubber, and from all liquids but water, and, in the "dry" experiments, oil of vitriol saturated with chlorine. The drying apparatus consisted of a "Winkler" worm, a series of bulbs, and a large U-tube packed with pumice, all filled with boiled oil of vitriol.

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[*] "On the Combination of Hydrogen and Chlorine, alone, and in Presence of other Gases." By H. B. DIXON and J. A. HARKER. 'Manch. Memoirs' [IV.], vol. 3, p. 118.

[†] 'Annalen der Physik,' N.F., vol. 32, p. 421, 1887.

[‡] 'Manch. Memoirs' [IV.], vol. 4, p. 3. (The numbers given in this paper were afterwards found to be affected by a constant error.)

[§] These glass tubes were made for me by Messrs. MOLINEAUX and WEBB, of Manchester. They were unannealed, but stood the shock of many explosions.

The following table gives the mean of each pair of experiments with the dry and with the moist gases saturated at 14°.

TABLE XXXVI.—Rate of Explosion of Hydrogen and Chlorine.

Dry.	Moist.
1748	1724
1745	1707
1737	1730
1748	
1750	
Mean 1745	Mean 1720

It appeared, therefore, that the explosion, once started, travelled at a slightly faster rate in the dry than in the wet gases, the steam appearing to act merely as a diluent.

These experiments, together with the analysis of the mixture which issued from the end of the explosion tube, showed that it was possible to fill a long tube directly from the electrolytic cell with hydrogen and chlorine in equal volumes, and that the mixture *detonated* in a glass tube of 10 mm. diameter. To prove the constancy of the rate of explosion, and at the same time to have a longer interval of time to measure, it was necessary to obtain a longer explosion tube. The difficulty of joining up long pieces of unannealed glass-tubing led me to try the glass-lined wrought-iron tubing manufactured by Messrs. RYLANDS, at Barnsley. Having found that this tubing answered the purpose, I made an explosion tube 40 yards long, and ½-inch diameter, by joining together a number of six-feet lengths. The junctions were made by fitting an asbestos washer on to the ends of each tube, and screwing them together with a nut. The tube was supported at intervals by wooden brackets fastened to the wall of a long corridor. The explosion tube so put together was perfectly gas-tight; it was unaffected by the chlorine passed into it during the experiments, and it stood the shock of many explosions without leaking. This tube was filled, as before, direct from the electrolytic cell of aqueous hydrochloric acid. The cell and drying vessels were all of glass with ground glass joints. Two short tubes charged with hydrogen and oxygen were used, as before, to carry the silver bridges. They were joined to the long tube immediately before the explosion. The mean rate of explosion of dry hydrogen and chlorine in this tube was found to be 1723 metres per second—as the result of three pairs of experiments :—

TABLE XXXVII.—Rate of Explosion of Hydrogen and Chlorine. 2nd Series.

	1718
	1727
	1725
Mean	1723

If the preliminary series of experiments with the dry gases is regarded as of equal value with *one* of these pairs, the mean rate of explosion is found to be 1729 metres per second.

The heat of combination of hydrogen and chlorine was found by FAVRE and SILBERMANN to be 23,780 calories; by J. THOMSEN it was found to be 22,000 calories. BERTHELOT gives one determination 22,100 calories. The mean of these numbers is 22,630 calories. Taking THOMSEN'S number for the heat of combination, the mean rate of translation of the products of combustion according to BERTHELOT'S formula (corrected to 13° C.) is found to be 1551 metres per second; taking the mean number 22,630 calories for the heat of combination, BERTHELOT'S formula gives 1571 metres for the mean rate. The rate of explosion of hydrogen and chlorine under ordinary conditions is, therefore, considerably faster than the rate given by BERTHELOT'S formula. The velocity of the sound-wave (Σ) calculated by my formula is 1805 or 1830, according as the lower or higher heat of combination is taken. The rate of the explosion thus falls appreciably below the calculated velocity of the sound-wave. Since BERTHELOT contends that his formula gives the maximum velocity of explosion, his hypothesis fails in the case of the explosion of hydrogen and chlorine. On the other hand, two questions arise: (1) Does dissociation occur in the explosion? (2) On diluting the mixture will the rate of explosion approach the velocity of the sound-wave?

When the hydrochloric acid gas, produced by the explosion in the long tube, was driven out by a stream of hydrogen through a washing apparatus containing potassium iodide, a quantity of iodide was liberated, showing the presence of free chlorine in the products of combustion. And when the gases were driven out by a stream of carbonic acid through a washing apparatus containing caustic soda, free hydrogen was found to be present also. Even after passing the current for many hours the gases coming from the electrolytic cell, and issuing from the further end of the explosion tube, were always found to contain a slight excess (amounting to between ·2 and ·5 per cent.) of hydrogen. The presence of free chlorine in the vessel after the explosion was thus a proof of dissociation occurring in the explosion, and the determination of this free chlorine by the liberation of iodine afforded a ready means of measuring the volume of gases left unburnt. About 1 per cent. of the combustible gases were found uncombined.

On diluting the electrolytic gas with hydrogen the rate of explosion was found to increase, and to approximate to the calculated velocity of the sound-wave. The mixtures were made by bringing the electrolytic gas from the cell, and the hydrogen from a holder, into a sulphuric acid wash-bottle through two tubes of the same size dipping side by side beneath the liquid. By regulating the flow of hydrogen the bubbles of each gas could be kept passing at an equal rate. In this way a mixture containing three volumes of hydrogen to one volume of chlorine was passed into the explosion tube. The correctness of the mixture was checked by collecting and analysing a sample of the gas escaping from the end of the tube. By passing the hydrogen at half the rate a mixture containing approximately two volumes of hydrogen to one of chlorine was obtained. While the addition of diluent hydrogen to the electrolytic gas makes little difference in the calculated rates, the observed velocities of explosion increased appreciably. On the assumption that the heat of combination of hydrogen and chlorine is 22,630 calories, the calculated and observed velocities are as follow :—

TABLE XXXVIII.—Rate of Explosion of Hydrogen and Chlorine with addition of Hydrogen.

Mixture.	H_2+Cl_2	H_4+Cl_2	H_6+Cl_2
BERTHELOT'S θ	1571	1581	1589
Rate of explosion . . .	1729	1849	1855
Σ	1830	1832	1832

These experiments show that in the explosion of hydrogen and chlorine as in the explosion of hydrogen and oxygen, in equivalent proportions, the rate falls below the calculated velocity of the sound-wave, but on diluting the gases the calculated and observed velocities come together. This divergence at the highest temperature may be explained either by the dissociation of the hydrochloric acid or by a rise in its specific heat.* As I have shown, we have direct evidence of *some* dissociation. On the other hand, the physical properties of hydrochloric acid more nearly resemble those of carbonic oxide and the elementary gases than those of steam or carbonic acid; we should expect, therefore, that the specific heat of hydrochloric acid should remain as constant as that of carbonic oxide at high temperatures.

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* MM. MALLARD and LE CHATELIER conclude from their experiments that some dissociation occurs when equal volumes of hydrogen and chlorine are exploded. When excess of hydrogen is added they find no dissociation.

The fact that the rate of explosion of hydrogen and chlorine exceeds the calculated velocity of the sound-wave, as calculated from THOMSEN'S Heat of Combination (22,000 cals.), is possibly due to the behaviour of the unburnt chlorine in the wave-front. The rates agree fairly with the sound-wave calculated from the *mean* heat of combination, which I have given in the table.

CAP. X.—THE INSTANTANEOUS PRESSURES PRODUCED IN THE EXPLOSION-WAVE.

There is, lastly, one important phenomenon in the propagation of explosions in gases which I have not touched on—the pressure produced in the reacting gases. BUNSEN was the first who attempted to measure this pressure. In his well-known work* he describes the explosion of mixtures of hydrogen and of carbonic oxide with oxygen in a strong vessel closed with a loaded lid. When the pressure produced in the vessel exceeded the pressure on the lid, the latter was raised in the manner of a safety valve, and some of the heated gas escaped; when the pressure was less than that on the lid, no gas escaped and but little noise was heard.

By successive trials he thus found the pressure produced by the explosion of carbonic oxide and oxygen to be about 10 atmospheres, and the pressure produced by the explosion of hydrogen and oxygen to be about 9·5 atmospheres. The comparison of these pressures with the numbers calculated from the heat of combination and the specific heats of the products led BUNSEN to the conclusion that, at the highest temperature reached in the explosion, only one-third of the gases entered into combination. BERTHELOT and VIEILLE, working in a manner essentially the same, only replacing BUNSEN'S loaded lid by a light piston moving against a spring, found the pressures produced in the explosion of carbonic oxide and hydrogen to be 10·1 and 9·8 atmospheres respectively. BERTHELOT rightly calls these pressures "*effective pressures.*"† On the other hand, MALLARD and LE CHATELIER, using the more delicate indicator of DEPREZ, found pressures produced in the explosion of hydrogen and oxygen, of far greater amount, but lasting an exceedingly short time. These fugitive pressures of great intensity were only obtained with mixtures which burnt very rapidly :—‡

"Cette pression très fugitive est d'autant plus grande que la vitesse de propagation est plus considérable. Elle peut devenir énorme avec l'onde explosive. C'est à ces pressions, si instantanées qu'elles soient, que sont dus les surprenants effets des substances explosives telles que la nitroglycérine. Ce sont précisément ces pressions passagères, sans relation directe avec le phénomène que nous voulions mesurer, qu'enregistrait notre appareil."

On substituting for the DEPREZ indicator a less sensitive Bourdon manometer,

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* 'Gasometrische Methoden,' 1857.

† 'Ann. Chim. et Phys.' [vi.] 4, p. 14, 1885.

‡ 'Combustions des Mélanges gazeux,' p. 123.

LE CHATELIER found "effective pressures" in agreement with those obtained by BUNSEN and by BERTHELOT, *i.e.*, about 10 atmospheres.

It is evident from these experiments that sudden pressures are produced in the explosion-wave greater than those shown by the experiments of BUNSEN and of BERTHELOT. If the explosion is propagated by compression after the manner of a sound-wave enormously high temperatures must be produced in the wave, as I have already shown, and correspondingly high pressures must also exist. This aspect of the matter has been well stated by MALLARD and LE CHATELIER :—

"A gas, increasing in volume by combustion, exerts a certain pressure on the unburnt gas next to it: it is possible to conceive that this pressure may be raised sufficiently high to bring the neighbouring layer to the temperature of ignition. The possibility of the propagation of the inflammation by the propagation of the pressure can thus be understood.

"Take for example the explosive mixture of hydrogen and oxygen. According to our experiments it inflames at 555°, and the pressure exerted after the combustion in a closed vessel is 10 atmospheres. An infinitely thin layer inflamed by heating, reaches this pressure at the greatest, and cannot exert any higher pressure on the uninflamed layer next to it.

"A compression of 10 atmospheres evolves a quantity of heat which can readily be calculated by thermo-dynamics : it is not sufficient to produce an elevation of temperature equal to that inflammation. To obtain this temperature a compression of 30 atmospheres would be needed. It would appear, then, that the explosion-wave could not be initiated under these conditions. . . . .

"Suppose, however, that the inflammation of the first layer is not brought about by heating, but by a suitable pressure, *i.e.*, one of not less than thirty atmospheres, which may readily be produced by the detonation of a little fulminate. After the combustion the pressure of the inflamed layer will be multiplied by ten, or at least will be far higher than the initial pressure of thirty atmospheres, and will be able, consequently, to exert on the next layer a pressure sufficient to inflame it. This will continue then to propagate itself from layer to layer by the same mechanism."—('Combustion des Mélanges Gazeux,' p. 88.)

The sudden state of pressure produced in each layer of burning gas ceases as suddenly as it is caused, by the expansion of the gas which compresses the unburnt layer in front; the high temperature of the burnt layer, therefore, rapidly falls by the transformation of heat into work. MALLARD and LE CHATELIER have shown by photographs of the flame that, in the explosion-wave, the luminosity of each layer does not last longer than one thousandth part of a second; while in the *régime* of ordinary combustion each layer remains luminous more than ten times as long. I can fully confirm these statements. By photographing the explosion-wave of hydrogen and oxygen on a fixed and on a rapidly moving plate at the same time, I have found that the interval during which the gas remains bright enough to effect a sensitive isochromatic plate cannot be more than *one five-thousandth of a second.*

The problem of measuring the pressure in a thin layer of gas, lasting for so small a fraction of a second, is one of the greatest difficulty. I believe the method tried by MALLARD and LE CHATELIER for explosions of carbon bisulphide and oxygen is probably the best that has yet been devised. This depends on the principle that if a pressure is produced in a glass vessel greater than the glass can stand, the vessel will be broken, although the pressure may endure for a very small interval of time. If the vessel is broken by the explosion, I believe that there must have been a pressure produced in the layer of gas next the glass *at least* equal to that which would break the vessel when gradually applied. By compressing gas in the vessel by means of a pump, it can be shown that the vessel is able to withstand a certain pressure. The vessel can then be connected with a long tube, the whole filled with an explosive mixture, and the explosion-wave set up in the tube. If the vessel is fractured, a pressure was exerted on its walls greater than the pressure applied by the pump.

To test this method, I have carried out a few preliminary experiments. Out of a piece of strong combustion tubing I made twelve short tubes, carefully sealed at one end, and slightly opened at the other. These tubes were then firmly attached to a metal piece by a collar and nut working against india-rubber rings—care being taken that the glass did not come anywhere into contact with the metal, but only with the rubber rings. The metal piece could be attached either to the pump or to a leaden firing tube. The twelve tubes were first tested with compressed air, which was slowly forced into the tube while an observer read the manometer. One tube (it was rather unevenly sealed) broke down between nineteen and twenty atmospheres. A second broke at twenty-three atmospheres. The other ten stood a pressure of twenty-five atmospheres for two minutes. These tubes were then in turn attached to one end of a firing-tube about twenty feet long. On filling the whole with a mixture of carbonic oxide and oxygen, and passing a spark through the gases at the other end of the tube, the explosion-wave was propagated through the leaden pipe into the glass vessel. All ten tubes were blown into small fragments by the explosion. Four stronger green glass tubes were tested by compressed air at fifty atmospheres. All these tubes stood the explosion of carbonic oxide and oxygen, and hydrogen and oxygen. Two of these green tubes were blown to pieces by the explosion of equal volumes of cyanogen and oxygen, a third was fractured by compressed air at seventy-eight atmospheres. These experiments confirm LE CHATELIER's results with the DEPREZ indicator, showing that very high pressures are produced for the moment in the explosion-wave; they do not, of course, *measure* the pressures, they only indicate a lower limit.

The quiet burning of carbonic oxide and oxygen, when this mixture is lighted at the open end at a short tube, such as a test-tube, is in striking contrast to the violence of the explosion when the "wave" is set up in the same gases. As a lecture experiment a test-tube full of the mixture may be ignited by a taper, when the quiet passage of the blue flame down to the tube can be followed by the eye; the tube is then refilled and screwed on to the end of a few feet of leaden pipe filled with the mixture. The test-tube is surrounded by metal gauze and a thick glass cylinder. On applying a flame to the open end of the pipe, or passing a spark near the extremity, a loud report is heard, and the test-tube is reduced to powder.[*] The explosion-wave in a glass tube produces a brilliant flash of orange light.

Professor A. SCHUSTER[†] has drawn my attention to a memoir, by RIEMANN, on the propagation of intense disturbances in gases, and has suggested a mode of calculating the momentary pressures produced in the explosion-wave from its known velocity of propagation, and the density of the unburnt gas. On the assumption that RIEMANN'S theory holds for an explosion, I have calculated, from the observed rates of the wave and the densities of the unburnt gas, the pressures existing in the wave in the explosion of different gaseous mixtures. In parallel columns I have given the "effective" pressures as registered in the apparatus used by BERTHELOT for the same mixtures. The pressures calculated for the wave are rather more than four times larger than these effective pressures. In the third column I have given the calculated pressures divided by 4·2.

The pressure in the wave is given by the equation

$$\pi = \frac{\nu^2 \rho}{100}$$

where π is the pressure in atmospheres, ν the rate in metres per second, and ρ is the weight in grammes of 1 c.c. of the unburnt gas.

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[*] Professor DEWAR has noticed the great violence with which carbonic oxide and oxygen explode under certain conditions. He says: "Sometimes the explosions were so violent as to break the plate closing the end of the tube, though this had resisted the explosions of the hydrogen mixture; while at the other times the wave of explosion passed slowly along the tube." 'Roy. Soc. Proc.,' April 3, 1884. Probably owing to differences in the nature of the electric sparks, the "explosion-wave" was sometimes set up in his tube and sometimes not.

[†] See note at end of cap.

TABLE XXXIX.—Pressures in the Explosion-wave calculated from RIEMANN'S Theory.

Explosive Mixture.	Calculated Pressure. Π	$\frac{\Pi}{4 \cdot 2}$	Effective Pressure measured by BERTHELOT.
$C_2N_2+O_2$	135·2	32·2	25·11
C_2N_2+2NO	123·0	29·3	23·34
$C_2N_2+2N_2O$	121·6	28·9	26·02
$C_2N_2+2O_2$	90·1	21·5	20·96
$C_2N_2+O_2+2N_2$	71·0	16·9	15·25
$C_2H_4+O_2$	81·5		
$C_2H_4+O_3$	94·3		
$C_2H_4+O_4$	88·4		
$C_2H_4+O_6$	75·2	17·9	16·13
CH_4+O_4	62·2	14·8	16·34
$C_2H_2+O_2$	110·0		
$C_2H_2+O_3$	94·5		
$C_2H_2+O_3+N_{12}$	45·3		
H_8+O	28·8	6·9	7·05
H_2+O	41·3	9·8	9·8
H_2+O+N_5	31·6	7·5	7·41
H_2+N_2O	52·9	12·6	13·6
$CO+O$	35·7	8·5	10·1
$H+Cl$	46·9	11·2	By LE CHATELIER. 10·0

It is to be noticed that these calculated pressures are in substantial agreement with my experiments on the fracturing of glass tubes by the explosion of carbonic oxide and of cyanogen with oxygen; and also that the minimum pressures calculated for the diluted electrolytic gas are close to 30 atmospheres—the pressure required to fire electrolytic gas by the heat of compression.

NOTE BY PROFESSOR ARTHUR SCHUSTER, F.R.S.

In 1859 RIEMANN* deduced an equation for the propagation of abrupt variations in the density and pressure of a gas, on the supposition that such an abrupt variation could be propagated without change of type. Lord RAYLEIGH† criticising this investigation, draws attention to the fact that a steady wave is only possible for a particular relation between the pressure and density of the gas, which is different from the one actually holding. In the case of the explosion-waves it seems possible, however, that the temperature, pressure, and density of the gas should so adjust themselves as to make RIEMANN's equations applicable. In fact, they must do so if the front of the wave keeps its type, which it probably does when the velocity has become constant.

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* 'Göttingen Abhandlungen,' vol. 8 (1860).

† RAYLEIGH, 'Sound,' vol. 2, p. 41.

If ρ_0 and ρ are the densities of the gas in the undisturbed state, and at some point in the wave respectively, then, referring the motion to a system of coordinates moving forward with the velocity V of the wave, we have the condition of steady motion

$$\rho u = \rho_0 V$$

where u is the velocity at the point at which the density is ρ;
also

$$\frac{dp}{dx} = - \rho u \, . \frac{du}{dx} .$$

As ρu is a constant quantity we may integrate the latter equation and obtain

$$p - p_0 = - \rho u \, (u - u_0) = \frac{\rho_0}{\rho} V^2 (\rho - \rho_0) ,$$

which gives for the velocity of the wave propagation

$$V = \sqrt{\frac{(p - p_0)}{(\rho - \rho_0)} . \frac{\rho}{\rho_0}} .$$

This equation gives a relation between the density ρ at any point of the wave and the pressure p at that point, the pressure and density in the initial state being p_0 and ρ_0. The equation can be simplified if ρ is large compared to ρ_0 for

$$\frac{\rho}{\rho_0 (\rho - \rho_0)} = \frac{1}{\rho_0} \Big/ (1 - \rho_0/\rho)$$

so that we may write

$$V = \sqrt{\frac{p - p_0}{\rho_0}} \quad \sqrt{\frac{1}{1 - \rho_0/\rho}}$$

Putting, for the sake of argument, $\rho = 10\rho_0$, the second square root in the above expression becomes 1·05, so that as a first approximation it may be put equal to one. Writing P for the *excess* of pressure over the atmospheric pressure, the equation now takes the simple form

$$V = \sqrt{\frac{P}{\rho_0}} .$$

In this expression the value of P has to be written in dynes per square centimetre, and the velocity would be given in centimetres. For more convenient numerical calculation we may adopt provisionally the suggestion of GUILLAUME that a pressure of 75 cms. of mercury under the normal conditions should be called an *atmosphere*. The atmosphere then would be exactly one megadyne per square centimetre, and if P is given in atmospheres, the equation becomes

$$V = 1000 \sqrt{\frac{P}{\rho_0}} ,$$

or if V is expressed in metres per second,

$$V = 10 \sqrt{\frac{P}{\rho_0}} .$$

The excess of pressure in the waves, therefore, will be, as far as the equations hold,

$$P = \frac{V^2 \rho_0}{100}.$$

If the factor

$$\sqrt{\frac{\rho}{\rho - \rho_0}}$$

be retained, the calculated pressure P would be rather smaller than that given; thus if the maximum density of the wave is only double that of the undisturbed state, the calculated value of P would have to be reduced in the ratio of 1·4 to 1.

In order to test the equation

$$V = \sqrt{\frac{P}{\rho_0}}$$

I have applied it to compare the rates of explosions in the mixtures $H_7 + O$, $H_2 + O + N_5$, $H_2 + O_6$. In these three cases the explosive mixture $H_2 + O$ is diluted with the same volume of inert gases, and according to the equation the rates should be inversely as the square roots of the densities of unburnt gases, for the pressure P may be expected to be the same in all three cases. The rates found by Professor Dixon are 3532; 1822; 1707. Assuming 3532 to be right, I calculate for the other two combinations 1806 and 1711, which so far confirms the equation. But I am doubtful whether this means much, as the same result could be deduced from arguments relating merely to the dimensions of the quantities involved.

I have not discussed the question whether a steady wave is really possible, but assuming it to be steady I believe that the equation

$$V = \sqrt{\frac{p - p_0}{\rho - \rho_0} \cdot \frac{\rho}{\rho_0}}$$

holds. In the strict sense of the word I do not think the explosion-wave can be steady, because if the motion is, as assumed, linear, compression must precede the explosion, and Lord Rayleigh's objection would hold for the front part of the wave in which no combination takes place.

But it seems possible to me that the motion may not strictly be a linear one, and that yet taking the average velocities over a cross-section of the tube the ordinary equations would apply. It seems probable that jets of hot gases are projected bodily forward from that part of the wave in which the combination takes place, and that these jets, which would correspond to the spray of a breaking wave, really fire the mixture.

Cap. XI.—Some Difficulties in the "Wave" Theory.

§ 1. In calculating a theoretical velocity for the rate of the explosion-wave in gases, I am aware that I have passed over many difficulties which demand consideration. For some of the problems raised here we have, I believe, no data which might enable us to solve them. Riemann's theory of the propagation of an intense disturbance in a gas requires, in order to calculate the velocity of propagation, a knowledge of the highest pressure reached in the wave. But concerning the pressures produced for extremely short periods in the explosion-wave, we have less exact knowledge than of the velocities, and possibly the rates of explosion may be used to calculate the pressures produced.

The question, indeed, may be raised whether the propagation of the ignition is due to a forward movement *en masse* of the burning molecules, or to the advance of a comparatively few molecules with extreme rapidity. In the latter case the explosion-wave might be propagated with a velocity far higher than the *average* velocity of the molecules of burning gas. I think the available evidence precludes the idea that the ignition can be propagated by a few molecules moving with velocities far above the mean. For, if this were the case, the presence of a diluent gas ought not to hinder the advance of the wave in the manner it is found to do. The heat evolved on the formation of a compound molecule is the same whether inert molecules are in the neighbourhood or not, and for the moment the temperature of the compound is the same. If, therefore, the ignition could be propagated by the advance of individual molecules, we might expect the explosion to be propagated at the same, or nearly the same, rate in the presence of an inert gas. My experiments have, however, conclusively shown that an inert gas retards the wave by an amount which depends upon its volume and density. It would seem to follow that each layer is heated as a whole and that the advance of the wave depends upon the average motion of the molecules. Two other facts appear to support this view. It is well known that small electric sparks may be sent through explosive mixtures of gases, *e.g.*, through electrolytic gas, without causing explosion. Yet the temperature in the spark must be extremely high, and combination occurs in the path of the spark. It would seem that the number of molecules heated by each spark was not sufficient to communicate ignition to the rest. Again, when a sound-wave is started in a mixture of gases, its rate of propagation depends upon the mean density of the gaseous mixture; there is no selective propagation of the wave by one of the constituents of the mixture.* This fact, of which I could find no experimental demonstration for gases of widely different density, has been verified, at my request, by Mr. F. J. SMITH in the MILLARD Laboratory, at Trinity College, Oxford. According to Mr. SMITH's careful experiments, sound is propagated at the same rate in ammonia gas (density 8·5) as in a mixture of equal volumes of hydrogen and oxygen of the same mean density as ammonia.

On the other hand, it may be that the wave does not advance perfectly regularly, but *breaks*, like a water-wave is broken by the quicker movement of the crest. In breaking, jets of heated gas may be projected in front of the wave into the unburnt gas beyond—such jets becoming the centres of fresh disturbances which coalesce to re-form a wave. If the jets were produced in sufficient number their average effect would be uniform.

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* FARADAY thought that gases of very different density would not mix perfectly. "As the atmosphere is now constituted, there exists a permanency of sonorous pitch; any tone being once generated remains the same tone until it dies away. . . . If, however, the atmosphere had been composed of two gases, each having widely different specific gravities, there would have been a difference. No permanency of tone could then have been depended upon—the pitch of every original note would have been continually varying as its transmission might be propagated by the vibrations of ever-varying mixtures of the two gases of dissimilar specific gravities." FARADAY'S Lectures, SCOFFERN, 1853.

§ 2. *The Specific Heat of Gases at High Temperatures.*

In calculating the theoretical velocity of the explosion-wave, I have assumed that the specific heats of the simple gases do not rise with increase of temperature. On the other hand, I have assumed that the specific heats of gases formed from their elements with a condensation of volume—such as steam, nitrous oxide, ethylene, etc.—are equal at a high temperature to the sum of the specific heats of their elements. Thus, I have taken the specific heat of steam at constant volume to be 7·2 instead of 6·5, as given by determinations at low temperatures. In making these assumptions, I have followed the example of BERTHELOT in his investigations on the explosion-wave. But in his later work on the specific heat of the gaseous elements at very high temperatures, BERTHELOT comes to the conclusion that these specific heats rise so rapidly with an increase of temperature that they are doubled between 1600° and 4400° C.* He arrives at this result by exploding cyanogen to carbonic oxide in a bomb, and observing the pressures registered by a piston moving against a spring when the cyanogen is burnt by oxygen alone, and when nitrogen is added or oxides of nitrogen are used instead of oxygen.

TABLE XL.—BERTHELOT's Experiments on the Specific Heats of Simple Gases.

Mixture.	Pressure developed.	Temperature.	Specific heat of N_2 or CO.
	At	°	
$C_2N_2 + O_2$	25·11	4394	9·60
$C_2N_2 + O_2 + 1\frac{1}{2}N_2$.	20·67	4024	8·39
$C_2N_2 + O_2 + 2N_2$ . .	15·26	3191	7·93
$C_2N_2 + O_2 + \frac{12}{31}N_2$.	11·78	2810	6·67
$C_2N_2 + 2NO$ . . .	23·34	4309	9·85
$C_2N_2 + 2N_2O$ . . .	26·02	3993	8·43

These values of the specific heats depend upon the accuracy with which the maximum pressure developed in the explosion is registered by the moving piston. Now the more rapidly the combustion takes place, the more nearly does the pressure exerted resemble a blow. I have found that a short column of mercury in a pressure-gauge suffers a far greater movement under the influence of a slow explosion than it does under the influence of a rapid explosion, although the quantity of heat evolved and the specific heats of the products were nearly the same in the two cases. No doubt this gauge was less sensitive to sudden pressures than the instrument used by M. BERTHELOT; but the experiment warns us that the pressures registered by a moving piston may fall below the true value, and the more rapid the explosion the greater this error may be. That the pressures registered, and, therefore, the specific heats deduced from them, vary with the instruments employed appears from a comparison of the numbers obtained by M. BERTHELOT, and those of MM. MALLARD and LE CHATELIER in the explosion of cyanogen. According to BERTHELOT's experiments, the specific heat of nitrogen and carbonic oxide is 6·2 at 2500°; according to MM. MALLARD and LE CHATELIER, it is 7·2 at this temperature. Again the latter have determined the specific heat of the simple gases by adding measured volumes of hydrogen, oxygen, and nitrogen to electrolytic gas, and measuring the pressures produced on explosion. From the seven experiments* made at temperatures near 2000° (calculated), they obtain, as a mean, the number 5·6 for the specific heat of the elementary gases at this temperature; in a former paper they gave the value 7·5 as the specific heat at this temperature. Neither of these values agrees with the 7·2 deduced from their cyanogen experiment at 2500°. The great difficulties attending determinations of this nature, in spite of the skill and patience with which the French chemists have attacked the problem, still leave it an open question whether the specific heats of the simple gases increase with the temperature.

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* 'Ann. Chim. et Phys.' (VI.), vol. 4, p. 66. 1885.

§ 3. The observations I have made on the effect of changes of initial pressure and temperature on the rate of explosion in gases are not such as might have been expected from the theory of ordinary sound-waves. There appears to be for each mixture a *crucial pressure*, above which variations of pressure do not affect the rate of explosion. Below this point the rates fall with diminution of pressure. In the case of electrolytic gas an increase of pressure from one to two atmospheres increases the rate of explosion from 2821 to 2872 metres per second. When the electrolytic gas is largely diluted, an increase in pressure from one to two atmospheres has less effect. The following rates were found at one and two atmospheres for electrolytic gas diluted with oxygen, nitrogen, and hydrogen respectively:—

TABLE XLI.

(1.) MIXTURE $H_2 + O_6$.

Pressure	1 At.	2 At.
Rate of explosion . . .	1696	1718

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* 'Combustion des Mélanges Gazeux,' p 259.

(2.) MIXTURE $H_2 + O + N_5$.

Pressure	1 At.	2 At.
Rate of explosion . . .	1815	1830

(3.) MIXTURE $H_8 + O$.

Pressure	1 At.	2 At.
Rate of explosion . . .	3532	3538

No doubt the numbers obtained point to a slight increase in the rate on doubling the pressure, but the differences are so small as to be confounded with the experimental error. I have not, therefore, given the higher rates as the true rates of the explosion, but have taken the mean of the two in each case.

The difference in the effect of pressure on electrolytic gas when pure, and when diluted, may be due to one of two causes. In the case of the pure gas increase of pressure may check dissociation, and thus increase the rate of explosion; whereas the diluted gas does not dissociate, and is consequently unaffected by changes of pressure. Or, more probably, since the diluted gas suffers less contraction in the chemical change, its crucial pressure is lower than that of the pure gas, and the change from one to two atmospheres being wholly, or in large part, *above* the crucial pressure, the effect on the rate is nil or very slight. In all the mixtures I have tried the rate of the explosion-wave is diminished by a sufficient reduction of pressure, even when the products of combustion are carbonic oxide and nitrogen alone. Changes of pressure appear, therefore, to affect the rate apart from any dissociation; and consequently may influence the rate of explosion of electrolytic gas independently of any effect on dissociation. Why changes of pressure affect the rate of explosion has yet received no satisfactory explanation.

As the initial pressure of a gaseous mixture is reduced, it is found that the flame travels further from the ignition-point before its rate becomes constant; it requires a longer run to attain its maximum velocity. By sufficiently reducing the pressure it is possible to stop the explosion of gases altogether; but the flame is still propagated slowly at pressures below that at which the explosion-wave dies out. For instance, with electrolytic gas, the explosion-wave is propagated at 200 mm. pressure, but not at 150 mm. At 200 mm. pressure the flame travels about 2 metres before attaining its maximum rate. According to my experiments* the flame is still propagated in electrolytic gas at 75 mm., but not at 70 mm., when an induction spark is passed through the mixture. LOTHAR MEYER and SEUBERT[*] found that a spark from a RUHMKORFF's coil caused partial ignition of electrolytic gas at 70 mm., and complete combustion at 72 mm., the exact pressure under which the gases unite varying with the strength of the spark.

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* 'Phil. Trans.,' 1884, p. 634.

It is conceivable, on the "breaking-wave" hypothesis, that the projected jets of heated gas do not under reduced pressure initiate their own explosion-waves so rapidly as under higher pressure.

Similarly with regard to changes of initial temperature, I do not know why a rise of temperature should diminish the rate of explosion. The limited range in which it is possible to conduct the experiments, and the small differences found within this range, do not permit of a curve being drawn which might indicate the general effect of changes of temperature upon the rate. In the case of electrolytic gas the effect of heating was found to lower the rate both at low and at high pressures. Experiments made at atmospheric pressure with ethylene, burning to carbonic oxide and steam, at 10° and 100°, showed a lowering of the rate rather greater than that of electrolytic gas; while similar experiments made with cyanogen, burning to carbonic oxide and nitrogen, showed a smaller difference—hardly, perhaps, to be distinguished from the experimental error.

§ 3. *The Rate of Explosion of Hydrocarbons with insufficient Oxygen.*

The explosion of hydrocarbons with insufficient oxygen has been studied by DALTON, J. DAVY, KERSTEN, and E. VON MEYER. In the 'New System of Chemical Philosophy,' Part II., p. 442 (1810), DALTON thus describes the explosion of equal volumes of ethylene and oxygen:—

"If 100 measures of oxygen be put to 100 of olefiant gas, and electrified, an explosion ensues, not very violent; but instead of a diminution, as usual, there is a great increase of gas; instead of 200 measures, there will be found about 360; some traces of carbonic acid are commonly observed, which disappear on passing two or three times through lime water; there will then remain perhaps 350 measures of a permanent gas, which is all combustible, yielding by an additional dose of oxygen, carbonic acid, and water, the same as if entirely burnt in the first instance. What, therefore, is this new gas in the intermediate state? The answer is clear. It is carbonic oxide and hydrogen mixed together, an equal number of atoms of each."

DALTON found that marsh gas could not be exploded with less than its own volume of oxygen, in which case half the hydrogen formed water and the other half was liberated:—

"If 100 measures of carburetted hydrogen be mixed with 100 measures of oxygen (the least that can be used with effect), and a spark passed through the mixture, there is an explosion without any material change of volume: after passing a few times through lime water, it is reduced a little, manifesting signs of carbonic acid. This residue is found to possess the characters of a mixture of equal volumes of carbonic oxide and hydrogen."

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[*] 'Chem. Soc. Journ.,' 1884, vol. 1. p. 587.

JOHN DAVY,[*] in 1832, described some experiments on the explosion of ethylene with insufficient oxygen, in which the volume was nearly doubled after detonation.

In 1861 KERSTEN[†] opposed the view, which he stated was common at that time, that in the combustion of hydrocarbons the hydrogen burns before the carbon. He describes experiments on the explosion of ethylene and of marsh gas with electrolytic gas, and with air. His conclusion is "that before a portion of hydrogen is burnt, all the carbon is burnt to carbonic oxide, and that then the excess of oxygen divides itself between the carbonic oxide and hydrogen."

In 1874, ERNST VON MEYER[‡] (who does not refer to the previous work on the subject) showed that equal volumes of ethylene and oxygen formed carbonic oxide and hydrogen when exploded in an eudiometer. Acetylene, diluted with hydrogen to diminish the force of the explosion, behaved in the same way.

On the other hand, in many of the text-books of the present day the doctrine is taught that in the combustion of hydrocarbons the hydrogen burns before the carbon.[§] BERTHELOT comes to the same conclusion from observations of the rate of explosion of ethylene mixed with hydrogen, and that of other hydrocarbons rich in hydrogen. Measuring the rate of explosion from the point of ignition through a space some 130 mm. long, BERTHELOT found that a mixture of hydrogen and ethylene burnt at a rate which was quicker than the mean of the rates of hydrogen and of ethylene when exploded with oxygen in the same apparatus:

Mixture.	Time taken by flame to travel 130 mm.	
$C_2H_4 + O_6$	2·86 thousandths of second	1·95 mean.
$H_2 + O$	1·04 ,, ,,	
$C_2H_4 + H_2 + O_7$ . .	1·37 ,, ,,	

He concludes from these experiments that the hydrogen burns before the ethylene in the mixture. It must again be pointed out that these velocities refer to the irregular period of vibration, and that everything depends on the quickness with which the explosion-wave is developed. Again, BERTHELOT compares the rate of explosion of hydrocarbons with that of hydrogen in the same apparatus :—

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[*] 'Edin. Journal of Science,' New Ser., vol. 6, p. 50.

[†] 'Journ. Prakt. Chem.,' vol. 84, p. 303.

[‡] 'Journ. Prakt. Chem.' [II.], vol. 10, p. 308.

[§] FARADAY, lecturing at the Royal Institution, said:—"The volatile matter raised by combustion from the tallow of a candle is a vapour composed of carbon and hydrogen; and our experiment teaches us that the forces which hold these elements together are so nicely balanced that the hydrogen is made to combine first, the carbon afterwards" ('Lectures at Royal Institution,' p. 280, J. SCOFFERN, 1853).

For a lucid discussion of this question, as far as it affects the liberation of carbon in hydrocarbon flames, see Professor A. SMITHELLS "On the Structure of Luminous Flames" ('Journ. Chem. Soc.,' 1892, vol. 1, p. 217).

Gas.	Time.
Acetylene	1·94
Ethylene	2·86
Ethane	·83
Marsh Gas	1·24
Hydrogen	1·04

"The rate of combustion of gases rich in hydrogen is very close to that of hydrogen. This seems to indicate that the hydrogen burns before the carbon, even in total combustions."

I do not think this conclusion is justified by the figures; nor do I see how the second argument can be reconciled with the first.

I have made experiments on the rate of the explosion-wave in ethylene and in acetylene mixed with their own volume of oxygen. In both cases the main reaction is the formation of carbonic oxide and hydrogen; but the rates found are not so near the calculated velocities as I have found in other cases. The explosions being made in a gas-tight tube surrounded with water, the increase in pressure gave the expansion due to the chemical changes. With ethylene and oxygen the volume was almost doubled; with acetylene and oxygen the volume was increased by one-half. In the following table I have given the observed rates of explosion for ethylene and acetylene, with an equal volume of oxygen, compared with the rate calculated by BERTHELOT's formula and with the sound-wave Σ :—

TABLE XLII.—Rates of Explosion of Ethylene and Acetylene with an equal volume of Oxygen.

1. $C_2H_4 + O_2 = 2CO + 2H_2$ ($Q = 67,000$ cals.)
2. $C_2H_2 + O_2 = 2CO + H_2$ ($Q = 112,700$ cals.)

Mixture.	$C_2H_4+O_2$	$C_2H_2+O_2$
BERTHELOT's θ	2134	2733
Rate of explosion	2507	2961
Σ	2330	3084

The rate of explosion of ethylene and an equal volume of oxygen was found to be

2505 metres per second as the mean of four double determinations. The rate of the sound-wave "Σ," calculated on the hypothesis that the mixture burns directly to carbonic oxide and hydrogen, is 2330. The calculated velocity is, therefore, about 7 per cent. too low. This is the only instance I have yet found where the observed rate is largely in excess of the calculated sound-wave. On the other hand, the rate of explosion of acetylene and an equal volume of oxygen was found to be 2961 metres per second as the mean of four double determinations. The rate of the sound-wave, calculated on the hypothesis that the mixture burns directly to carbonic oxide and hydrogen, is 3084. The calculated velocity is, therefore, about 4 per cent. too high. This difference between the explosion of ethylene and acetylene, with an equal volume of oxygen, is somewhat striking; it seems to point to some different mode of decomposition in the two cases.

As stated by DALTON, a mixture of marsh gas with half its volume of oxygen will not explode; but a mixture of equal volumes of marsh gas and oxygen explodes. I have determined the rate of explosion of this mixture, and find the velocity is 2½ per cent. faster than the sound-wave calculated for the reaction—

$$CH_4 + O_2 = CO + H_2 + H_2O.$$

TABLE XLIII.—Rate of Explosion of Marsh Gas with an equal volume of Oxygen.

Mixture.	$CH_4 + O_2$.
BERTHELOT'S θ	2184
Rate of explosion . .	2528
Σ	2466

At the temperature produced in this reaction the steam must be partly dissociated —lowering the rate. The observed velocity, therefore, is undoubtedly too high to reconcile with the calculated sound-wave. Marsh gas must, therefore, be classed with ethylene as giving exceptionally high rates.

At this point a difficulty must be pointed out which I have hitherto avoided. I have endeavoured to show that the explosion-wave is in part propagated by the movements of the yet unburnt molecules. The question arises—Are these unburnt molecules ever broken up by impact before they are burnt? In the case of an endothermic compound, such as cyanogen or acetylene, is the propagation of the flame partly due to the explosive decomposition of the compound, previous to the combustion of its constituents? If so, the unburnt gas would be diminished in density, and one might expect to find the explosion of endothermic compounds travelling at velocities greater than those calculated for sound in the undecomposed gas. But this is exactly the reverse of what is observed. The formula, which gives the correct velocity for explosions of cyanogen, gives numbers which are rather too high for acetylene (also a strongly endothermic compound), and too low for ethylene and for marsh gas (an exothermic compound). It would appear as if ethylene and marsh gas were unstable bodies, and acetylene and cyanogen were stable bodies, in the particular condition of the explosion-wave. Let me recall the words used by Professor OSTWALD in his address last year at Halle, to the sections of Physics and Chemistry, when speaking on a cognate subject—the spectra of gases at high temperatures :—

"It is held as an undoubted dogma that at the highest temperatures, as, for example, in the electric arc, all compounds must be dissociated into their elements. This view is certainly not justified. What we do know about the stability of compounds is, on the contrary, that all compounds which are formed with absorption of heat become *more stable* with rising temperature, and the reverse. Because the majority of the compounds known to us are formed from the elements with evolution of heat, and correspondingly become more unstable with rising temperature, the conclusion has been drawn that this is in general the case. But if we reflect that cyanogen and acetylene, two compounds formed with great absorption of energy, are readily formed in quantity, at the highest temperatures, in the blast furnace and in the arc light, we become conscious that spectra occurring at high temperatures may belong to compounds which have a fleeting existence confined to those temperatures only."*

We know, then, that cyanogen and acetylene are formed from their elements at the highest temperature; but BERTHELOT has shown that these gases are decomposed by the shock of a fulminate detonator.† Is it possible that there is an essential difference between the two acts—that molecules of acetylene, charged by highly heated molecules, do not break up, but that they are decomposed when they are dashed against cold molecules, or the cold walls of the containing vessel? I have found that the decomposition of acetylene by a fulminate is confined to the neighbourhood of the detonator. When a charge of fulminate is fired in a steel bomb filled with acetylene, the decomposition of the acetylene is not propagated along a tube attached to the bomb and filled with the gas. The flame of the detonator is "carried on" for some little distance, but *no explosion-wave is set up in the acetylene.* Similar results were found with carbon bisulphide vapour, though in this case the decomposition is propagated to a greater distance from the detonator. These experiments show that "the decomposition by shock of acetylene and carbon bisulphide is not propagated like the explosion-wave at a constant velocity as far as the gas extends, but the decomposition set up by the fulminate dies out at a distance from the detonator, depending on the nature of the gas, and also probably on the intensity of the initial shock and the cooling power of the walls."*

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* See 'Nature,' April 21, 1892.

† 'La Force des Matières Explosives,' vol. 1, p. 109. See also THORPE, 'Journ. Chem. Soc.,' 1889, p. 220.

On the other hand, there is some evidence of the breaking up of ethylene in the explosion-wave. When ethylene is exploded with an equal volume of oxygen some carbon is deposited, and an appreciable quantity of marsh gas is found in the products of combustion. It is difficult to account for this formation of marsh gas except by the decomposition of the ethylene, either directly or by partial oxidation :—

$$\text{(i.)}\ C_2H_4\ .\ \ .\ = C + CH_4.$$
$$\text{(ii.)}\ C_2H_4 + O = CO + CH_4.$$

When ethylene is exploded with less than its own volume of oxygen, the proportion of marsh gas found in the residue increases as the volume of oxygen is diminished.† This points to the formation of marsh gas as an intermediate product in the explosion of ethylene.

Again, it is a matter of frequent observation in the analysis of coal gas, when the residue of marsh gas, hydrogen, and carbonic oxide is exploded in an eudiometer, with a large excess of oxygen, that carbon is deposited towards the bottom of the tube. When, however, the mixture is sufficiently diluted with air, the marsh gas is completely burnt. This fact is easily explained on the assumption that in the explosion-wave (which is set up in marsh gas with oxygen but not with air) the marsh gas is more or less decomposed in the wave-front, and some of the liberated carbon escapes combustion by cooling.‡

I do not forget that the question raised concerning the breaking up of yet unburnt compounds in the explosion-wave, may also be asked concerning the elementary molecules. What happens, for instance, to the molecules of oxygen in the wave? When a molecule of oxygen reacts with a molecule of hydrogen, is an oxygen atom liberated, and, if so, how far does its existence influence the rate? It was with the object of obtaining some answer to these questions that experiments were made on the rate of explosion of hydrogen with nitrous oxide, in which reaction it is possible that no free atoms are liberated :—

$$H_2 + N_2O = H_2O + N_2.$$

So far as the experiments go, I cannot trace the influence of liberated atoms of oxygen.

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* DIXON and HARKER, "On the Decomposition by Shock of Endothermic Compounds." 'Manchester Memoirs' [IV.], vol. 5, 1892.

† LEAN and BONE, "The Behaviour of Ethylene on Explosion with less than its own Volume of Oxygen." 'Chem. Soc. Journ.,' 1892.

‡ Results which might be explained on the same hypothesis—viz., the decomposition of unburnt compounds in the wave-front, are afforded by the explosion of ammonia with nitrous oxide, and of chlorine with hydrogen. In both cases the rate is somewhat higher than might be expected.

§ 4. *Conclusion.*

The difficulties I have touched on show how far we are from a complete theory of explosions.

The tentative formula which I have proposed must be taken for nothing more than a "working hypothesis." It has been found so far *parallel to the truth* that it may be useful to suggest further experiments on the nature of explosions. As BERTHELOT's formula helped me to interpret the reactions of gaseous carbon, and suggested experiments which led me to modify the hypothesis I was using; so I will venture to hope that this modified hypothesis may in turn lead other workers to a truer knowledge of explosions.

In concluding, I wish to thank three former pupils of mine, who have rendered me devoted service in the conduct of these experiments—Mr. J. E. MARSH, of Balliol College, Oxford, and Messrs. H. W. SMITH and J. A. HARKER, of Owens College, Manchester.

GENERAL CONCLUSIONS.

1. BERTHELOT's measurements of the rates of explosion of a number of gaseous mixtures have been confirmed. The rate of the explosion-wave for each mixture is constant. It is independent of the diameter of the tube above a certain limit.

2. The rate is not absolutely independent of the initial temperature and pressure of the gases. With rise of temperature the rate falls; with rise of pressure the rate increases; but above a certain *crucial pressure* variations in pressure appear to have no effect.

3. In the explosion of carbonic oxide and oxygen in a long tube the presence of steam has a marked influence on the rate. From measurements of the rate of explosion with different quantities of steam the conclusion is drawn that at the high temperature of the explosion-wave, as well as in ordinary combustion, the oxidation of the carbonic oxide is effected by the interaction of the steam.

4. Inert gases are found to retard the explosion-wave according to their volume and density. Within wide limits an excess of one of the combustible gases has the same retarding effect as an inert gas (of the same volume and density) which can take no part in the reaction.

5. Measurements of the rate of explosion can be employed for determining the course of some chemical changes.

In the explosion of a volatile carbon-compound with oxygen, the gaseous carbon burns first to carbonic oxide, and afterwards, if oxygen is present in excess, the carbonic oxide first formed burns to carbonic acid.

6. The theory proposed by BERTHELOT—that in the explosion-wave the flame travels at the mean velocity of the products of combustion—although in agreement with the rates observed in a certain number of cases, does not account for the velocities found in other gaseous mixtures.

7. It seems probable that in the explosion-wave—
   (1) The gases are heated at *constant volume*, and not at *constant pressure;*
   (2) Each layer of gas is raised in temperature *before* being burnt;
   (3) The wave is propagated not only by the movements of the burnt molecules, but also by those of the heated but yet unburnt molecules;
   (4) When the permanent volume of the gases is changed in the chemical reaction an alteration of temperature is thereby caused which affects the velocity of the wave.

8. In a gas, of the mean density and temperature calculated on these assumptions, a sound-wave would travel at a velocity which nearly agrees with the observed rate of explosion in those cases where the products of combustion are perfect gases.

9. With mixtures in which steam is formed, the rate of explosion falls below the calculated rate of the sound-wave. But when such mixtures are largely diluted with an inert gas, the calculated and found velocities coincide. It seems reasonable to suppose that at the higher temperatures the lowering of the rate of explosion is brought about by the dissociation of the steam, or by an increase in its specific heat, or by both these causes.

10. The propagation of the explosion-wave in gases must be accompanied by a very high pressure lasting for a very short time. The experiments of MM. MALLARD and LE CHATELIER, as well as my own, show the presence of these fugitive pressures. It is possible that data for calculating the pressures produced may be derived from a knowledge of the densities of the unburnt gases and of their rates of explosion.

APPENDIX.

Received July 22, 1892.

Part I.—*The Preparation of the Gases.*

Hydrogen.—The hydrogen used in most of the experiments was prepared by the action of pure dilute sulphuric acid on re-distilled zinc, free from arsenic. The gas was washed with potash and silver nitrate solutions, and was then passed through a scrubbing tower and a series of bulbs filled with a strong solution of alkaline permanganate. The gas so prepared had only the faintest odour.

For a few experiments the gas was prepared by the electrolysis of pure dilute sulphuric acid with platinum electrodes.

Marsh Gas.—The marsh gas was prepared by GLADSTONE and TRIBE'S method,* in which pure methyl iodide is decomposed by the zinc-copper couple in presence of absolute alcohol. The "scrubber," packed with the zinc-copper couple, was kept moistened by alcohol dropping from a tap-funnel, and the gas was further washed with strong alcoholic potash and finally with oil of vitriol. The generating flask (of 2 litres capacity) was kept at 30° C.

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* 'Chem. Soc. Journ.,' 1884, p. 1.

An analysis of the gas by explosion with air in McLeod's gas apparatus showed 99·3 per cent. of marsh gas.

Ethylene.—The ethylene was prepared by the method given by Erlenmeyer and Bunte. 25 grms. of absolute alcohol are mixed with 150 grms. of pure sulphuric acid, and the mixture is heated in a large flask, on a sand bath, to 160°–170° C. A mixture of one part alcohol and two parts sulphuric acid is then slowly added by a tap-funnel. Great difficulty was experienced in getting rid of all acid gases from the ethylene. The gas was passed (1) through two wash-bottles containing oil of vitriol, (2) through a tower packed with pieces of solid potash, (3) through two wash-bottles containing strong potash solution, (4) through a second tower packed with pieces of solid potash. The large quantity of air contained in the flask and washing apparatus rendered it difficult to obtain the gas quite free from air. The gas was analysed and found to be free from sulphurous acid, but it contained small quantities of nitrogen. Allowance was made for the presence of this diluent gas in making up the mixtures of ethylene with oxygen, &c. The percentage impurity in the ethylene could be reduced to 1 per cent.; no other gas used in these experiments presented the same difficulty in making pure.

Acetylene. —The acetylene was prepared by the action of strong alcoholic potash or ethylene dibromide. The potash was dissolved in amyl alcohol. The gas evolved from the boiling solution was passed through a second flask containing boiling potash solution, and then into an ammoniacal solution of cuprous chloride. The precipitated cuprous acetylide was washed with dilute ammonia solution and kept till required under this solution. On warming the red precipitate with strong hydrochloric acid to 50° C. a regular stream of acetylene was given off. It was washed with water, dried with caustic potash, and collected over mercury. Analysis showed that the gas was pure.

Ammonia.—The ammonia was distilled from a strong aqueous solution, was dried by caustic potash, and was collected over mercury.

Carbonic Oxide.—This gas was prepared by gently warming formic acid with pure sulphuric acid diluted with an equal volume of water. It was washed with potash solution.

Cyanogen.—Re-crystallised mercuric cyanide was powdered and carefully heated to drive off moisture. The dry powder was introduced into a steel retort with a long neck. After a considerable quantity of cyanogen had been allowed to escape, the gas was collected directly in a mercury holder. The retort was raised to dull redness in a fireclay furnace, heated by gas.

Oxygen.—The oxygen was prepared by heating powdered chlorate of potash in a hard glass flask. It was washed by a solution of caustic potash.

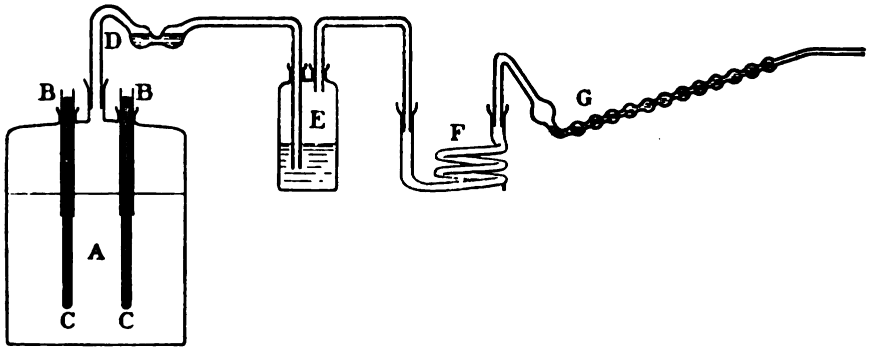
Nitrous Oxide.—This gas was prepared by heating pure ammonium nitrate in a hard glass flask. It was washed with a solution of ferrous sulphate, dried by sulphuric acid, and collected over mercury.

Nitric Oxide.—The nitric oxide was made by mixing eight parts of crystallised ferrous sulphate with one part of crystallised nitre, and pouring on to this mixture sulphuric acid diluted with three volumes of water. If the acid is warm the action starts at once, afterwards the flask is gently warmed.

Nitrogen.—The nitrogen was prepared by Vernon Harcourt's method, in which a mixture of air and ammonia is passed through a heated tube containing copper and copper oxide. The nitrogen issuing from the tube was washed with sulphuric acid.

Chlorine.—The chlorine employed in determining the rates of explosion of hydrogen and chlorine was prepared by the electrolysis of aqueous hydrochloric acid, and was passed, mixed with hydrogen, into the explosion-tube without standing in a holder. The gas was washed first with water, and then with sulphuric acid, and was dried in a glass worm and a series of bulbs filled with sulphuric acid. The electrolytic cell was made entirely of glass, and the washing bottles and drying vessels were all fitted together with ground glass joints with cups.

Fig. 1.



Apparatus for preparing chlorine and hydrogen.

The electrolytic vessel A was made with three tubulures, into which ground glass tubes were fitted. Two of these, BB, passed into the vessel, and dipped about an inch below the level of the acid. Through these tubes passed the gas-carbon electrodes CC, the small annular space between the rods and tubes being made tight with paraffin. A little mercury poured on top of the carbon rods served to connect them with the battery wires. Into the third tubulure was ground a gas delivery tube D, containing washing bulbs filled with water, and passing through a tubulure in the sulphuric acid wash-bottle E. From E the gas passed through the worm F and through the bulbs G to a long flexible glass tube leading into the explosion-tube.

For mixtures containing an excess of hydrogen, the wash-bottle E was replaced by a vessel fitted with three tubulures similar in shape to A. Two of the tubulures were fitted with tubes of the same size dipping beneath the acid, the third tubulure serving for the exit tube. The electrolytic gas entered through one tube, while hydrogen was driven from a holder through the other.

At the further end of the explosion tube the gases passed through (1) a wash bottle; (2) a long graduated tube fitted with taps at either end; (3) a thin glass bulb; (4) a tower filled with lime and powdered charcoal. The thin bulb could be readily detached and submitted to a "magnesium flash." The graduated tube served to analyse the gas. It was connected with a movable reservoir from which a solution of caustic potash could be introduced into the tube. The volume of chlorine present could thus be readily estimated, the residual hydrogen being measured under atmospheric pressure by bringing the level of the liquid in the tube and reservoir to the same height.

The current for the electrolysis was given by 7 secondary cells (ELWELL and PARKER), E.M.F. 13–15 volts; current 4–6 amperes. The preliminary saturation occupied many hours. It was found best to leave a small current (about ½ ampere) passing through the cell for two days before making an experiment. When the liquids were all saturated it took about four hours to fill the explosion tube. An analysis of the issuing gas at the end of that time gave—

Chlorine	49·82
Hydrogen	50·18

In another experiment an analysis at the end of 4½ hours gave—

Chlorine	49·90
Hydrogen	50·10

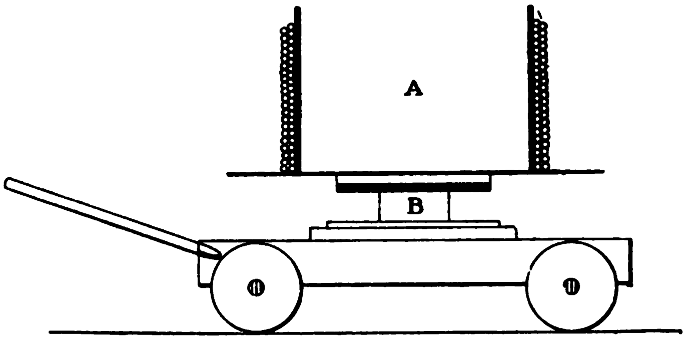
Part II.—*Arrangement of the Firing Tubes.*

The leaden pipes used for the explosion tubes were 5, 6½, 9, and 13 mm. in internal diameter. The pipe was wound on a galvanised iron drum about 2 feet in diameter, provided with a flange at the bottom.

In order to wind the pipe it was first laid straight along a corridor. The drum A was attached to a turntable B fixed to a low trolley, which was pushed forward as the pipe was wound on the revolving drum.

The drum with the pipe wound on it was placed in an iron water-bath, which could be heated by a large ring burner. The ends of the pipe passed out through two openings near the top of the bath. To each end of the explosion tube was soldered a short steel tube, ending in a cap with a polished steel face 30 mm. in diameter, perforated in the centre by a hole 10 mm. in diameter. The steel face of each cap is pressed by a clamp against a similar steel face, as in REGNAULT's gas apparatus, from which these joints are taken. One end of the explosion tube is thus joined to the "firing piece," the other end to a shorter tube, called the "end piece," carrying the second "bridge."

Fig. 2.

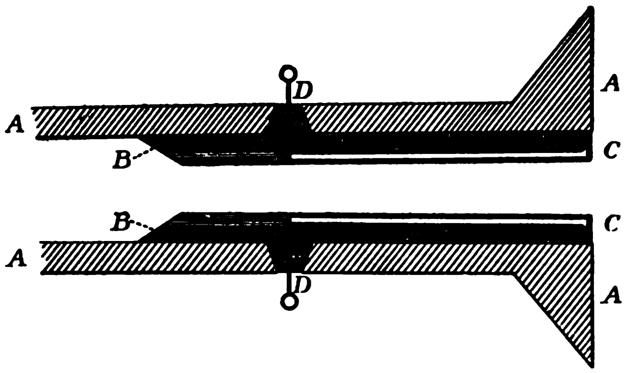


Section of drum mounted on turntable, and trolley for winding the explosion tube.

The caps which carried the bridges to be broken by the flame were constructed in the following way:—

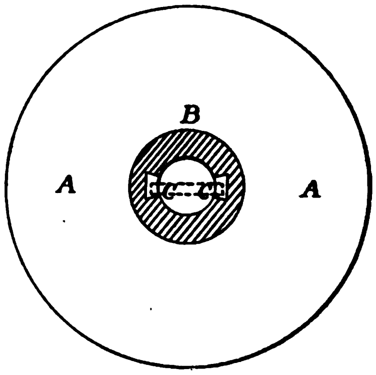
The steel tube AA was pierced at a distance of 30 mm. from the face by two conical holes opposite each other. The interior of the tube was lined with a cylinder of vulcanite, which also filled the conical holes. Two thick platinum wires, DD, passed through the holes, and were soldered to two pieces of silver, CC, embedded in the vulcanite, and thus insulated from the steel tube. The silver pieces were broadened out at the face end into two small plates, on to which the narrow strip of silver foil forming the bridge was soldered. The silver plates were brought not quite flush with the polished steel face, so that there was no danger of breaking the bridge when clamping up the two caps.

Fig. 3.



Section of bridge-piece.

Fig. 4.

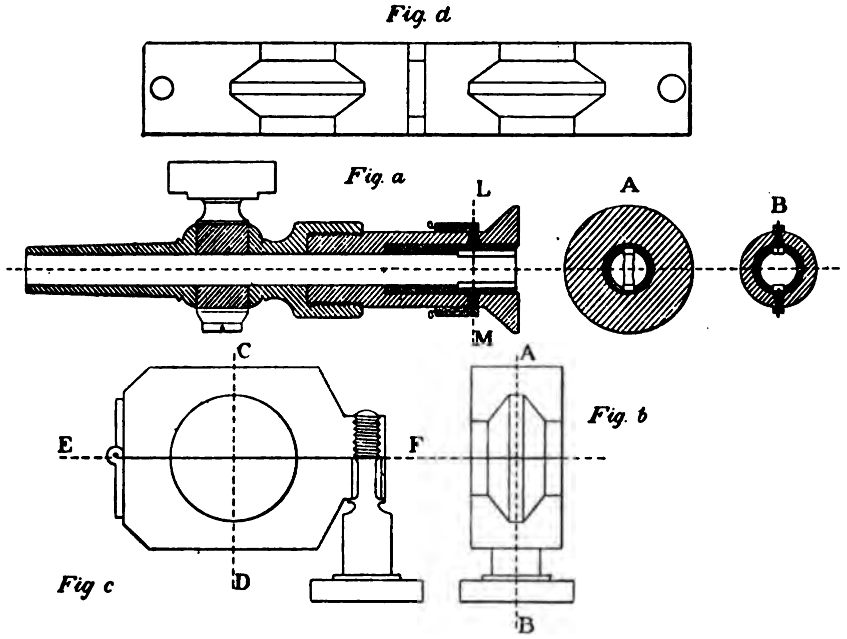


Front view.

For the experiments with chlorine and hydrogen a glass tube was fixed inside the vulcanite so as to protect it from the action of the chlorine. The silver bridge, after being soldered to the silver plates, was warmed and coated with a layer of paraffin as a protection from the chlorine.

The apparatus being arranged as shown in fig. 7, the explosive mixture from the holder was driven through the washing vessel A, partly filled with boiled oil of vitriol, and through the two drying towers B, B, packed with pieces of asbestos saturated with boiled oil of vitriol. The gas then passed through a short length of black rubber tubing to the end piece C carrying the second bridge. From the end piece the mixture passed into the explosion tube coiled on a drum standing in the large iron water-bath D. A second clamp united the explosion tube to the firing piece E, carrying the first bridge at one end, and the firing wires *f*, *f*, sealed through a stout glass tube, and a tap G at the other. The taps were made of hardened steel. After leaving the firing piece the gas passed by a glass tube M to a three-way tap H, by which it could be sent either to the testing tube I or by the pipe K to the window. In many of the experiments a sulphuric acid wash-bottle L was joined to the tube at M, and connected with a holder containing electrolytic gas. Before starting the experiment the air was displaced from the vessel L by a stream of electrolytic gas. When the explosion tube and firing piece were full of the explosive mixture, the tap H was shut, and a few bubbles of electrolytic gas were driven into the firing piece so as to fill about one-third of its length. This insured the explosion-wave being set up in the firing piece before the flame reached the first bridge, and was a convenient mode of firing mixtures—such as those containing cyanogen—which do not readily ignite.

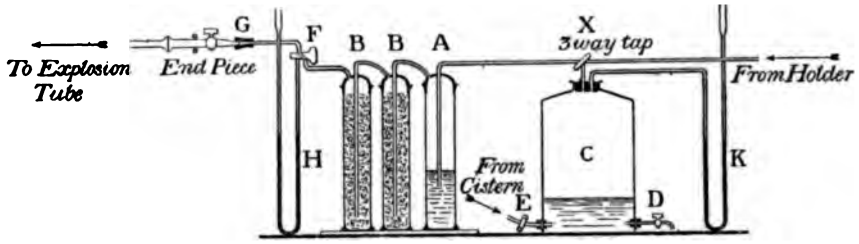
Fig. 5.



Shows the construction of a second pair of bridge pieces and clamps used during the last two years.

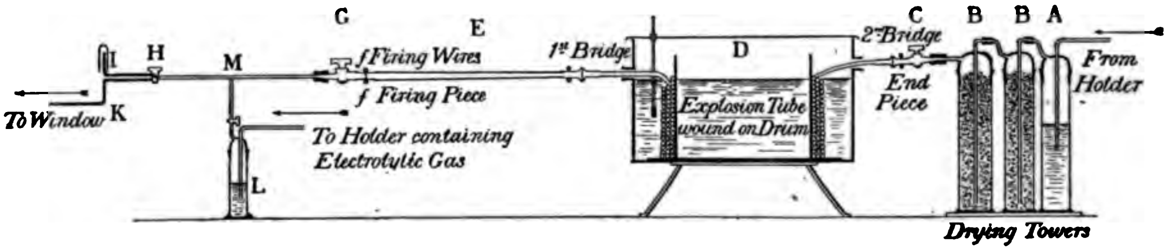
For experiments under increased pressure the arrangement shown in fig. 6 was adopted. The strong glass cylinder C, filled with water from a high cistern through the tube E, or (when necessary) with mercury from a second movable cylinder, was filled with the mixture from the holder by allowing the liquid to run out at the tap D. After the cylinder was filled, the three-way tap X was turned so as to send the gas from the holder through the drying vessels A, B, B, and through the three-way tap F, into the explosion tube. When the explosion tube was full, the tap X was turned so as to put it in communication with the cylinder C. The tap at the further end of the explosion tube being closed, water or mercury was admitted to C, forcing the gas into the explosion tube through the drying vessels. When the manometer K showed the desired pressure had been reached, the taps were closed and the connections made with the chronograph and firing wires. Just before firing, the taps were opened to the manometer K and the pressure in the explosion tube was read off. The taps were then shut and the mixture fired.

Fig. 6.



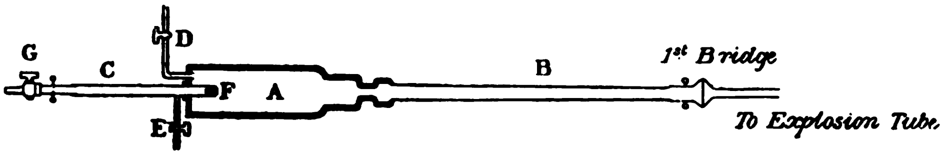
For experiments under reduced pressure the explosion tube was filled in the usual way. The gas was then sucked from the apparatus by a pump attached to the tube K (Fig. 7), a manometer sealed in at M (in place of the wash bottle L) giving the pressure in the explosion tube.

Fig. 7.



For firing the mixture of cyanogen with nitric oxide the following arrangement was adopted. The firing piece B was firmly screwed to a stout steel cylinder A. Through the other end of this cylinder a tube C was screwed so as to project 2 inches into the cylinder. Into the end of this tube a copper detonator holding fulminate of mercury was attached, by a caoutchouc stopper with a large hole, the open end of the detonator pointing inwards. The explosive mixture passed from the explosion tube through the firing piece and cylinder, escaping at the tube D which was also screwed into the base of the cylinder. The tube C was then filled with electrolytic gas through the tap G, the air escaping at E. On firing the gas in C by the firing wires the explosion travelled to F, and there set off the detonator in the middle of the cyanogen mixture. The explosion thus started was propagated through the tube B, into the explosion tube which it traversed to the end.

Fig. 8.



Part III.—*Chronographic Apparatus.*

The arrangement of the chronograph and electrical connections are shown in fig. 9. The chronograph, made by Messrs. Elliot, consisted of a heavy pendulum carrying a smoked glass plate. The electro-magnetic styli used were made by Pfeil, of Berlin, and by Mr. F. J. Smith,* of Trinity College, Oxford, to whom I am greatly indebted for several of these instruments. In most of the experiments recorded in this paper, one of Pfeil's instruments and one of Mr. Smith's were employed. These two were chosen out of several, for the great constancy of their latent period, and for the sharpness of their action. The styli were attached to an upright bar on a heavy iron stand, with three pointed feet. The bar could be tilted back on two feet and exactly replaced.

At the lowest point of its swing the pendulum acted on two breaks, X and X′, which by a rack and pinion could be brought exactly in a line. By bringing the pendulum slowly against the uprights the position of the two breaks could be adjusted so that they were broken at the same instant.

In the preliminary experiment the wires were arranged as shown in the figure, except that the wires to the firing battery were detached at C′C′, and the wires from *bb*, *b′b′* to the two bridges were not connected. The stylus S was put in circuit through a resistance RR, and the break X; the stylus S′ was put in circuit through a resistance R′R′ and the break X′. On letting the pendulum fall the two styli were released, and registered their marks on the moving plate. The resistances RR and R′R′ were now removed, the firing battery and coil put into circuit at C′C′, the wires from *bb* and *b′b′* attached to the two bridges, and the wires from the coil attached to the firing piece. The pendulum, being again let fall, broke the primary circuit of the coil on striking X′, causing a spark to pass between the platinum wires in the firing piece. The flame travelling down the tube broke both bridges in turn, and the styli connected with them registered their marks on the moving plate. The connections were immediately adjusted as in the preliminary experiment, and the pendulum was again let fall. If the "confirmatory" marks passed directly over the "preliminary" it was a proof that nothing had been displaced during the experiment.*

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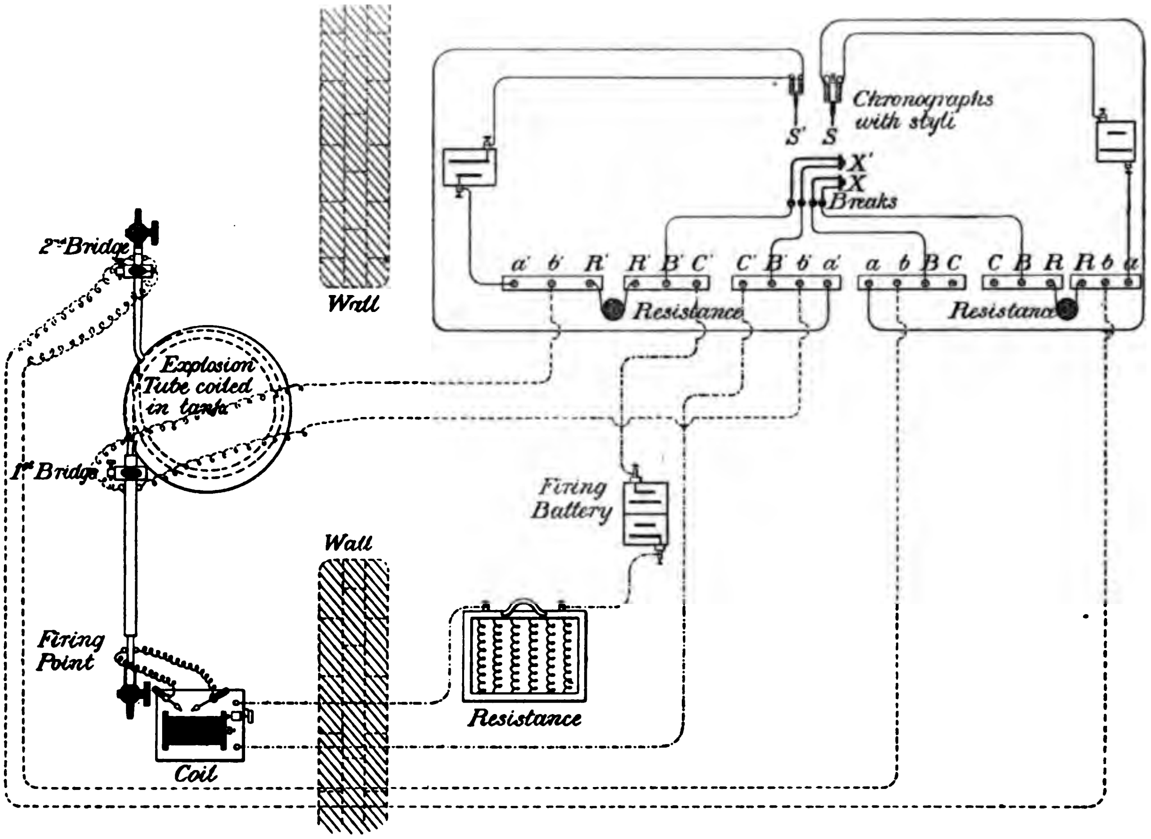
* 'Phil. Mag.,' 1890, p. 377.

Fig. 9.



In the companion experiment, which was made as soon as possible after the first, the wires from the break X were removed from BB to B′B′, and the wires from X′ from B′B′ to BB. X was now the firing break. Any difference between the two breaks which might affect the moment of firing was thus eliminated by taking the mean of the pair of experiments.

The breaks were constructed as shown in fig. 10. The contact was made between a platinum plate A and a platinum point B joined to a copper strip. The platinum plate was fastened to a piece of ebonite turning on a horizontal axis; a spiral spring pressed the plate against the platinum point until the arm C of the pendulum struck the copper plate D attached to the ebonite. The whole piece could be moved to the right or left by a pinion working in the rack E.

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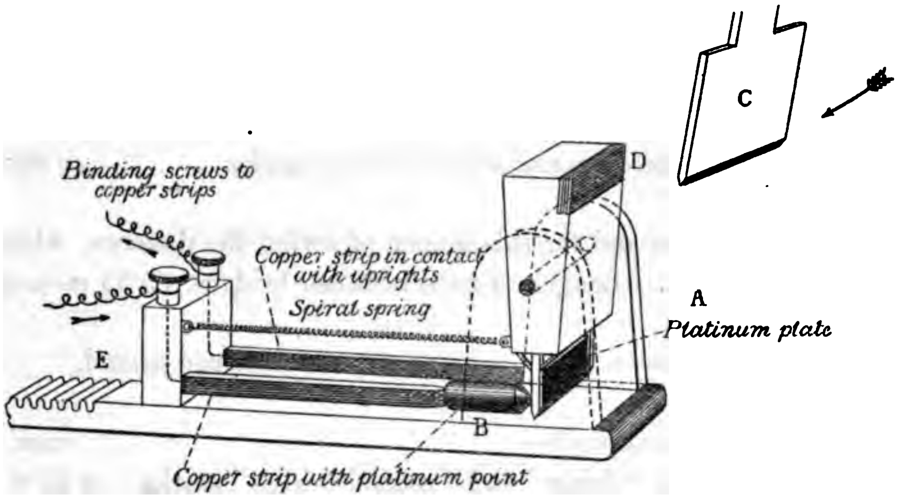
* Mr. F. J. SMITH has pointed out that the difference in the latencies of two electro-magnetic styli may be eliminated by making two experiments, and reversing the connections for the second experiment. I think my plan of making a "preliminary experiment" (without firing the gases) is quicker. In both cases it is assumed that the latencies remain constant during the experiment: the quicker the operation the more likely is this condition to be attained.

Fig. 10.


C
Binding screws to copper strips
D
Copper strip in contact with uprights
Spiral spring
A
Platinum plate
E
B
Copper strip with platinum point

Side view of one of the breaks.

The tuning fork was by Koenig. It gave 100 complete vibrations per second at 15° C. Its temperature correction was determined by Mr. R. H. M. Bosanquet in the Laboratory of St. John's College, Oxford. By measuring the length of the wave traced by the tuning fork on the plate at different heights, a table of wave-lengths was constructed, giving the value of $\frac{1}{100}$ second at different heights on the plate, and for different lengths of swing. After measuring the distance between two marks of a stylus and the height of the tracing on the plate, a reference to the table gave the value of the distance in time. But, to make sure that the pendulum was moving at the same rate as when the table was constructed, a tuning fork trace was taken after each experiment—everything being in the same state except that the tuning fork was substituted for the electro-magnetic styli. From the length of the observed trace, compared with that given in the table, a correction could readily be applied to the table readings. A small correction was also made, when necessary, for the effect of temperature on the tuning fork.

The distance between the marks of the styli was measured by cutting the rising trace 2 mm. above the datum line with a pointed style, and drawing a perpendicular line through the points of intersection, as shown in fig. 11. At *a* the upper stylus began to rise in the preliminary experiment, and the lower stylus at *b*. In the explosion the lower stylus was released by the breaking of the first bridge, and began to rise at *b′*; the upper stylus was released by the breaking of the second bridge, and began to rise at *a′*. The distances between the lines AA′ and between the lines BB′ being measured, as well as the position of *aa′* and *bb′* on the plate, the value of BB′

(corrected for position) is subtracted from AA′, the remainder giving the time of explosion.

I take one example from my note-book of the actual measurements obtained in a pair of experiments :—

mm.
A.—Distance between "firing" and "second bridge" marks = 70·5
Distance between "firing" and "first bridge" marks . = 4·7

Distance between first and second bridge marks . . . = 65·8

At this point on the plate and for this length of swing the distance, which is equal to $\frac{1}{100}$ second, is 24·62 mm. Length of tube between bridges, 75·35 metres.

Rate of explosion . . . . = 2820 metres per second.

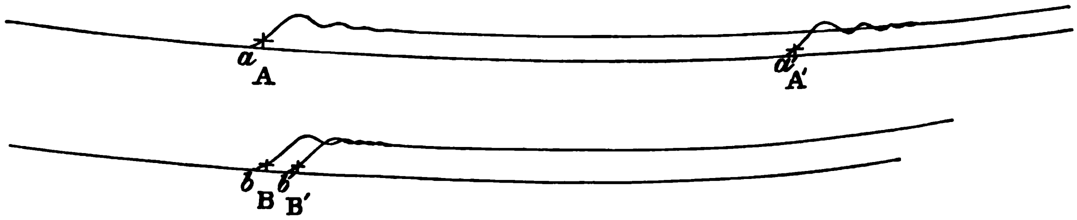
B. (with connections reversed)— mm.
Distance between "firing" and "second bridge" marks = 68·6
Distance between "firing" and "first bridge" marks . = 2·4

Distance between first and second bridge marks . . . = 66·2

At this point on the plate and for this length of swing, the distance, which is equal to $\frac{1}{100}$ second, is 24·70 mm.

Rate of explosion . . . . = 2811 metres per second.
Mean rate . . . . = 2815 „ „

Fig. 11.



Traces made by electro-magnetic styli on the moving plate.

Part IV.—*The Rates of Explosion as Measured.*

(1.) HYDROGEN and Oxygen under Ordinary Temperature and Pressure.

The rate of explosion of the mixture $H_2 + O$ was measured in a tube 9 mm. in diameter and 55 metres long.

Rates Found.

SERIES A.
Metres per sec. 2804 2812 2835
Mean 2817

In a tube of same diameter and 100 metres long :—

SERIES B.
2822 2847 2795
2821

In a tube 13 mm. diameter and 100 metres long :—

SERIES C.
2805 2840 2831 2800
2819

In 1888–90 these experiments were repeated. The numbers which follow are all the means of pairs of measurements with the connections of the chronograph reversed:—

Series D.
2818 2820 2831 2828
2824

Series E.	Series F.	Series G.
In 9 mm. tube.	In 6½ mm. tube.	In 6½ mm. tube.
2852 2814 2825 2825 2830	2795 2781 2814 2807	2827 2840 2803 2836
2828	2799	2826

Most of the experiments given in these three series, E, F, and G, were *control* experiments interpolated in other series to check the working of the whole apparatus.

In the following series the mixture was fired alternately in a straight tube and in the same tube coiled on a drum. Each number is the mean of a pair of measurements with connections reversed:—

Series H.	
9 mm. tube 100 metres long.	
2828	straight
2829	coiled
2820	straight
2812	coiled
2818	straight
2821	

(2.) HYDROGEN and Oxygen under Varying Pressure.

1. At 10° C.				
200 mm.	300 mm.	500 mm.	1100 mm.	1500 mm.
2636 2619 2626	2716 2693 2680 2713 2723	2784 2764 2777	2869 2850 2864 2844 2852 2845	2869 2871 2873 2883 2861 2873
2627	2705	2775	2856	2872

2. At 100° C.				
390 mm.	500 mm.	760 mm.	1000 mm.	1450 mm.
2704 2689	2730 2746	2784 2793 2799 2796 2789 2800 2782 2774	2833 2825 2824 2832	2831 2837 2849 2849
2697	2738	2790	2828	2842

(3.) HYDROGEN and Oxygen in Different Proportions, under Ordinary Temperature and Pressure.

In 13 mm. tube.	In 9 mm. tube.				In 13 mm. tube.
H_8+O.	H_6+O.	H_4+O.	H_2+O_2.	H_2+O_4.	H_2+O_6.
3547 3518 3537 3526	3517 3526 3538 3527	3265 3281 3261 3266	2315 2331 2344 2322	1945 1905 1943 1914	1703 1728 1708 1691
3532	3527	3268	2328	1927	1707

(4.) HYDROGEN and Oxygen in Different Proportions under 1500 mm.

$H_8 + O$	$H_2 + O_6$
3534 3542	1723 1713
3538	1718

(5.) HYDROGEN and Oxygen Diluted with Nitrogen.

At 760 mm. and 10°.		
In 9 mm. tube.		In 13 mm. tube.
H_2+O+N	H_2+O+N_3	H_2+O+N_5
2429 2408 2443 2425	2065 2045 2065 2063 2035	1822 1805 1809 1824
2426	2055	1815

Under 1500 mm. and 10°.
H_2+O+N_5
1821 1838
1830

MARSH Gas and Oxygen in Varying Proportions.

At 760 mm. and 10°.				
CH_4+O_3	CH_4+O_3	CH_4+O_4	CH_4+O_6	CH_4+O_8
2531 2510 2552 2520	2479 2469 2463	2322 2318 2326	2147 2145	1954 1972
2528	2470	2322	2146	1963

MARSH Gas and Oxygen diluted with Nitrogen.

At 760 mm. and 10°.		
CH_4+O_3+N	$CH_4+O_3+N_3$	$CH_4+O_3+N_5$
2349 2364 2348 2334	2146 2162 2155	1872 1888
2349	2154	1880

MARSH Gas and Oxygen under Varying Pressures.

CH_4+O_3		CH_4+O_4	
500 mm.	1000 mm.	500 mm.	1000 mm.
2414 2422	2480 2496	2273 2287	2324 2314
2413	2488	2280	2319

ETHYLENE and Oxygen in Different Proportions.

At 760 mm. and 10° C.						
$C_2H_4+O_2$	$C_2H_4+2O_2$	$C_2H_4+3O_2$	$C_2H_4+4O_2$	$C_2H_4+6O_2$	$C_2H_4+8O_2$	$C_2H_4+10O_2$
2516 2527 2472 2515	2606 2578 2562 2571 2586	2364 2359 2380 2369	2239 2255	2131 2105	1974 1986	1852 1860
2507	2581	2368	2247	2118	1980	1856

ETHYLENE and Oxygen Diluted with Nitrogen.

$C_2H_4+2O_2+N_2$	$C_2H_4+2O_2+2N_2$	$C_2H_4+2O_2+4N_2$	$C_2H_4+2O_2+6N_2$	$C_2H_4+2O_2+8N_2$
2430 2401 2408	2202 2220	2009 2039	1881 1874	1737 1731
2413	2211	2024	1878	1734

ETHYLENE and Oxygen.

At 760 mm. and 100° C.
$C_2H_4+2O_2$
2541 2532 2532 2548
2538

ACETYLENE and Oxygen in Different Proportions.

At 760 mm. and 10° C.		
$C_2H_2+O_2$	$C_2H_2+O_3$	$C_2H_2+O_5$
2961	2698	2389
2950	2747	2399
2971	2691	2382
	2703	
	2722	
	2733	
2961	2716	2391

ACETYLENE and Oxygen Diluted with Nitrogen.

$C_2H_2+O_3+N_2$	$C_2H_2+O_3+N_6$	$C_2H_2+O_3+N_8$	$C_2H_2+O_3+N_{10}$	$C_2H_2+O_3+N_{12}$
2410	2197	2110	2028	1903
2418	2221	2122	2009	1913
2415				
2414	2209	2116	2019	1908

ACETYLENE and Oxygen under Varying Pressures.

At 500 mm.
2685
2693
2689

CYANOGEN and Oxygen in Varying Proportions.

At 760 mm. and 10°.		
$C_2N_2+O_2$	$C_2N_2+2O_2$	$C_2N_2+3O_2$
2714	2319	2108
2742	2318	2112
2727	2326	
2728	2321	2110

CYANOGEN and Oxygen Diluted with Nitrogen.

$C_2N_2+O_2+N_2$	$C_2N_2+O_2+2N_2$
2394 2401	2156 2175
2398	2166

CYANOGEN and Oxygen under Varying Pressures.

In 5 mm. tube. 10° C.		
$C_2N_2+O_2$		
At 500 mm.	At 760 mm.	At 1000 mm.
2533 2539	2663 2686	2667 2675
2536	2677	2671

At 760 mm. and 100° C.
In 9 mm. tube.
2697 2725
2711

CYANOGEN and Nitrous Oxide.

In 9 mm. tube.		
$C_2N_2+2N_2O$	$C_2N_2+2N_2O+N_2$	$C_2N_2+2N_2O+2N_2$
2451 2458	2273 2292	2101 2095
2454	2283	2098

CYANOGEN and Nitric Oxide.

In 9 mm. tube. (Fired with fulminate detonator.)
C_2N_2+2NO
2766 2740 2773
2760

HYDROGEN and Nitrous Oxide in Varying Proportions.

At 760 mm. and 10° C.			
In 13 mm. tube.	In 9 mm. tube.		
H_7+N_2O	H_6+N_2O	H_4+N_2O	H_3+N_2O
2728 2737	2700 2711	2552 2538	2307 2302
2732	2705	2545	2305

HYDROGEN and Nitrous Oxide Diluted with Nitrogen.

In 9 mm. tube.	In 13 mm. tube.	
$H_3+N_2O+N_2$	$H_3+N_2O+N_3$	$H_3+N_2O+N_4$
2001 1980	1886 1875	1650 1416 went out ditto
1991	1880	irregular

HYDROGEN with Nitrous Oxide under Varying Pressures.

H_2+N_2O	
At 500 mm.	At 1000 mm.
2089 2098	2306 2298
2094	2302

AMMONIA and Oxygen.

At 760 mm. and 10° C.
$4NH_3 + 3O_2$.
2381 2399
2390

AMMONIA and Nitrous Oxide.

$2NH_3 + 3N_2O$.
2199 2202
2200

CARBONIC Oxide and Oxygen with Varying Proportions of Steam.

CO + O at 760 mm.				
Well dried.	Dried.	Saturated at 10°.	Saturated at 20°.	Saturated at 28°.
1267 1262	1289 1297 1320 1316	1670 1673 1679 1674 1687 1675	1710 1704 1685 1718 1700 1701	1711 1726 1702
1264	1305	1676	1703	1713

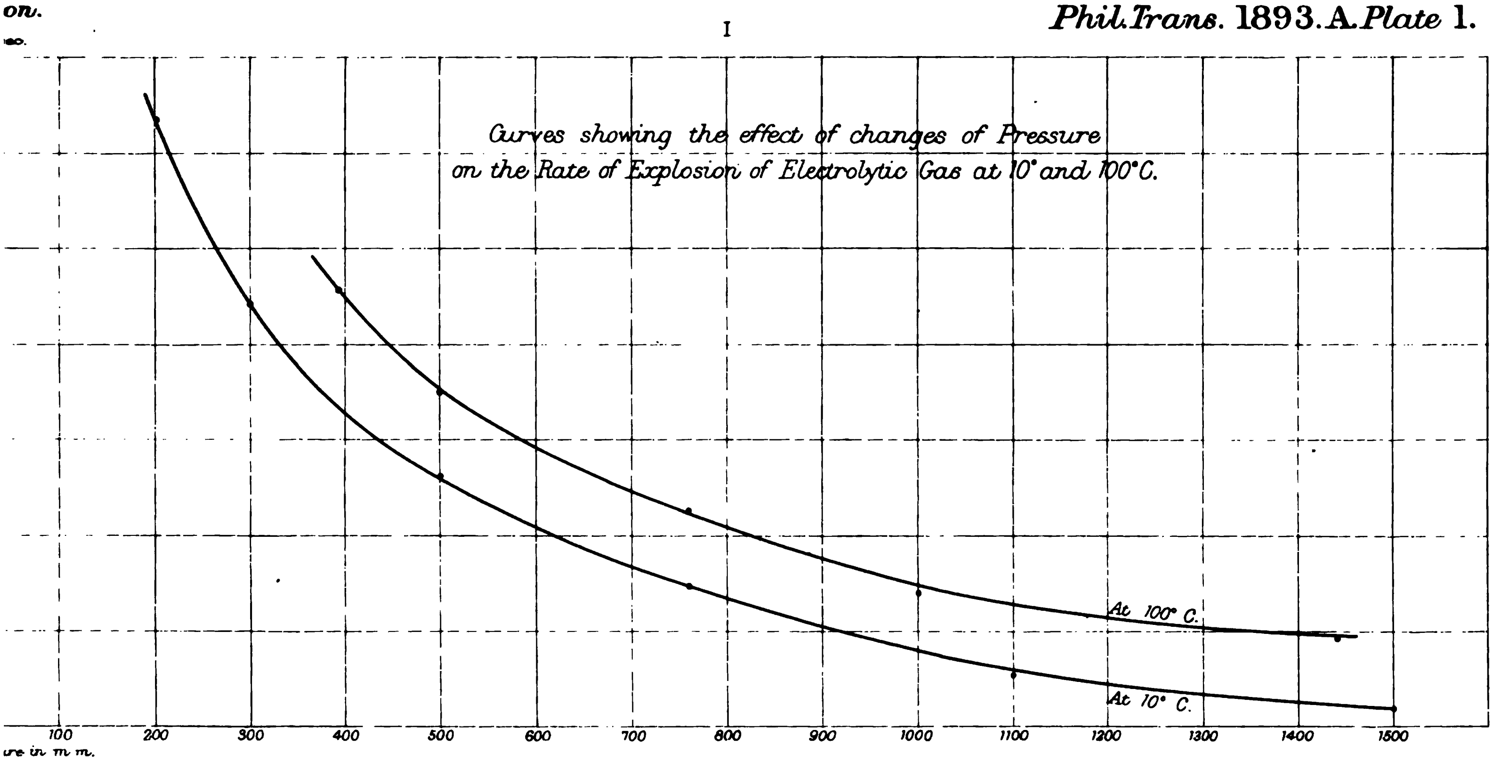
CO + O at 760 mm.				
Saturated at 35°.	Saturated at 45°.	Saturated at 55°.	Saturated at 65°.	Saturated at 75°.
1741 1731 1726 1750 1742	1700 1681 1697	1673 1650 1674	1519 1534	1260 1272
1738	1693	1666	1526	1266

At 400 mm.		
Saturated at 10°.	Saturated at 25°.	Saturated at 33°.
1573 1582 1574	1607 1617 1625	1580 1568 1562
1576	1616	1570

At 1100 mm.		
Saturated at 26°.	Saturated at 43°.	Saturated at 53°.
1747	1767	1730
1737	1799	1751
1728	1779	1745
1737	1782	1742

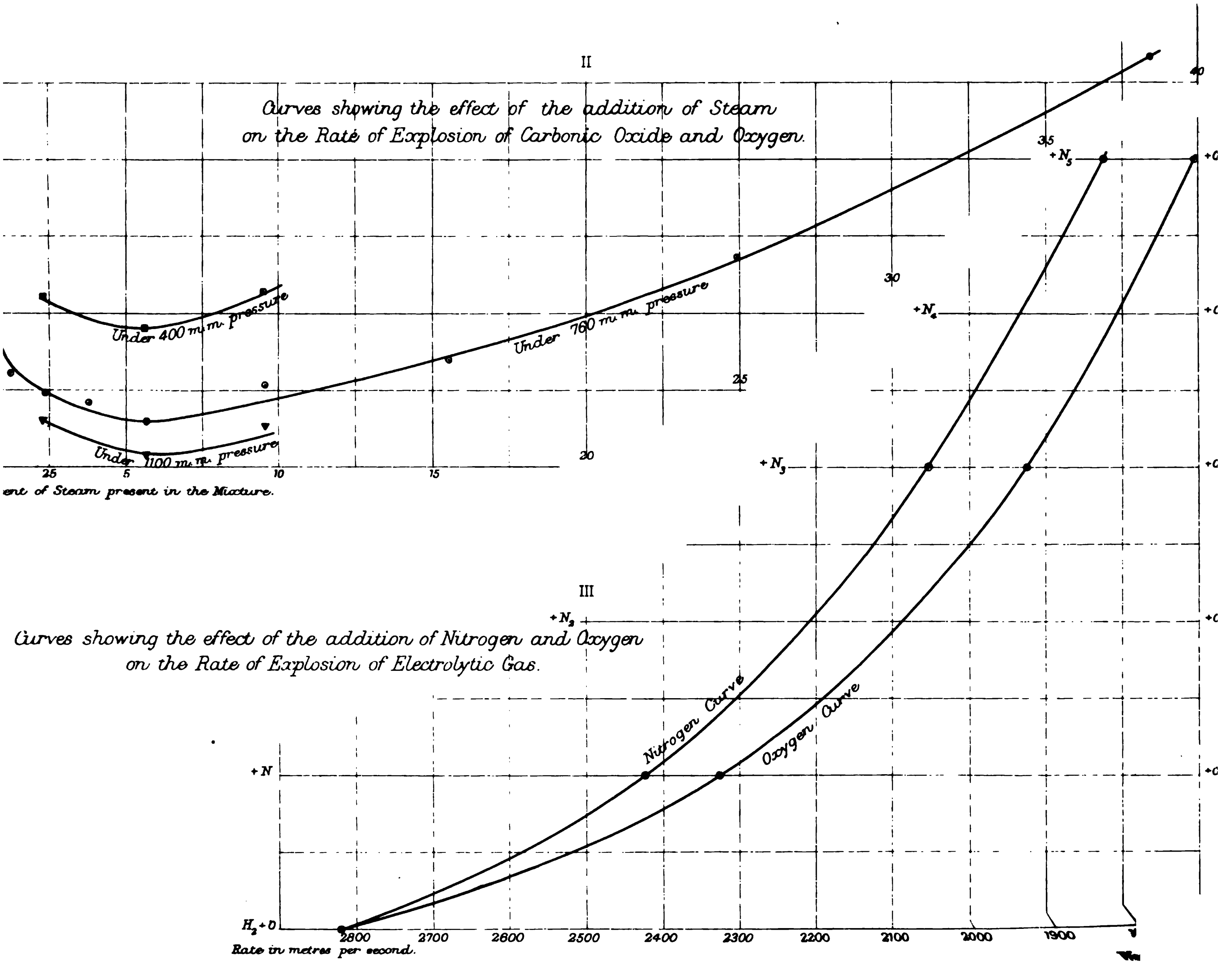
HYDROGEN and Oxygen Diluted with Carbonic Oxide [K = 1 vol. CO.].

$H_2 + O + K$.	$H_2 + O + K_3$.	$H_2 + O_2 + N_2$.	$H_2 + O_2 + K_2$.
2467	2086	1993	2144
2446	2070	2014	2142
2458	2084		
2448			
2455	2080	2003	2143



I

Curves showing the effect of changes of Pressure on the Rate of Explosion of Electrolytic Gas at 10° and 100° C.



II

Curves showing the effect of the addition of Steam on the Rate of Explosion of Carbonic Oxide and Oxygen.

III

Curves showing the effect of the addition of Nitrogen and Oxygen on the Rate of Explosion of Electrolytic Gas.

IV. *On the Residues of Powers of Numbers for any Composite Modulus, Real or Complex.*

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THE present work consists of two parts, with an appendix to the second. Part I. deals with real numbers, Part II. with complex.

In the simple cases when the modulus is a real number, which is an odd prime, a power of an odd prime, or double the power of an odd prime, we know that there exist primitive roots of the modulus; that is, that there are numbers whose successive powers have for their residues the complete set of numbers less than and prime to the modulus. A primitive root may be said to generate by its successive powers the complete set of residues. It is also known that, in general, when the modulus is any composite number, though primitive roots do not exist, there may be laid down a set of numbers which will here be called *generators*, the products of powers of which give the complete set of residues prime to the modulus.

The principal object of Part I. is to investigate the relations which must subsist among any such set of generators; to determine the most general form that they can take; to show how to form any such set of generators, and conversely to furnish tests for the efficiency, as generators, of any given set of numbers. Other results which are obtained as instrumental in effecting these objects, such as the determination of the number of numbers that belong to any exponent, may also possess independent interest.

The object of Part II. is to make, for complex numbers, an investigation which shall be as nearly as possible parallel to that of Part I. for real numbers. Much of the work of Part I. may be applied immediately to complex numbers; of the rest some will need slight modification, and some will need replacing by propositions leading to corresponding results. Of those cases which thus call for independent treatment, the most noticeable is that of the modulus $(1 + i)^\lambda$, which is the complex analogue of the real modulus 2^λ.

The work is put in the form of a series of propositions, and is started almost from first principles. The early part is consequently elementary, but the advantages of completeness and ease of reference may be more than sufficient to compensate for this. A large number of illustrative examples are given. These will sometimes, perhaps,

8.3.93

assist in elucidating the symbolical proofs which they follow; in all cases they will help to maintain clearly the actual arithmetical meaning of the results arrived at, a meaning which may easily seem obscure if it be noticed only in its symbolical and generalised form.

The Appendix contains tables of indices for complex numbers for all moduli whose norms do not exceed 100.

PART I.—ON THE RESIDUES OF POWERS OF NUMBERS FOR ANY COMPOSITE REAL MODULUS.

In what follows (except when the contrary is explicitly stated) we shall be treating of the residues of powers of numbers which are *prime* to the modulus with regard to which those residues are taken; and the modulus will be taken to be any composite number whatever. In this first part, moreover, all the numbers dealt with are real.

(1.) The residues (modulus m) of the successive powers of a number a prime to m form a recurring series of periods of terms, the first period beginning with the first term.

Consider the series of numbers

$$a, a^2, a^3 \dots$$

Since a is prime to m, therefore any power of a is prime to m, and therefore the residue of a^s for modulus m is prime to m.

Hence each term of the series of residues,

$$a, a^2, a^3 \dots (\text{mod } m),$$

is one of the numbers less than m and prime to it.

There are $\phi(m)$ numbers less than m and prime to it.

Hence in the above infinite series there are only $\phi(m)$ *different* terms.

Suppose that the first term which occurs for the second time is a^{t+s} (mod m), and suppose that this is congruent to a^s.

Then

$$a^{t+s} \equiv a^s (\text{mod } m),$$

where s and t are both to have as small values as possible.

$$a^s(a^t - 1) \equiv 0 (\text{mod } m),$$

and since a^s is prime to m,

$$a^t - 1 \equiv 0 (\text{mod } m),$$

and so

$$a^{t+\sigma} \equiv a^\sigma (\text{mod } m),$$

for *every* value of σ

Hence the term which first appears a second time is the first term a; which appears next as a^{t+1}, t being the least number for which

$$a^t \equiv 1 \pmod{m}.$$

Definition.—The smallest number t which makes $a^t \equiv 1 \pmod{m}$, where a is prime to m, is called the *exponent* of a for modulus m. CAUCHY uses the word "*indicator*" in the same sense. He also uses "*maximum indicator*" to denote what will be called the *highest exponent*.

Thus the infinite series of residues of $a, a^2, a^3, \ldots$ consists of a repetition of the period of t terms beginning from the first term.

(2.) If t be the exponent of a and $a^s \equiv 1 \pmod{m}$ then t divides s.

Let

$$s = qt + r \text{ where } r < t.$$

Then

$$a^s = a^{qt+r} \equiv 1 \pmod{m},$$
$$(a^t)^q \, . \, a^r \equiv 1 \pmod{m},$$
$$a^r \equiv 1 \pmod{m},$$

whereas t is the smallest value (not zero) which makes $a^t \equiv 1$. Therefore

$$r = 0,$$

therefore

$$t \text{ divides } s.$$

(3.) FERMAT'S *Theorem.*

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Let $a_1, a_2, a_3, \ldots a_{\phi(m)}$ be the $\phi(m)$ numbers less than m and prime to it.

Take any one of them a.

Then since $aa_1, aa_2, aa_3, \ldots aa_{\phi(m)} \pmod{m}$ are all prime to m, and no two congruent, they must be the same set of numbers as $a_1, a_2, \ldots a_{\phi(m)}$; therefore

$$a_1a_2a_3 \ldots a_{\phi(m)} a^{\phi(m)} \equiv a_1a_2 \ldots a_{\phi(m)} \pmod{m},$$

and, therefore,

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

Corollary.—It follows from this proposition and proposition (1) that the exponent of any number (modulus m) is always a divisor of $\phi(m)$. The propositions which follow will determine that divisor.

(4.) If t be the exponent of a then the exponent of a^s is τ: where $t = \kappa\tau$ and κ is the G.C.M. of s and t.

Let T be the exponent of a^s.

We have

$$a^t \equiv 1 \pmod{m},$$

therefore

$$a^{\kappa\tau} \equiv 1 \pmod{m},$$
$$a^{\kappa\sigma\tau} \equiv 1 \pmod{m},$$
$$(a^s)^\tau \equiv 1 \pmod{m},$$

therefore

$$\mathrm{T} \text{ divides } \tau. \quad \text{(Prop. 2.)}$$

Again,

$$(a^s)^{\mathrm{T}} \equiv 1 \pmod{m},$$

therefore

$$a^{s\mathrm{T}} \equiv 1 \pmod{m},$$

therefore

$$t \text{ divides } s\mathrm{T},$$

therefore

$$\kappa\tau \text{ divides } \kappa\sigma\mathrm{T},$$

therefore

$$\tau \text{ divides } \sigma\mathrm{T},$$

therefore

$$\tau \text{ divides } \mathrm{T},$$

therefore

$$\mathrm{T} = \tau.$$

Example.—The exponent of 3 for modulus 308 is 30: the residues of its successive powers are given in the following table:—

Number . . .	3	9	27	81	243	113	31	93	279	221
Power of 3 . .	1	2	3	4	5	6	7	8	9	10
Exponent . .	30	15	10	15	6	5	30	15	10	3

Number . . .	47	141	115	37	111	25	75	225	59	177
Power of 3 . .	11	12	13	14	15	16	17	18	19	20
Exponent . .	30	5	30	15	2	15	30	5	30	3

Number . . .	223	53	159	169	199	289	251	137	103	1
Power of 3 . .	21	22	23	24	25	26	27	28	29	30
Exponent .	10	15	30	5	6	15	10	15	30	1

where the exponents are all immediately deducible from the proposition.

(5.) The exponent of a is t and of a' is t' and t and t' are co-prime; then the exponent of aa' is tt'.

Let T be the exponent of aa'.

Then,

$$a^{t} \equiv 1 \pmod{m} \qquad a'^{t'} \equiv 1 \pmod{m},$$

therefore

$$a^{tt'} \equiv 1 \pmod{m} \qquad a'^{tt'} \equiv 1 \pmod{m},$$

therefore

$$(aa')^{tt'} \equiv 1 \pmod{m},$$

therefore

$$\text{T divides } tt'. \quad \text{(Prop. 2.)}$$

Again,

$$(aa')^{\mathrm{T}} \equiv 1 \pmod{m},$$
$$(aa')^{\mathrm{T}t} \equiv 1 \pmod{m},$$
$$a'^{\mathrm{T}t} \equiv 1 \pmod{m},$$

therefore

$$t' \text{ divides } \mathrm{T}t \text{ (Prop. 2)},$$

therefore

$$t' \text{ divides T}.$$

Similarly,

$$t \text{ divides T},$$

and therefore (since t and t' are co-prime),

$$tt' \text{ divides T},$$

therefore

$$\mathrm{T} = tt'.$$

Corollary.—It follows that if the numbers a, a', a'', . . . have exponents (for modulus m) t, t' t'', . . . these exponents being all co-prime, then the exponent of $aa'a''$ . . . is $tt't''$ . . .

Example.—The exponent of

23 mod 308 is 6; (23. 221. 155. 177. 67. 1),

and the exponent of

113 mod 308 is 5; (113. 141. 225. 169. 1).

Since 5 and 6 are co-prime, it follows that the exponent of

$$135 \equiv 23.\ 113 \text{ is } 30.$$

Corollary. Example.—The exponent of

$$\left.\begin{array}{l}111 \text{ is } 2\\ 221 \text{ is } 3\\ 113 \text{ is } 5\end{array}\right\}\text{therefore the exponent of } 111.\ 221.\ 113 \equiv 3 \text{ is } 30.$$

(6.) Let a have exponent t and a' exponent t' and suppose that t and t' are not co-prime. Then, if t and t' contain no prime factor raised to the same power in both, the exponent of aa' is the L.C.M. of t and t'.

Let

$$\left.\begin{array}{r}t = \kappa\tau\\ t' = \kappa\tau'\end{array}\right\}\text{where } \tau \text{ and } \tau' \text{ are co-prime,}$$

then

$$a^{\kappa} \text{ has exponent } \tau \text{ (Prop. 4)},$$
$$a'^{\kappa} \text{ has exponent } \tau' \text{ (Prop. 4)},$$

therefore,

$$(aa')^{\kappa} \text{ has exponent } \tau\tau' \text{ (Prop. 5)}.$$

So if aa' has exponent T, then

$$\frac{\mathrm{T}}{\kappa} = \frac{\tau\tau'}{z}$$

where $\tau\tau'$ divides T, z divides κ and z is prime to $\tau\tau'$. (Prop. 4.)

Now since t and t' contain no prime raised to the same power in each, therefore $\tau\tau'$ contains every prime factor which occurs in t and t' and therefore contains every prime factor which occurs in κ. Hence z cannot divide κ and be prime to $\tau\tau'$ unless it be unity.

Therefore,

$$\mathrm{T} = \kappa\tau\tau' = \text{L.C.M. of } t \text{ and } t'.$$

(7.) If a has exponent t, a' has exponent t', a'' — t'', &c., for modulus m, and if, of the $tt't'' \ldots$ numbers $a^r a'^{r'} a''^{r''} \ldots$ (mod m) formed by giving to r all values modulus t, to r' all values modulus $t' \ldots$, no two are congruent; and if

$$a^s a'^{s'} a''^{s''} \ldots \equiv 1 \pmod{m};$$

then we must have

$$a^s \equiv 1, \quad a'^{s'} \equiv 1, \quad a''^{s''} \equiv 1, \ldots \pmod{m}.$$

For suppose that at least one of these congruences is violated.

Say

$$a^s \not\equiv 1 \pmod{m},$$

and, therefore,

$$s \not\equiv 0 \pmod{t}.$$

Then, because

$$a^s a'^{s'} a''^{s''} \dots \equiv 1 \pmod{m},$$

therefore (multiplying by a^{t-s}),

$$a^t a'^{s'} a''^{s''} \dots \equiv a^{t-s} \pmod{m},$$

or

$$a^0 a'^{s'} a''^{s''} \dots \equiv a^{t-s} \pmod{m},$$

or

$$a^0 a'^{s'} a''^{s''} \dots \equiv a^{t-s}\, a'^0 a''^0 \dots \pmod{m},$$

which is contrary to the supposition that no two numbers of the form $a^r a'^{r'} \dots \pmod{m}$ are congruent; for the last congruence obtained shows that if we make

$$\left.\begin{array}{lll} r = 0 & r' = s' & r'' = s'' \dots \\ r = t - s & r' = 0 & r'' = 0 \;\dots \end{array}\right\}$$

where

$$t - s \not\equiv 0 \pmod{t},$$

the two numbers are congruent.

Hence, we must have

$$s \equiv 0 \pmod{t},$$

and, therefore,

$$a^s \equiv 1 \pmod{t}.$$

Similarly,

$$a'^{s'} \equiv 1 \pmod{t'}, \text{ \&c.}$$

Definition.—If a, a', a'' . . . have exponents t, t', t'', . . . and if no two of the $tt't''$ . . . numbers that can be formed by products of their powers are congruent modulus m (as in the last proposition), then these numbers, a, a', a'', . . . which generate the $tt't''$ . . . incongruent numbers, will be called *independent* generators.

The last proposition may then be stated thus:—If a product of powers of a set of independent generators be congruent to unity, then each of those powers is itself congruent to unity.

(8.) If a, a', a'', . . . independent generators, have exponents t, t', t'', . . . then the exponent of $aa'a''$ . . . is the L.C.M. of t, t', t'' . . .

Let T be the L.C.M. of t, t', t'', . . . and τ the required exponent of $aa'a''$ . . .

$$a^t \equiv 1 \pmod{m},$$

therefore,

$$a^{\mathrm{T}} \equiv 1 \pmod{m}.$$

Similarly,

$$a'^{\mathrm{T}} \equiv 1 \pmod{m}, \text{ \&c.},$$

therefore,

$$(aa'a'' \ldots)^{T} \equiv 1 \pmod{m},$$

therefore,

$$\tau \text{ divides } T. \quad \text{(Prop. 2.)}$$

Again,

$$(aa'a'' \ldots)^{\tau} \equiv 1 \pmod{m},$$

therefore,

$$a^{\tau}a'^{\tau}a''^{\tau} \ldots \equiv 1 \pmod{m},$$

therefore,

$$\left.\begin{array}{l} \tau \equiv 0 \pmod{t} \\ \tau \equiv 0 \pmod{t'} \\ \tau \equiv 0 \pmod{t''} \end{array}\right\} \text{(Prop. 7)},$$

therefore,

$$T \text{ divides } \tau,$$

therefore,

$$T = \tau.$$

Corollary.—If the exponents t, t', t'', be all of them powers of the same prime, and a, a', a'', . . . are independent generators, then the exponent of the product $aa'a''$ . . . is equal to the greatest of the exponents t, t,′ t'' . . .

(9.) If the exponent of a for modulus m is t, and for modulus n is t', and if m and n are co-prime, then the exponent of a for modulus mn is the L.C.M. of t and t'.

Let $t'' = t\tau = t'\tau'$ where τ and τ' are co-prime, so that t'' is the L.C.M. of t and t'.

Let the exponent of a for modulus mn be T.

Thus we have

$$a^{t} \equiv 1 \pmod{m},$$

and, therefore, raising to the power τ,

$$a^{t''} \equiv 1 \pmod{m}.$$

Similarly,

$$a^{t''} \equiv 1 \pmod{n},$$

and, therefore (since m and n are co-prime),

$$a^{t''} \equiv 1 \pmod{mn},$$

therefore

$$T \text{ divides } t''. \quad \text{(Prop. 2.)}$$

Again,

$$a^{T} \equiv 1 \pmod{mn},$$

therefore

$$a^{T} \equiv 1 \pmod{m},$$

therefore

$$t \text{ divides } T.$$

Similarly,

$$t' \text{ divides T},$$

therefore

$$t'' \text{ divides T}.$$

Therefore

$$\mathrm{T} = t''.$$

Corollary.—If the exponents of a for moduli m, m', m'', . . . are respectively $t, t', t'', \ldots$ and the moduli are co-prime, then the exponent of a for modulus $mm'm''$ . . . is the L.C.M. of $t, t', t'' \ldots$

Examples.—The exponent of

$$\left.\begin{array}{l} 3 \text{ for mod } \ 4 \text{ is } 2 \\ 3 \text{ for mod } \ 7 \text{ is } 6 \\ 3 \text{ for mod } 11 \text{ is } 5 \end{array}\right\} \qquad \begin{array}{l} (3.\ 1.) \\ (3.\ 2.\ 6.\ 4.\ 5.\ 1.) \\ (3.\ 9.\ 5.\ 4.\ 1.) \end{array}$$

Therefore the exponent of

$$3 \text{ for mod } 4.\ 7.\ 11 = 308 \text{ is } 30.$$

The exponent of

$$\begin{array}{ll} 5 \text{ for mod } \ 4 \text{ is } 1. & \\ 5 \text{ for mod } \ 7 \text{ is } 6 & (5.\ 4.\ 6.\ 2.\ 3.\ 1.) \\ 5 \text{ for mod } 11 \text{ is } 5 & (5.\ 3.\ 4.\ 9.\ 1.) \end{array}$$

and, therefore, the exponent of

$$5 \text{ for mod } 308 \text{ is } 30.$$

The exponent of

$$\begin{array}{ll} 9 \text{ for mod } \ 4 \text{ is } 1. & \\ 9 \text{ for mod } \ 7 \text{ is } 3 & (2.\ 4.\ 1.) \\ 9 \text{ for mod } 11 \text{ is } 5 & (9.\ 4.\ 3.\ 5.\ 1.) \end{array}$$

and, therefore, the exponent of

$$9. \text{ mod } 308 \text{ is } 15.$$

(10.) If the exponent of a is t, and $t = pqr \ldots$ where p, q, r are co-prime factors of t, to express a as a product of numbers whose exponents are p, q, $r \ldots$

Let

$$\left.\begin{array}{l} \mathrm{P} \equiv 0 \ (\text{mod}\, qr \ldots) \\ \quad \equiv 1 \ (\text{mod}\, p) \end{array}\right\} \quad \left.\begin{array}{l} \mathrm{Q} \equiv 0 \ (\text{mod}\, pr \ldots) \\ \quad \equiv 1 \ (\text{mod}\, q) \end{array}\right\} \ \&\text{c.}$$

These congruences determine one value each and one only (mod t) for P, Q, . . .

$$\begin{array}{l} a^{P} \text{ has exp } p \text{ (Prop. 4),} \\ a^{Q} \quad ,, \quad ,, \quad q \\ \qquad \&\text{c.} \end{array}$$

From the above congruences we obtain

$$\begin{array}{l} P \equiv 1 \pmod{p}, \\ Q \equiv 0 \pmod{p}, \\ R \equiv 0 \pmod{p}, \\ \qquad \&\text{c.} \end{array}$$

therefore, by addition,

$$P + Q + R + \ldots \equiv 1 \pmod{p}.$$

Similarly,

$$P + Q + R + \ldots \equiv 1 \pmod{q},$$
$$\&\text{c.},$$

and, therefore, since p, q, r, . . . are co-prime,

$$P + Q + R + \ldots \equiv 1 \pmod{t},$$

therefore

$$a^{P}.\, a^{Q}.\, a^{R} \ldots \equiv a \pmod{m}.$$

And so a is expressible (in one way only) as the product of numbers a^{P}, a^{Q} . . . whose exponents are p, q . . .

Example.—3 has exponent 30 mod 308.

To express it as a product of 3 numbers with exponents 2. 3. 5,

$$\begin{array}{lll} P \equiv 0 \pmod{15} & Q \equiv 0 \pmod{10} & R \equiv 0 \pmod{6} \\ \;\;\; \equiv 1 \pmod{2} & \;\;\; \equiv 1 \pmod{3} & \;\;\; \equiv 1 \pmod{5}, \end{array}$$

therefore

$$P = 15 \qquad\qquad Q = 10 \qquad\qquad R = 6,$$

and, therefore,

$$3 \equiv 3^{15}.\, 3^{10}.\, 3^{6} \equiv 111.\ 221.\ 113 \pmod{308}$$

where

$$\left.\begin{array}{l} 111 \text{ has exp } 2 \\ 221 \;\; ,, \quad ,, \;\; 3 \\ 113 \;\; ,, \quad ,, \;\; 5 \end{array}\right\}$$

Similarly 79 has exponent 30 (mod 308), and

$$79 \equiv 79^{15}.\, 79^{10}\ 79^{6} \equiv 43.\ 177.\ 141 \pmod{308}$$

where

$$\left.\begin{array}{r} 43 \text{ has exp } 2 \\ 177 \text{ ,, ,, } 3 \\ 141 \text{ ,, ,, } 5 \end{array}\right\} .$$

(11.) The number of numbers that belong to any exponent t when the modulus is a prime p is $\phi(t)$.

If there are any numbers which have exponent t, mod p, let α be one.

Then because

$$\alpha^t \equiv 1 \pmod{p},$$

therefore each of the t incongruent numbers

$$\alpha, \alpha^2, \alpha^3, \ldots \alpha^t$$

when raised the t^{th} power is $\equiv 1$.

Hence each is a root of $x^t \equiv 1 \pmod{p}$ which has only t incongruent roots.

Therefore every number β which has exponent t, and which is therefore such that $\beta^t \equiv 1$ is included in the above set. Hence every number with exponent t is to be found in the above set. Now, of the powers of α there are $\phi(t)$ which have their index prime to t, and which, therefore, have exponent t (Prop. 4).

Therefore if there is *one* number with exponent t there are $\phi(t)$, and no more.

Now if $t_1, t_2, \ldots$, are all the divisors of $p - 1$, and, therefore, all possible exponents (Prop. 3, corollary),

$$\phi(t_1) + \phi(t_2) + \ldots = \phi(p).$$

Corresponding to each value t there are either $\phi(t)$ numbers or none with t as exponent. The number of numbers altogether is $\phi(p)$. Hence in no case can there fail to be $\phi(t)$ numbers with exponent t.

Corollary.—In particular there are $\phi(p-1)$ numbers with exponent $p-1$, modulus p, *i.e.*, any odd prime p has $\phi(p-1)$ primitive roots.

(12.) The exponents to which a number a belongs for successive powers of a prime p as moduli.

We suppose that a is prime to p and that $p \neq 2$.

Let the exponents to which a belongs for the moduli $p, p^2, p^3, \ldots p^\lambda \ldots$ be $t_1, t_2, t_3, \ldots$ respectively.

Then because

$$a^{t_{\lambda+1}} \equiv 1 \pmod{p^{\lambda+1}},$$

therefore

$$a^{t_{\lambda+1}} \equiv 1 \pmod{p^\lambda},$$

therefore

$$t_\lambda \text{ divides } t_{\lambda+1}. \quad \text{(Prop. 2.)}$$

Again

$$a^{t_\lambda} \equiv 1 \pmod{p^\lambda},$$

therefore (raising to the p^{th} power $a^{t_\lambda} = 1 + \kappa p^\lambda$)

$$a^{pt_\lambda} \equiv 1 \pmod{p^{\lambda+1}},$$

therefore

$$t_{\lambda+1} \text{ divides } pt_\lambda. \quad \text{(Prop. 2.)}$$

It follows from these two results that either $t_{\lambda+1} = t_\lambda$ or $t_{\lambda+1} = pt_\lambda$.

Each exponent in the series $t_1, t_2, \ldots t_\lambda, \ldots$ is either equal to that immediately preceding it or is p times that value.

We can show, however, that after the first set of equal exponents $t_1 = t_2 = \ldots$ comes to an end, that each exponent is p times that which immediately precedes it.

For suppose that, if possible, after

$$t_{\lambda+1} = pt_\lambda$$

we can have

$$t_{\lambda+2} = t_{\lambda+1}.$$

We thus have

$$t_{\lambda+2} = t_{\lambda+1} = pt_\lambda.$$

Now

$$a^{t_\lambda} \equiv 1 \pmod{p^\lambda} = 1 + xp^\lambda + yp^{\lambda+1},$$

say, where $x < p$.

Also,

$$a^{pt_\lambda} = a^{t_{\lambda+2}} \equiv 1 \pmod{p^{\lambda+2}},$$

therefore

$$(1 + xp^\lambda + yp^{\lambda+1})^p \equiv 1 \pmod{p^{\lambda+2}}$$

$$1 + xp^{\lambda+1} \equiv 1 \pmod{p^{\lambda+2}}$$

therefore

$$x = 0,$$

and, therefore,

$$a^{t_\lambda} = 1 + yp^{\lambda+1}$$

$$\equiv 1 \pmod{p^{\lambda+1}}.$$

Therefore

$$t_{\lambda+1} \text{ divides } t_\lambda,$$

which is not so. Therefore we cannot have

$$t_{\lambda+2} = t_{\lambda+1}$$

if

$$t_{\lambda+1} = pt_\lambda,$$

and therefore we must have

$$t_{\lambda+2} = pt_{\lambda+1}.$$

If, then, the exponent of a for modulus p be $t_1 = t$ and $a^t - 1$ contain p^s as the highest power of p, we have

$$a^t \equiv 1 \pmod{p^s},$$

and

$$t_1 = t_2 = \ldots = t_s = t.$$

And after these

$$t_{s+1} = pt_s,$$
$$t_{s+2} = pt_{s+1},$$
$$\&c.$$

Hence the exponent of a for modulus p^λ is

$$t \text{ if } \lambda \leqq s,$$
$$tp^{\lambda - s} \text{ if } \lambda > s,$$

where t is the exponent of a for modulus p, and p^s is the greatest power of p that will divide $a^t - 1$.

Corollary.—The greatest value that t can have is $p - 1$. This is so when a is congruent to a primitive root of p (Proposition 11, corollary). The greatest value that $p^{\lambda - s}$ can have is got by making s as small as possible, viz., by making $s = 1$, *i.e.*, by taking a so that $a^{p-1} - 1$ (though necessarily divisible by p) is not divisible by p^2.

Therefore the greatest possible exponent that a number can have for modulus p^λ is $(p - 1) \cdot p^{\lambda - 1}$, and as this is equal to $\phi(p^\lambda)$ it follows that primitive roots exist for a modulus a power of a prime.

Examples.—Exponent of 3 for mod 5^6.

The exp of 3 for mod 5 is 4 (3. 4. 2. 1).

$3^4 - 1 = 80$ is divisible by 5^1.

Therefore

exp of 3 is $4.\ 5^5$ (mod 5^6).

Exponent of 24 for mod 5^6.

The exp of 24 for mod 5 is 2 ($t = 2$).

$24^2 - 1 = 575$ which is divisible by 5^2. ($s = 2$).

Therefore

exp of 24 mod $5^6 = 2.\ 5^4$.

(13.) The exponent to which a number a belongs for a power of 2 as modulus.

The number a is now to be considered odd.

Let $t_1, t_2, t_3, \ldots t_\lambda \ldots$ be the exponents of a for moduli $2, 2^2, \ldots 2^\lambda \ldots$

We have

$$a^{t_{\lambda+1}} \equiv 1 \pmod{2^{\lambda+1}}.$$

Therefore

$$a^{t_{\lambda+1}} \equiv 1 \pmod{2^{\lambda}};$$

therefore

$$t_{\lambda} \text{ divides } t_{\lambda+1}. \quad \text{(Prop. 2.)}$$

Again

$$a^{t_{\lambda}} \equiv 1 \pmod{2^{\lambda}} = 1 + x.2^{\lambda} + y.2^{\lambda+1} \text{ (where } x = 1 \text{ or } 0).$$

Therefore

$$a^{2t_{\lambda}} \equiv 1 \pmod{2^{\lambda+1}},$$

and therefore

$$t_{\lambda+1} \text{ divides } 2t_{\lambda}.$$

Therefore either

$$\left.\begin{array}{l} t_{\lambda+1} = t_{\lambda} \\ t_{\lambda+1} = 2t_{\lambda} \end{array}\right\}.$$

or

Suppose now that after

$$t_{\lambda+1} = 2t_{\lambda}$$

we can have

$$t_{\lambda+2} = t_{\lambda+1}.$$

Then

$$a^{t_{\lambda}} \equiv 1 \pmod{2^{\lambda}} = 1 + x.2^{\lambda} + y.2^{\lambda+1}.$$

Therefore

$$a^{2t_{\lambda}} \equiv 1 + x.2^{\lambda+1} + x^2 2^{2\lambda} \pmod{2^{\lambda+2}}.$$

Now if $\lambda \geqq 2$, then

$$2\lambda \geqq \lambda + 2,$$

and therefore

$$a^{2t_{\lambda}} \equiv 1 + x.2^{\lambda+1} \pmod{2^{\lambda+2}}.$$

Now

$$a^{2t_{\lambda}} = a^{t_{\lambda+2}} \equiv 1 \pmod{2^{\lambda+2}},$$

therefore

$$x = 0.$$

Therefore

$$a^{t_{\lambda}} = 1 + x.2^{\lambda} + y2^{\lambda+1} \equiv 1 \pmod{2^{\lambda+1}},$$

therefore

$$t_{\lambda+1} \text{ divides } t_{\lambda},$$

which is not so.

Hence we have the result, when $\lambda \geqq 2$ and $t_{\lambda+1} = 2t_{\lambda}$, then $t_{\lambda+2} = 2t_{\lambda+1}$.

The first exponent t_1, of the series, is equal to unity.

If this be followed by a set of 1's (at least one), then by what has been proved they will be followed by the series $2, 2^2, 2^3 \ldots$

If the second exponent be a 2, the third may be also, and so on; the series then continues with $2^2, 2^3, 2^4, \ldots$ Of the 2's there are at least two; for otherwise the first three exponents would be $t_1 = 1$, $t_2 = 2$, $t_3 = 4$, making 4 an exponent for mod $2^3 = 8$, which is impossible.

Hence the series of exponents run either thus—

$$1.\ 1.\ 1.\ \ldots\ 1.\ 2.\ 2^2.\ 2^3.\ 2^4 \ldots$$

or thus

$$1.\ 2.\ 2.\ \ldots\ 2.\ 2.\ 2^2.\ 2^3.\ 2^4 \ldots$$

These results can be expressed thus—

Let the highest power of 2 which divides $a^2 - 1$ be 2^{s+1}. (Since a is an odd number, s is at least equal to 2.)

Then the exponent of a is

$$\text{if } \lambda > s,\ 2^{\lambda - s}$$

$$\text{if } \lambda \leqq s, \begin{cases} 2 \text{ if } a^2 \equiv 1 \pmod{2^\lambda} \text{ and } a \not\equiv 1 \pmod{2^\lambda}. \\ 1 \text{ if } a \equiv 1 \pmod{2^\lambda}. \end{cases}$$

Corollary.—The greatest exponent possible for mod 2^λ is $2^{\lambda-2}$; and as $\phi(2^\lambda) = 2^{\lambda-1}$, primitive roots do not exist.

Examples.—Exponent of 3 for mod 2^6.

Exp of 3 for mod 2 is 1,
„ „ 2^2 is 2,
„ „ 2^3 is 2,
„ „ 2^4 is 4,

and therefore

exp of 3 for mod 2^6 is 16

Exponent of 35 for mod $64 = 2^6$ $(\lambda = 6)$.

$$35^2 - 1 = 1224 = 153.2^3,$$

therefore

$$s = 2,$$

therefore

exp of 35 is 2^4.

Exp of 41 for mod $128 = 2^7$ $(\lambda = 7)$.

$$41^2 - 1 = 1680 = 105.2^4,$$

therefore

$$s = 3,$$

therefore

$$\text{exp of } 41 = 2^{7-3} = 2^4.$$

(14.) The numbers which belong to exponent 2 for modulus 2^κ ($\kappa \geqq 3$).

Let a have exponent 2, modulus 2^κ.

Then

$$a^2 \equiv 1 \pmod{2^\kappa}$$

and

$$a \not\equiv 1 \pmod{2^\kappa}.$$

The congruence

$$a^2 \equiv 1 \pmod{2^\kappa}$$

gives

$$(a - 1)(a + 1) \equiv 0 \pmod{2^\kappa}.$$

Since a is odd, if 2^s is the highest power of 2 that divides $a - 1$ and $s > 1$, then the highest power of 2 that divides $a + 1$ is 2, and *vice versâ*.

Hence either

$$a + 1 \equiv 0 \pmod{2^{\kappa-1}},$$

or

$$a - 1 \equiv 0 \pmod{2^{\kappa-1}}.$$

Therefore (excluding $a \equiv 1$ (modulus 2^κ)) we have three numbers whose exponents are 2 for modulus 2^κ, viz.,

$$2^{\kappa-1} + 1, \quad 2^\kappa - 1, \quad 2^{\kappa-1} - 1. \quad \pmod{2^\kappa}.$$

The product of each pair of these is congruent to the third.

$$\left.\begin{array}{l} (2^{\kappa-1} + 1)(2^\kappa - 1) \equiv 2^{\kappa-1} - 1 \\ (2^\kappa - 1)(2^{\kappa-1} - 1) \equiv -2^{\kappa-1} + 1 \equiv 2^{\kappa-1} + 1 \\ (2^{\kappa-1} - 1)(2^{\kappa-1} + 1) \equiv 2^\kappa - 1 \end{array}\right\} \pmod{2^\kappa}.$$

(15.) The numbers which have exponent 2^s for modulus 2^κ.

We have already seen that $s \not> \kappa - 2$, and we have already treated (Prop. 14) of the case when $s = 1$.

Let a have exponent 2^s.

The exponents to which it belongs for successive powers of 2 as moduli are given either by

$$\left.\begin{array}{llllllll} \text{exp} & 1. & 1. & 1. \ldots & 1 & 2. & 2^2. & \ldots 2^s \\ \text{mod} & 2. & 2^2. & 2^3. \ldots & 2^{\kappa-s}. & 2^{\kappa-s+1}. & 2^{\kappa-s+2}. & \ldots 2^\kappa \end{array}\right\},$$

or by

$$\left.\begin{array}{l} \text{Exp } 1.\ 2.\ 2.\ \ldots\ \ 2.\quad 2^2.\ \ldots\ 2^s \\ \text{Mod } 2.\ 2^2.\ 2^3.\ \ldots\ 2^{\kappa-s+1}.\ 2^{\kappa-s}.\ \ldots\ 2^\kappa \end{array}\right\}$$

In either case

$$a^2 \equiv 1 \pmod{2^{\kappa-s+1}}$$
$$a^2 \not\equiv 1 \pmod{2^{\kappa-s+2}},$$

therefore

$$a^2 \equiv 1 + 2^{\kappa-s+1} \pmod{2^{\kappa-s+2}},$$

and therefore

$$a \equiv \pm 1 + 2^{\kappa-s} \pmod{2^{\kappa-s+1}}.$$

Thus the numbers (modulus 2^κ) which have exponent 2^s are the 2^s numbers given by

$$\pm 1 + 2^{\kappa-s} \pmod{2^{\kappa-s+1}}.$$

In particular, the numbers with the greatest exponent $2^{\kappa-2}$ are the $2^{\kappa-2}$ numbers

$$\pm 1 + 2^2 \pmod{2^3}.$$

We have seen that there is one number (viz., unity) with exponent 1.

$$\begin{array}{llllll} \text{there are } 3 & \text{numbers with exponent } 2. \\ \text{,, \ ,, } 2^2 & \text{,, \quad ,, \quad ,, } \quad 2^2. \\ \quad\vdots & \\ \text{,, \ ,, } 2^{\kappa-2} & \text{,, \quad ,, \quad ,, } \quad 2^{\kappa-2}. \end{array}$$

In all, $1 + 3 + 2^2 + 2^3 + \ldots + 2^{\kappa-2} = 2^{\kappa-1} = \phi(2^\kappa)$, the number of odd numbers less than 2^κ, as it should be.

(16.) If we take any number g which has exponent $2^{\kappa-2}$ for modulus 2^κ, the successive powers of g give $2^{\kappa-2}$ incongruent numbers, one half of the complete set of odd numbers less than 2^κ.

Of these, one and only one, has exponent 2, viz., $g^{2^{\kappa-3}}$ (Prop. 4).

Now

$$g \equiv \pm 1 + 2^2 \pmod{2^3}$$

squaring,

$$g^2 \equiv 1 + 2^3 \pmod{2^4}$$

and successively squaring

$$g^{2^2} \equiv 1 + 2^4 \pmod{2^5}$$
$$\vdots$$
$$g^{2^{\kappa-3}} \equiv 1 + 2^{\kappa-1} \pmod{2^\kappa}.$$

So of the three numbers $2^\kappa - 1$, $2^{\kappa-1} + 1$, $2^{\kappa-1} - 1$, it is always $2^{\kappa-1} + 1$ to which the power $2^{\kappa-3}$ of any number with exponent $2^{\kappa-2}$ is congruent.

Let f be either of the two numbers $2^\kappa - 1$, $2^{\kappa-1} - 1$; so that $g^{2^{\kappa-3}} \not\equiv f \pmod{2^\kappa}$.

Consider the $2^{\kappa-1}$ numbers

$$\left.\begin{array}{l} g\,, g^2\,, g^3\ \dots\ g^{2^{\kappa-2}} \\ fg, fg^2, fg^3 \dots\ fg^{2^{\kappa-2}} \end{array}\right\} \pmod{2^\kappa}.$$

Clearly no two in the first row are congruent, nor in the second.

The supposition

$$fg^r \equiv g^{r+t} \pmod{2^\kappa}$$

leads to

$$g^r\,(g^t - f) \equiv 0 \pmod{2^\kappa}$$
$$\therefore\ g^t \equiv f \pmod{2^\kappa},$$

which is contrary to the supposition that $g^{2^{\kappa-3}}$, and therefore *no* power of g is congruent to f.

Therefore no two of the above $2^{\kappa-1}$ odd numbers are congruent mod 2^κ, and hence they are the $2^{\kappa-1}$ numbers (mod 2^κ).

If for f we take $2^\kappa - 1 \equiv -1 \pmod{2^\kappa}$, the $2^{\kappa-1}$ numbers may be written

$$\pm g,\ \pm g^2,\ \pm g^3, \dots \pm g^{2^{\kappa-2}} \pmod{2^\kappa}.$$

Whichever number be taken for f, any number (mod 2^κ) is expressible in the form

$$a \equiv f^i g^j \pmod{2^\kappa},$$

where

$$\left.\begin{array}{l} i \text{ is referred to mod } 2, \\ j \text{ is referred to mod } 2^{\kappa-2}. \end{array}\right\}$$

Note.—In the last propositions relating to mod 2^κ, κ is supposed to be $\geqq 3$.

In the case $\kappa = 2$, when the modulus is $2^2 = 4$, there are two odd numbers, less than the modulus, viz., 1 and 3, and 3 (having exponent 2) is a primitive root.

In the case $\kappa = 1$ when the modulus is 2, the only odd number is unity.

We see that the case when the modulus is a power of 2, differs very much from the case when the modulus is a power of an odd prime.

When the mod is 2^κ ($\kappa \geqq 3$).

(i.) The highest exponent is not $\phi(2^\kappa) = 2^{\kappa-1}$, but $2^{\kappa-2}$, and hence there are no primitive roots.

(ii.) The form of the numbers which belong to any exponent is known, and the numbers can be at once written down when κ is known. In particular the numbers $\pm 1 + 2^2$ (mod 8) always belong to the highest exponent.

When the modulus is a power of an odd prime.

(i.) Primitive roots always exist.

(ii.) The determination of primitive roots depends on a knowledge of those of the prime in question.

Examples.—For modulus $2^5 = 32$ the numbers which belong to the different exponents are

Exp 1. 1.
Exp 2. 15. 17. 31.
Exp 4. 7. 23. 9. 25. . . . . ($\pm 1 + 8 \pmod{16}$).
Exp 8. 3. 11. 19. 27. 5. 13. 21. 29 ($\pm 1 + 4 \pmod{8}$).

The residues of powers of 3 are

3. 9. 27. 17. 19. 25. 11. 1.

Multiplying each by $2^5 - 1 = 31$ we get

29. 23. 5. 15. 13. 7. 21. 31,

and, multiplying by $2^4 - 1 = 15$, we get

13. 7. 21. 31. 29. 23. 5. 15,

the same set.

THE Table of Indices for Generators 3 and 15 is—

..	0	1	2	3	4	5	6	7
0	1	3	9	27	17	19	25	11
1	15	13	7	21	31	29	23	5

(Index of 3.)

(Index of 15.)

(17). From propositions (9), (12) and (13) the exponent to which any number a belongs for modulus $m = 2^\kappa p^\lambda q^\mu \ldots$ is readily determined.

For by (9) the exponent is the L.C.M. of the separate exponents of a for moduli $2^\kappa, p^\lambda, q^\mu, \ldots$ separately.

These exponents are separately determined by propositions (12) and (13).

The greatest possible separate exponents are

$$\left.\begin{array}{lll} 2^{\kappa-2} & \text{if} & \kappa \geqq 3 \\ 2 & \text{if} & \kappa = 2 \\ 1 & \text{if} & \kappa = 1 \end{array}\right\} \text{for mod } 2^\kappa,\ (p-1)p^{\lambda-1} \text{ for mod } p^\lambda, \text{ \&c.,}$$

and, hence, the greatest exponent possible for modulus m is the L.C.M. of these.

The value of

$$\phi(m) \text{ is } 2^{\kappa-1}(p-1)p^{\lambda-1}(q-1)q^{\mu-1}\ldots$$

and, since $p-1$, $q-1$, . . . are all even, the L.C.M. can only equal $\phi(m)$ when

(i.) there is the only one odd prime p present; and
(ii.) $\kappa = 1$ or 0.

Hence the only moduli which admit of primitive roots (which exist only when the highest exponent is equal to $\phi(m)$), are powers of odd primes, and double the powers of odd primes.

Examples.—What is the exponent of 3 for mod $1000000 = 2^6 . 5^6$?

The exp of 3 for mod 2^6 is 2^4.
" " 5^6 is $4 . 5^5$.

Therefore

$$\text{exp of } 3 \text{ mod } 10^6 = 2^4 . 5^5 = 50000.$$

How many decimal places are there in the period of the product of $\cdot\dot{0}1$ and $\cdot\dot{0}1$, *i.e.*, what is the exponent of 10 for mod $99 \times 99 = 3^4 . 11^2$?

Exp of 10 for mod $3 = 1$.
" " $3^2 = 1$.
" " $3^3 = 3$.
" " $3^4 = 9$.
" " $11 = 2$.
" " $11^2 = 22$.

Therefore

$$\text{exp of } 10 \text{ for mod } 3^4 . 11^2 = 9 \times 22 = 198.$$

Hence there are 198 figures in the period of $(\cdot\dot{0}1)^2$.

How many decimals are there in the period of $(\cdot\dot{0}01)^n$, *i.e.*, what is the exponent of 10 for mod $(999)^n = 37^n . 3^{3n}$?

The exp of 10 mod 3 is 1,

and

$$10 - 1 \text{ is divisible by } 3^2,$$

therefore

$$\text{exp of } 10 \text{ mod } 3^{3n} \text{ is } 3^{3n-2}.$$

$$\text{Exp of } 10 \text{ mod } 37 = 3,$$

and

$$37^1 \text{ divides } 10^3 - 1,$$

therefore

$$\text{exp of } 10 \text{ mod } 37^n = 3 . 37^{n-1},$$

therefore

$$\text{exp of } 10 \text{ mod } 37^n . 3^{3n} = 3^{3n-2} . 37^{n-1},$$

the number of figures in the period.

(18.) We will next briefly consider the residues of successive powers of a number not prime to the modulus.

Let a be the number, m the modulus.

Let $m = p\text{P}$: where p consists of powers of those primes which occur as factors of a, and P is prime to a.

Consider the series of residues

$$a . a^2 . a^3 \ldots (\text{mod } m).$$

Suppose that the first term which is repeated is a^r and suppose that

$$a^{r+t} \equiv a^r (\text{mod } m),$$

for which we seek the smallest values of r and t.

Then after the first $r - 1$ terms we shall have a period of t terms constantly repeated.

We have

$$a^r (a^t - 1) \equiv 0 \, (\text{mod P}p),$$

and P is prime to a. Therefore

$$a^t - 1 \equiv 0 \, (\text{mod P}) \quad \ldots\ldots\ldots\ldots \quad (\text{i.}).$$

Each prime factor of p is a factor of a, therefore $a^t - 1$ is prime to p. Therefore

$$a^r \equiv 0 \, (\text{mod } p) \quad \ldots\ldots\ldots\ldots \quad (\text{ii.}).$$

(i.) Shows that t is the exponent of a for modulus P.

(ii.) Shows that r is the least number that makes a^r divisible by p.

Corollaries.—(i.) When $a \equiv 1$ (mod P) $t = 1$ and the period consists of one term only. (ii.) When a is divisible by p the period starts from the first term. (iii.) When both these hold good, then every power of a is $\equiv a$ (mod m).

Examples.—Residues of powers of 2 mod 100.

$$100 = 5^2 . 2^2.$$

$$\text{Exp of } 2 \text{ mod } 5 = 4$$

$$\text{,, \quad ,, } \quad 5^2 = 20$$

therefore

$$t = 20,$$

and

$$2^2 \text{ is divided by } 2^2$$

therefore

$$r = 2.$$

$$2.\ \dot{4}.\ 8.\ 16.\ 32.\ 64.\ 28.\ 56.\ 12.\ 24.\ 48.\ 96.\ 92.\ 84.\ 68.\ 36.\ 72.\ 44.\ 88.\ 76.\ 5\dot{2},$$

the points indicating the period.

Powers of 5 mod $1000 = 2^3 \,.\, 5^3$.

$$5 \text{ has exp } 2 \text{ mod } 2^3,$$

therefore

$$t = 2,$$

$$5^3 \text{ is divided by } 5^3,$$

therefore

$$r = 3.$$

$$5.\ 25.\ 1\dot{2}5.\ 6\dot{2}5.$$

Powers of 5 mod $1000000 = 2^6 \,.\, 5^6$.

$$5 \text{ has exp } 2^4 = 16 \text{ mod } 2^6,$$

therefore

$$t = 16,$$

and

$$5^6 \text{ is divisible by } 5^6,$$

therefore

$$r = 6.$$

$$5.\ 25.\ 125.\ 625.\ 3125.\quad \begin{array}{r} \left.\begin{array}{r} 15625 \\ 78125 \\ 390625 \\ 953125 \\ 765625 \\ 828125 \\ 140625 \\ 703125 \\ 515625 \\ 578125 \\ 890625 \\ 453125 \\ 265625 \\ 328125 \\ 640625 \\ 203125 \end{array}\right\} \\ \hline 15625 \end{array}$$

Powers of $57 \,.\, 5^6 = 890625$ mod 1000000.

$$\text{Exp of } 57 \,.\, 5^6 \text{ mod } 2^6 \text{ is } 1 \ (890625 \equiv 1 \pmod{2^6}),$$

and

$$57 \,.\, 5^6 \text{ is divisible by } 5^6,$$

therefore

$$t = 1$$

and

$$r = 1,$$

and so all powers of 890625 are $\equiv 890625 \pmod{1000000}$.

$$\begin{array}{l} 890625 \\ \underline{890625} \\ 453125 \\ 81250 \\ 3750 \\ \underline{25} \\ 890625 \end{array}$$

(19.) Let $m = p_1 p_2 p_3 \ldots$, where $p_1, p_2, p_3, \ldots$ are co-prime.

Take any number a prime to m.

Suppose that

$$\begin{aligned} a &\equiv \alpha_1 \pmod{p_1}, \\ &\equiv \alpha_2 \pmod{p_2}, \\ &\equiv \alpha_3 \pmod{p_3}, \\ &\quad \&\text{c.} \end{aligned}$$

Since a is prime to m and therefore to p_1, it follows that α_1 is prime to p_1. So α_2 is prime to p_2, α_3 to p_3, &c.

Suppose now that, conversely, $\alpha_1, \alpha_2, \alpha_3, \ldots$ prime to $p_1, p_2, p_3, \ldots$ are given, and we wish to find a, such that it is congruent to $\alpha_1 \pmod{p_1}$, $\alpha_2 \pmod{p_2}$, &c.

Let x_1 be determined from

$$x_1 p_2 p_3 \ldots \equiv 1 \pmod{p_1},$$

which can be done, and in one way only, since $p_2 p_3 \ldots$ is prime to p_1; and when x_1 is found let $x_1 p_2 p_3 \ldots \equiv \xi_1$. Determine similarly $\xi_2, \xi_3 \ldots$ Then the number a is given by

$$a \equiv \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 + \ldots \pmod{m}.$$

For

$$\xi_2, \xi_3, \ldots \text{ are all congruent to zero mod } p_1,$$

and

$$\xi_1 \text{ is congruent to unity mod } p_1.$$

Therefore

$$a \equiv \alpha_1 \pmod{p_1},$$

and similarly

$$\equiv \alpha_2 \pmod{p_2},$$
$$\text{\&c.,}$$

and therefore

$$a \equiv \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3 + \ldots \pmod{m},$$

a formula which gives the value of a corresponding to given values of the α's, the coefficients ξ being independent of the α's and depending only on the moduli $p_1, p_2 \ldots$

Note.—The principal use to be made of this proposition and the next will be for the special case when $p_1, p_2, \ldots$ are the powers of primes of which the modulus m is the product.

(20.) Let us take two numbers expressed in this form,

$$a \equiv \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m},$$
$$b \equiv \beta_1\xi_1 + \beta_2\xi_2 + \ldots \pmod{m},$$

in which

$$\left.\begin{aligned} \xi_1 &\equiv 1 \pmod{p_1} \\ &\equiv 0 \left(\text{mod } \frac{m}{p_1}\right) \end{aligned}\right\}$$

with similar relations for $\xi_2, \xi_3, \ldots$

Let us form the product of a and b.

By Proposition (18), Corollary (iii.), all powers of ξ_1 are congruent to ξ_1, mod m, therefore

$$\alpha_1\beta_1\xi_1^2 \equiv \alpha_1\beta_1\xi_1 \pmod{m}.$$

Taking any cross-term such as $\alpha_1\beta_2\xi_1\xi_2$, since

$$\xi_1 \equiv 0 \pmod{p_2p_3 \ldots},$$
$$\xi_2 \equiv 0 \pmod{p_1p_3 \ldots},$$

therefore

$$\xi_1\xi_2 \equiv 0 \pmod{m}.$$

Hence

$$ab \equiv \alpha_1\beta_1\xi_1 + \alpha_2\beta_2\xi_2 + \alpha_3\beta_3\xi_3 + \ldots \pmod{m},$$

i.e., the multiplication of numbers expressed thus is simply effected by forming the products of the coefficients of ξ_1, $\xi_2 \ldots$

Corollary (i.). If

$$a \equiv \alpha_1 \xi_1 + \alpha_2 \xi_2 + \ldots + \pmod{m},$$

then

$$a^s \equiv \alpha_1^s \xi_1 + \alpha_2^s \xi_2 + \ldots \pmod{m}.$$

Corollary (ii.). If

$$a \equiv \alpha_1 \xi_1 + \alpha_2 \xi_2 + \ldots \pmod{m},$$

then a is congruent to the product of the numbers

$$\begin{cases} a_1 \equiv \alpha_1 \xi_1 + \xi_2 + \ldots \pmod{m}, \\ a_2 \equiv \xi_1 + \alpha_2 \xi_2 + \ldots \pmod{m}, \\ \text{\&c.} \end{cases}$$

Or, since $\xi_1 + \xi_2 + \ldots \equiv 1 \pmod{m}$ (*cf.* Prop. 10),

$$\begin{cases} a_1 \equiv (\alpha_1 - 1)\, \xi_1 + 1 \pmod{m}, \\ a_2 \equiv (\alpha_2 - 1)\, \xi_2 + 1 \pmod{m}, \\ \ldots\ldots\ldots\ldots \end{cases}$$

Examples.—To find a, modulus 308, so that

$$\begin{aligned} a &\equiv \alpha_1 \pmod{4} \\ &\equiv \alpha_2 \pmod{7} \\ &\equiv \alpha_3 \pmod{11} \end{aligned}$$

$$\begin{array}{rrr} 7.\ 11.\ x_1 \equiv 1 \pmod{4} & 4.\ 11.\ x_2 \equiv 1 \pmod{7} & 4.\ 7.\ x_3 \equiv 1 \pmod{11} \\ 3.\ \ 3.\ x_1 \equiv 1 \pmod{4} & 2.\ x_2 \equiv 1 \pmod{7} & 6.\ x_3 \equiv 1 \pmod{11} \\ x_1 \equiv 1 \pmod{4} & x_2 \equiv 4 \pmod{7} & x_3 \equiv 2 \pmod{11} \\ \xi_1 \equiv 77 \pmod{308} & \xi_2 \equiv 176 \pmod{308} & \xi_3 \equiv 56 \pmod{308} \end{array}$$

and so

$$a \equiv 77\, \alpha_1 + 176\, \alpha_2 + 56\, \alpha_3 \pmod{308},$$

e.g.,

$$\begin{aligned} a &\equiv 3 \pmod{4} \\ &\equiv 6 \pmod{7} \\ &\equiv 1 \pmod{11} \end{aligned}$$

then

$$\begin{aligned} a &\equiv 77.\ 3 + 176.\ 6 + 56.\ 1 \pmod{308} \\ &\equiv 111 \ (\text{mod. } 308), \end{aligned}$$

and

$$\begin{aligned} 111 &\equiv (77.\,3 + 176 + 56)\,(77 + 176.\,6 + 56)\,(77 + 176. + 56) \\ &\equiv 155.\,265 \pmod{308}. \end{aligned}$$

(21.) The number of numbers which belong to a given exponent when the modulus is a power of a prime.

I. Let the prime be an odd prime, and p^λ the modulus.

In Proposition (12), Corollary, we saw that for mod p^λ primitive roots exist.

Let g be a primitive root.

The numbers $g, g^2, g^3, \ldots g^{\phi(p^\lambda)} (\mathrm{mod}\, p^\lambda)$ are congruent to the complete set of numbers less than p^λ and prime to it.

The exponent to which any one of these numbers, g^s, belongs is t, where

$$\left.\begin{aligned} \phi(p^\lambda) &= \kappa t \\ \text{and} \qquad s &= \kappa\sigma \end{aligned}\right\} \text{ and } t \text{ and } \sigma \text{ are co-prime (Prop. 4).}$$

For any given value of t the value of $\kappa = \phi(p^\lambda)/t$ is given.

σ may then have any value prime to t such that

$$s \ngtr \phi(p^\lambda),$$

i.e.,

$$\kappa\sigma \ngtr \kappa t$$

$$\sigma \ngtr t.$$

Hence σ may have each of the $\phi(t)$ values of the numbers less than t and prime to it.

Therefore there are $\phi(t)$ numbers having t as their exponent (mod m).

II. Let the modulus be 2^κ. ($\kappa \geqq 3$).

In this case we have seen (Proposition 15) that there are 2^s numbers with exponent 2^s (if $s > 1$): and 3 numbers with exponent 2; and 1 number with exponent unity.

When the modulus is 2^2 there is one number with exponent 2 and one with unity.

When the modulus is 2 there is one number (unity) with exponent unity.

Definition.—When a number m is expressed in the form $m = 2^\kappa P_1^{\lambda_1} P_2^{\lambda_2} \ldots$ where $P_1, P_2, \ldots$ are different odd primes, it will be convenient to speak of $2^\kappa, P_1^{\lambda_1}, P_2^{\lambda_2} \ldots$ as the *principal factors* of m.

(22.) The number of numbers, each of which has, as exponent, some power of a prime p, for modulus m, p being a divisor of $\phi(m)$.

Let

$$m = 2^\kappa P_1^{\lambda_1} P_2^{\lambda_2}, \ldots$$

$$\phi(P_1^{\lambda_1}) = 2^{\kappa_1} p^{l_1} q^{m_1}, \ldots$$

$$\phi(P_2^{\lambda_2}) = 2^{\kappa_2} p^{l_2} q^{m_2}, \ldots$$

&c.

Any number has for its exponent, modulus m, the L.C.M. of its separate exponents for moduli 2^κ, $P_1^{\lambda_1}$, $P_2^{\lambda_2}$, . . . the principal factors of m. (Proposition (9), Corollary.)

Hence, when the exponent, modulus m, is a power of a prime p, the exponent for each of the principal factors of m as moduli, must each be either unity or some power of p.

Conversely, if we take a set of numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$ one for each of the moduli 2^κ, $P_1^{\lambda_1}$, . . . such that the exponent of each, for its own modulus, is unity or some power of p, then the number $a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \ldots \pmod{m}$, will have a power of p as exponent, modulus m.

The numbers ξ are given by

$$\begin{array}{lll} \xi_0 \equiv 1 \pmod{2^\kappa}, & \xi_1 \equiv 1 \pmod{P_1^{\lambda_1}}, & \\ \quad \equiv 0 \left(\operatorname{mod} \dfrac{m}{2^\kappa}\right), & \quad \equiv 0 \left(\operatorname{mod} \dfrac{m}{P^{\lambda_1}}\right), & \&c. \end{array}$$

Thus, by giving to the α's all possible sets of values consistently with each having unity or a power of p as exponent, we shall obtain all the numbers (mod m) which have (unity or) a power of p as exponent.

There are

$$\left\{\begin{array}{llllll} \phi\,(p^{l_1}) & \text{numbers} & \text{which} & \text{have exp} & p^{l_1}, & \text{mod } P_1^{\lambda_1}. \quad \text{(Prop. 21.)} \\ \phi\,(p^{l_1-1}) & ,, & ,, & ,, & p^{l_1-1} & ,, \\ \quad \&c. & & & & & \\ \quad 1 & ,, & ,, & ,, & 1 & ,, \end{array}\right.$$

Hence there are $\phi\,(p^{l_1}) + \phi\,(p^{l_1-1}) + \ldots + 1 = p^{l_1}$ numbers, mod $P_1^{\lambda_1}$, which have unity or a power of p as exponent.

Similarly, there are p^{l_2} for mod $P_2^{\lambda_2}$, and so on.

Hence, in

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m},$$

$$\left.\begin{array}{l} \text{we can give 1 value (viz., unity) to } \alpha_0 \\ \text{any one of } p^{l_1} \text{ values to } \alpha_1 \\ \quad ,, \qquad p^{l_2} \quad ,, \qquad \alpha_2 \\ \qquad \&c., \qquad\qquad \&c. \end{array}\right\}$$

and then a has unity or a power of p as exponent, mod m; moreover, in this way, all such numbers are obtained.

We thus obtain $p^{l_1+l_2+\cdots} = p^{\Sigma l}$ incongruent numbers, each of which has a power of p (or unity) for exponent.

In the case when $p = 2$

for mod 2^κ, all $2^{\kappa-1}$ numbers have powers of 2 as exponents,

„ $P_1^{\lambda_1}$, $\phi(2^{\kappa_1}) + \phi(2^{\kappa_1-1}) + \ldots + 1 = 2^{\kappa_1}$ have powers of 2 as exponents,

„ $P_2^{\lambda_2}$, $\phi(2^{\kappa_2}) + \phi(2^{\kappa_2-1}) + \ldots + 1 = 2^{\kappa_2}$ „ „ „

&c. &c. &c.

Hence, in all, there are $2^{\kappa-1+\kappa_1+\kappa_2+\cdots}$ numbers, which have unity or a power of 2 as exponent.

Note.—Any number a, mod m, with any exponent, must be congruent to a product of one number from each set of $2^{\kappa-1+\kappa_1+\kappa_2+\cdots}$ numbers, with powers of 2 as exponent, $p^{\Sigma l}$ with powers of p as exponent, . . . &c. (Proposition 10.)

Hence in all we get from these

$$\begin{aligned} & 2^{\kappa-1+\kappa_1+\cdots} p^{\Sigma l} q^{\Sigma m} \ldots \text{ numbers,} \\ = {} & \phi(2^\kappa)\, \phi(P_1^{\lambda_1}) \ldots \quad \text{„} \\ = {} & \phi(m) \text{ numbers, } \textit{i.e.}\text{, the complete set of numbers prime to } m. \end{aligned}$$

(23.) The number of numbers having exponent p^s, mod m, p^s being a divisor of the greatest exponent.

Let a be such a number and let

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m},$$

then the exponent of

$$\left.\begin{array}{c} \alpha_0 \text{ mod } 2^\kappa \\ \alpha_1 \text{ mod } P_1^{\lambda_1} \\ \alpha_2 \text{ mod } P_2^{\lambda_2} \\ \text{\&c.} \end{array}\right\}$$

must each be unity or some power of p, and the greatest power of p must be p (Proposition (9), Corollary); p being an odd prime α_0 is necessarily unity.

There are

$$\begin{array}{l} \phi(p^{l_1}) \text{ numbers, mod } P_1^{\lambda_1} \text{ with exp } p^{l_1}, \\ \phi(p^{l_1-1}) \quad \text{„} \quad \text{„} \quad \text{„} \quad \text{„} \quad p^{l_1-1}, \\ \qquad \text{\&c}, \qquad\qquad \text{\&c.}, \end{array}$$

and therefore there are

$$\begin{array}{l} p^{l_1} \text{ numbers with exp a power of } p \leqq p^{l_1}, \\ p^{l_1-1} \quad \text{„} \quad \text{„} \quad \text{„} \quad \text{„} \quad \leqq p^{l_1-1}, \\ \qquad \text{\&c.}, \qquad\qquad \text{\&c.} \end{array}$$

Hence if

$$\left.\begin{array}{ll} & l_1 < s \text{ there are } p^{l_1} \text{ numbers, mod } \mathrm{P}_1^{\lambda_1}, \text{ with exp a power of } p \not> p^s \\ \text{and if} & \\ & l_1 \overset{=}{>} s \text{ there are } p^{s} \text{ numbers, mod } \mathrm{P}_1^{\lambda_1}, \text{ with exp a power of } p \not> p^s \end{array}\right\}$$

Suppose that $(l_1)_s$ stands for s if $l_1 \overset{=}{>} s$ and for l_1 if $l_1 < s$, then in either case there are $p^{(l_1)_s}$ numbers, mod $\mathrm{P}_1^{\lambda_1}$, whose exponents are powers of $p \not> p^s$.

Similarly there are $p^{(l_2)_s}$ numbers mod $\mathrm{P}_2^{\lambda_2}$, whose exponents are powers of $p \not> p^s$ and so for $\mathrm{P}_2^{\lambda_2}$, &c.

Giving any one of these $p^{(l_1)_s}$ values to α_1, $p^{(l_2)_s}$ values to α_2, &c., we obtain

$$p^{(l_1)_s + (l_2)_s + \dots} = p^{(\Sigma l)_s} \text{ numbers, mod } m,$$

whose exponents are powers of $p \not> p^s$; where in $p^{(\Sigma l)_s}$ each number l in $l_1 + l_2 + \dots$ is to be replaced by s if it exceeds s.

Similarly, there are $p^{(\Sigma l)_{s-1}}$ numbers, mod m, whose exponents are powers of $p \not> p^{s-1}$ &c., &c., $p^{(\Sigma l)_1}$ numbers with exponents p or unity, and 1 number with exponent 1.

Hence the number of numbers whose exponent is p^s, mod m, is

$$p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}}, \text{ which when } s = 1 \text{ is } p^{(\Sigma l)_1} - 1.$$

Corollary.—If p^s be the highest power of p which can be an exponent for mod m, *i.e.*, if p^s is the highest power of p that divides the greatest exponent, *i.e.*, if s is the greatest of the numbers $l_1, l_2, \dots$ then

$$1 + (p^{(\Sigma l)_1} - 1) + (p^{(\Sigma l)_2} - p^{(\Sigma l)_1}) + \dots + (p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}}) = p^{(\Sigma l)_s}$$

is the number of numbers having as exponent (mod m) a power of p (or unity) as exponent.

Now since s is equal to the greatest of the quantities $l_1, l_2, \dots$ (or rather is not less than any) $(\Sigma l)_s = \Sigma l$ and therefore $p^{(\Sigma l)_s} = p^{\Sigma l}$: the result which was obtained in the last proposition.

(23A.) To find the number of numbers having exponent 2^σ, mod m.

(It will be convenient to write now $\kappa = \kappa' + 2$, so that $m = 2^{\kappa'+2}\mathrm{P}_1^{\lambda_1}\mathrm{P}_2^{\lambda_2}\dots$).

If a be any such number, and

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \dots (\text{mod } m),$$

the exponents of

$$\left.\begin{array}{c} \alpha_0 \text{ mod } 2^\kappa \\ \alpha_1 \text{ mod } \mathrm{P}_1^{\lambda_1} \\ \alpha_2 \text{ mod } \mathrm{P}_2^{\lambda_2} \\ \text{\&c.} \end{array}\right\}$$

must each be unity or a power of 2, and the greatest power of 2 must be 2^σ.

First suppose that $\kappa \geqq 3$, and so $\kappa' \geqq 1$.

Then there are, mod $2^{\kappa'+2}$,

$$\left.\begin{array}{llll}
2^{\kappa'} & \text{numbers with exponent} & & 2^{\kappa'} \\
2^{\kappa'-1} & \text{,,} \quad \text{,,} \quad \text{,,} & & 2^{\kappa'-1} \\
\vdots & & & \\
2^{2} & \text{,,} \quad \text{,,} \quad \text{,,} & & 2^{2} \\
3 & \text{,,} \quad \text{,,} \quad \text{,,} & & 2 \\
1 & \text{number} \quad \text{,,} \quad \text{,,} & & 1
\end{array}\right\}$$

Hence there are

$$\begin{array}{lll}
2^{\kappa'+1} & \text{numbers with exponent a power of 2} & \not> 2^{\kappa'} \\
2^{\kappa'} & \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} & \not> 2^{\kappa'-1} \\
\vdots & & \\
2^{2} & \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} & \not> 2 \\
1 & \text{number} \quad \text{,,} \quad \text{,,} \quad \text{,,} & \not> 2^{0}
\end{array}$$

If

$\sigma > \kappa'$ then there are $2^{\kappa'+1}$ numbers with exponent a power of 2 $\not> 2^{\sigma}$, mod $2^{\kappa'+2}$,

and if

$\sigma \leqq \kappa'$ then ,, ,, $2^{\sigma+1}$,, ,, ,, ,, ,, ,,

(this holds unless $\sigma = 0$, and then there is one number (unity) with exponent 1).

In either case, then, there are $2^{(\kappa')_{\sigma}+1}$ numbers with powers of 2 $\not> 2^{\sigma}$ as exponents, where $(\kappa')_{\sigma}$ is to be replaced by

$$\left.\begin{array}{l}
\sigma \text{ if } \kappa' > \sigma \\
\kappa' \text{ ,, } \kappa' \leqq \sigma
\end{array}\right\}.$$

Next for mod $P_1^{\lambda_1}$ there are

$$\begin{array}{lll}
2^{\kappa_1-1} & \text{numbers with exponent a power of 2} & \not> 2^{\kappa_1-1} \\
2^{\kappa_1-2} & \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} & \not> 2^{\kappa_1-2} \\
\text{\&c.} & &
\end{array}$$

Hence (using the same notation) there are $2^{(\kappa_1)_{\sigma}}$ numbers (mod $P_1^{\lambda_1}$) having exponent a power of 2 $\not> 2^{\sigma}$.

Thus α_0 may have each of $2^{(\kappa')_{\sigma}}$ values, α_1 each of $2^{(\kappa_1)_{\sigma}}$ values, &c., and the corresponding value of a has exponent a power of 2 (mod m), $\not> 2^{\sigma}$.

Hence the number of these is $2^{(\Sigma\kappa)_{\sigma}+1}$, where in $\Sigma\kappa = \kappa' + \kappa_1 + \kappa_2 + \ldots$ each number κ is to be replaced by σ if it exceeds σ.

Similarly the number of numbers whose exponent, mod m, is a power of $2 \ngtr 2^{\sigma-1}$ is

$$2^{(\Sigma\kappa)_{\sigma-1}+1}.$$

Hence the number of numbers with exponent 2^{σ} (mod m) is

$$2^{(\Sigma\kappa)_{\sigma}+1} - 2^{(\Sigma\kappa)_{\sigma-1}+1}.$$

This holds for $\sigma \geqq 2$. When $\sigma = 1$ the number of numbers with exponent 2 is

$$2^{(\Sigma\kappa)_1+1} - 1.$$

Secondly suppose that $\kappa = 2$, and so $\kappa' = 0$.

Then, if σ is > 1, α_0 may be either 1 or 3, two values.

Hence the number of numbers with exponent 2^{σ} is

$$2^{(\Sigma\kappa)_{\sigma}+1} - 2^{(\Sigma\kappa)_{\sigma-1}+1},$$

which agrees with the above when κ' is omitted.

If $\sigma = 1$, the number of numbers with exponent 1 or 2, mod 2^2, is 2.

Therefore the number of numbers with exponent 2, mod m, is

$$2^{(\Sigma\kappa)_1+1} - 1,$$

which agrees with the above when κ' is put $= 0$.

Thirdly, let $\kappa = 1$. Then α_0 must be unity and the number of numbers with exponent 2^{σ} is $2^{(\Sigma\kappa)_{\sigma}} - 2^{(\Sigma\kappa)_{\sigma-1}}$ where $\Sigma\kappa = \kappa_1 + \kappa_2 + \ldots$

Fourthly, when ($\kappa = 0$) m is odd, then again the number required is $2^{(\Sigma\kappa)_{\sigma}} - 2^{(\Sigma\kappa)_{\sigma-1}}$.

To collect the results :—

When

$$\begin{array}{ll} & m = 2^{\kappa'+2}P_1^{\lambda_1}P_2^{\lambda_2}\ldots \\ \text{or} & \\ & \phantom{m = {}}2^2P_1^{\lambda_1}P_2^{\lambda_2}\ldots \end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}\begin{array}{r} \text{there are } 2^{(\Sigma\kappa)_1+1} - 1 \text{ numbers with exp } 2, \\ \\ 2^{(\Sigma\kappa)_{\sigma}+1} - 2^{(\Sigma\kappa)_{\sigma-1}+1} \text{ numbers with exp } 2^{\sigma}. \end{array}$$

And when

$$\begin{array}{ll} & m = 2P_1^{\lambda_1}P_2^{\lambda_2}\ldots \\ \text{or} & \\ & \phantom{m = {}}P_1^{\lambda_1}P_2^{\lambda_2}\ldots \end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}\begin{array}{r} \text{there are } 2^{(\Sigma\kappa)_1} - 1 \text{ numbers with exp } 2, \\ \\ 2^{(\Sigma\kappa)_{\sigma}} - 2^{(\Sigma\kappa)_{\sigma-1}} \text{ numbers with exp } 2^{\sigma}. \end{array}$$

(23B.) The last two propositions give us the number of numbers belonging to any exponent $t = 2^a p^b q^c \ldots$

For the complete set of these numbers is formed by taking all possible products of a number with exponent 2^a, one with exponent p^b, &c.

Hence the number of numbers with exponent t is the product of the number of numbers that belong to each of its principal factors as exponent.

Examples.—The number of numbers that belong to any exponent for mod $m = 2^4. 13. 17. 19.$

The greatest exponent is the L.C.M. of

$$\left.\begin{array}{l} \phi(2^4) = 2^3 \\ \phi(13) = 2^2. 3 \\ \phi(17) = 2^4 \\ \phi(19) = 2. 3^2 \end{array}\right\} = 2^4. 3^2. \qquad \phi(m) = 2^{10}. 3^3.$$

For the 2-power exponent numbers we have for the κ's

$$\kappa' = 2 \qquad \kappa_1 = 2 \qquad \kappa_2 = 4 \qquad \kappa_3 = 1.$$

$$\left.\begin{array}{l} (\Sigma\kappa)_1 = 4 \\ (\Sigma\kappa)_2 = 7 \\ (\Sigma\kappa)_3 = 8 \\ (\Sigma\kappa)_4 = 9 \end{array}\right\}.$$

Therefore

$$\left.\begin{array}{llll} \text{belonging to exp } & 2^4 & \text{there are } 2^{10} - 2^9 & \text{numbers} \\ \quad ,, \quad ,, & 2^3 & \text{there are } 2^9 - 2^8 & \text{numbers} \\ \quad ,, \quad ,, & 2^2 & \text{there are } 2^8 - 2^5 & \text{numbers} \\ \quad ,, \quad ,, & 2 & \text{there are } 2^5 - 1 & \text{numbers} \end{array}\right\}.$$

For the 3-power exponent numbers we may take $l_1 = 1$, $l_2 = 2$,

$$(\Sigma l)_1 = 2,$$
$$(\Sigma l)_2 = 3.$$

$$\left.\begin{array}{llll} \text{Belonging to exp } & 3^2 & \text{there are } 3^3 - 3^2 & \text{numbers} \\ \quad ,, \quad ,, & 3 & \text{there are } 3^2 - 1 & \text{numbers} \end{array}\right\}.$$

The number of numbers belonging to any other exponent is at once got from these by multiplying, *e.g.*, the number of numbers with the greatest exponent $2^4. 3^2$ is $(2^{10} - 2^9)(3^3 - 3^2)$.

(24.) We shall now establish a particular set of independent generators which generate the $\phi(m)$ numbers (mod m) prime to m.

Any number a, mod m, is expressible in the form

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m}.$$

Let

and let

$$\left.\begin{array}{l} g_1 \text{ be a primitive root of } P_1^{\lambda_1} \\ g_2 \quad ,, \qquad\qquad ,, \quad P_2^{\lambda_2} \\ \quad \&c. \qquad\qquad \&c., \\ \\ g_0 \text{ have exp } 2^{\kappa-2} \text{ mod } 2^{\kappa} \\ f \text{ be} \equiv 2^{\kappa} - 1 \text{ or } 2^{\kappa-1} - 1 \text{ mod } 2^{\kappa} \end{array}\right\}$$

(In the case when $\kappa = 2$, f does not occur, when $\kappa = 1$ or 0, neither f nor g_0.)

Then

and

$$\left.\begin{array}{l} \alpha_1 \equiv g_1^{i_1} (\text{mod } P_1^{\lambda_1}) \\ \alpha_2 \equiv g_2^{i_2} (\text{mod } P_2^{\lambda_2}) \\ \quad \&c., \\ \\ \alpha_0 \equiv g_0^{i_0} f^j (\text{mod } 2^{\kappa}) \end{array}\right\} \text{say.}$$

Therefore

$$\begin{aligned} a &\equiv g_0^{i_0} f^j \xi_0 + g_1^{i_1} \xi_1 + \ldots (\text{mod } m) \\ &\equiv (f\xi_0 + \xi_1 + \ldots)^j (g_0\xi_0 + \xi_1 + \ldots)^{i_0} (\xi_0 + g_1\xi_1 + \ldots)^{i_1} \ldots (\text{mod } m) \\ &\equiv [(f-1)\,\xi_0 + 1]^j [(g_0 - 1)\,\xi_0 + 1]^{i_0} [(g_1 - 1)\,\xi_1 + 1]^{i_1} \ldots (\text{mod } m), \end{aligned}$$

where

$$\begin{array}{l} j \text{ is referred to mod } 2, \\ i_0 \quad ,, \qquad ,, \quad \text{mod } 2^{\kappa-2}, \\ i_1 \quad ,, \qquad ,, \quad \text{mod } \phi(P_1^{\lambda_1}), \\ \qquad\qquad \&c., \end{array}$$

and so the set of indices j, i_0, $i_1 \ldots$ correspond uniquely to the number a: and the numbers

$$(f-1)\,\xi_0 + 1, \quad (g_0 - 1)\,\xi_0 + 1, \quad (g_1 - 1)\,\xi_1 + 1, \ldots$$

are a set of independent generators, with exponents

$$2, \qquad 2^{\kappa-2}, \qquad \phi(P_1^{\lambda_1}), \ldots$$

which generate completely the $\phi(m)$ numbers, mod m, which are prime to m.

Example.—Take the modulus

$$m = 112 = 2^4.\ 7. \qquad \phi(m) = (2^3)\,(2.\ 3) = 2^4.\ 3 = 48.$$

If

$$a \equiv \alpha_0 \,(\text{mod } 2^4),$$

and

$$a \equiv \alpha_1 \pmod{7},$$
$$a \equiv \alpha_0 \xi_0 + \alpha_1 \xi_1 \pmod{112}.$$

$$\begin{array}{ll} 7x_0 \equiv 1 \pmod{2^4} & 2^4 x_1 \equiv 1 \pmod{7} \\ x_0 \equiv 7 \pmod{2^4} & 2x_1 \equiv 1 \pmod{7} \\ \xi_0 \equiv 49 \pmod{112} & x_1 \equiv 4 \pmod{7} \\ & \xi_1 \equiv 64 \pmod{112}, \end{array}$$

$$a \equiv 49\alpha_0 + 64\alpha_1 \pmod{112}.$$

We will take

$$f = 7 \ (= 2^3 - 1)$$
$$g_0 = 3 \text{ (having exp } 2^2\text{)},$$

and

$$g_1 = 3 \text{ (a primitive root of 7)}.$$

Then

$$(f - 1)\,\xi_0 + 1 = 6.\,49 + 1 = 295 \equiv 71 \pmod{112},$$
$$(g_0 - 1)\,\xi_0 + 1 = 2.\,49 + 1 = \ \ 99 \pmod{112},$$
$$(g_1 - 1)\,\xi_1 + 1 = 2.\,64 + 1 = 129 \equiv 17 \pmod{112};$$

and

$$\left.\begin{array}{l} 71 \text{ with exp } 2 \\ 99 \text{ with exp } 4 \\ 17 \text{ with exp } 6 \end{array}\right\} \text{generate the 48 numbers prime to 112.}$$

The following table gives the indices corresponding to any number :—

Numbers	1.	3.	5.	9.	11.	13.	15.	17.	19.	23.	25.	27.	29.	31.	33.	37.	39.	41.	43.	45.	47.	51.	53.	55.	57.	59.
Ind. of 17	0	1	5	2	4	3	0	1	5	2	4	3	0	1	5	2	4	3	0	1	5	2	4	3	0	1
Ind. of 99	0	1	1	2	3	3	2	0	1	0	2	3	3	2	0	1	0	2	3	3	2	1	1	0	2	3
Ind. of 71	0	0	1	0	0	1	1	0	0	1	0	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0

Numbers	61.	65.	67.	69.	71.	73.	75.	79.	81.	83.	85.	87.	89.	93.	95.	97.	99.	101.	103.	107.	109.	111
Ind. of 17	5	2	4	3	0	1	5	2	4	3	0	1	5	2	4	3	0	1	5	2	4	3
Ind. of 99	3	0	1	1	0	2	3	2	0	1	1	0	2	3	2	0	1	1	0	3	3	2
Ind. of 71	1	0	0	1	1	0	0	1	0	0	1	1	0	1	1	0	0	1	1	0	1	1

We proceed now to the point we have been approaching from the beginning, viz., the investigation of the mode of formation of and the relations among the most general set of numbers capable of generating the $\phi(m)$ numbers (modulus m) which are prime to the modulus m.

(25.) Suppose $G_1, G_2, G_3, \ldots$ with exponents $t_1, t_2, \ldots$ are capable of acting as complete generators of the $\phi(m)$ numbers.

Then

$$\text{(i.)}\ t_1 t_2 t_3 \ldots = \phi(m)$$

and

(ii.) the generators must be independent.

Since the numbers G generate the complete set of $\phi(m)$ numbers they can generate in particular the numbers which have a power of a prime p (which is a factor of $\phi(m)$) as exponent.

Suppose the highest powers of p which occur in $t_1, t_2, \ldots$ are $p^{s_1}, p^{s_2}, \ldots$ respectively.

Say

$$\left.\begin{array}{c} t_1 = p^{s_1} t'_1, \\ t_2 = p^{s_2} t'_2, \\ \text{\&c.} \end{array}\right\}$$

Then we can express the numbers G thus (Proposition 10) :—

$$\begin{array}{c} G_1 \equiv g_1 h_1 \ (\text{mod } m), \\ G_2 \equiv g_2 h_2 \ (\text{mod } m), \\ \text{\&c.,} \end{array}$$

when g_1 has exponent p^{s_1} and h_1 has exponent t'_1, &c.

Suppose now that

$G_1^{i_1} G_2^{i_2} G_3^{i_3} \ldots$ is congruent to a number with a power of p as exponent,

and so

$(g_1^{i_1} g_2^{i_2} \ldots)(h_1^{i_1} h_2^{i_2} \ldots)$,, ,, ,, ,, ,,

g_1, and therefore $g_1^{i_1}$, has a power of p as exponent (Proposition 4), and so for $g_2^{i_2}$, &c.

Therefore

$g_1^{i_1} g_2^{i_2} \ldots$ has a power of p as exponent.

The exponent of h_1, and therefore of $h_1^{i_1}$, is prime to p, and so for $h_2^{i_2}, \ldots$ &c.

Therefore the exponent of $h_1^{i_1} h_2^{i_2} \ldots$ which divides the L.C.M. of the exponents of $h_1^{i_1}, h_2^{i_2}, \ldots$ is prime to p.

Therefore the exponent of $(g_1^{i_1} g_2^{i_2} \ldots)(h_1^{i_1} h_2^{i_2} \ldots)$ is the L.C.M. of a power of p and of a number (the exponent of $h_1^{i_1} h_2^{i_2} \ldots$) prime to p. Now the exponent is to be a power of p. Hence the exponent of $h_1^{i_1} h_2^{i_2} \ldots$ must be unity.

Therefore

$$h_1^{i_1} h_2^{i_2} \ldots \equiv 1 \ (\text{mod } m).$$

Now we can show that this cannot be unless

$$h_1^{i_1} \equiv 1,\ h_2^{i_2} \equiv 1, \ldots \pmod{m}.$$

For if not suppose that at least $i_1 \not\equiv 0$ (modulus t'_1).
Then

$$h_1^{i_1} h_2^{i_2} \ldots \equiv h_1^{t'_1} \pmod{m},$$
$$h_2^{i_2} h_3^{i_3} \ldots \equiv h_1^{t'_1 - i_1} \pmod{m}.$$

Let

$$\begin{array}{lll} \mathrm{I}_2 \equiv i_2 \pmod{t'_2} & \text{and the same for } \mathrm{I}_3, \text{\&c.} : \text{ and} & \mathrm{I}_1 \equiv - i_1 \pmod{t'_1} \\ \quad \equiv 0 \pmod{p^{s_2}}, & & \quad \equiv \quad 0 \pmod{p^{s_1}}, \\ \text{so that } \mathrm{I}_2 < t_2 & & \text{so that } \mathrm{I}_1 < t_1 \\ \text{and } \left. \begin{array}{l} g_2^{\mathrm{I}_2} \equiv 1 \\ h_2^{\mathrm{I}_2} \equiv h_2^{i_2} \end{array} \right\} \pmod{m} & & \begin{array}{l} g_1^{\mathrm{I}_1} \equiv 1 \pmod{m} \\ h_1^{\mathrm{I}_1} \equiv h_1^{t'_1 - i_1} \pmod{m}, \end{array} \end{array}$$

then

$$\mathrm{G}_2^{\mathrm{I}_2} \mathrm{G}_3^{\mathrm{I}_3} \ldots \equiv \mathrm{G}_1^{\mathrm{I}_1} \pmod{m},$$

where

$$\mathrm{I}_1 \not\equiv 0 \pmod{t_1},$$

which is contrary to the supposition that the generators G are *independent*, and that therefore no two numbers of the $t_1 t_2 \ldots$ that they generate shall be congruent.

Therefore we must have

$$i_1 \equiv 0 \pmod{t'_1},$$
$$i_2 \equiv 0 \pmod{t'_2},$$
$$\text{\&c.}$$

and so

$$h_1^{i_1} \equiv 1 \pmod{m},$$
$$h_2^{i_2} \equiv 1 \pmod{m},$$
$$\text{\&c.}$$

and so

$$\mathrm{G}_1^{i_1} \equiv g_1^{i_1} \pmod{m},$$
$$\text{\&c.}$$

We have shown, then, that when a product of powers of the generators G, $\mathrm{G}_1^{i_1} \mathrm{G}_2^{i_2} \ldots$ is congruent to a number with a power of a prime p as exponent, then each factor, $\mathrm{G}_1^{i_1}$, $\mathrm{G}_2^{i_2}$, . . . must have a power of p for its exponent.

Thus, of the factors of the product to which G_1 is congruent, each factor having a principal factor of t_1 as exponent (Proposition 10), only one factor g_1 (viz., that which has, as exponent, a power of p) is effective in any power of G_1 which can be used to form a number with exponent a power of p, the product of the remaining factors h_1 being necessarily raised to such a power i_1 that $h_1^{i_1} \equiv 1 \pmod{m}$.

(It will be convenient to call the numbers which have a power of p as exponent, the "p-power-exponent numbers").

Suppose then that we take the generators G, and express each as a product in the manner of Proposition 10. Then take from each the factor (if any) which has a power of the prime p as exponent. Then, since the numbers G can generate all the numbers, modulus m, which have a power of p as exponent, and, since in the number G the factor g is alone effective in so doing, it follows that the set of numbers g generate completely the p-power-exponent numbers.

If we take from each generator G the factor which has a power of any other prime q as exponent, we obtain a set of numbers g', which generate the q-power-exponent numbers, and so on for each prime which divides $\phi(m)$.

We see now, that any set of complete independent generators, G, must be formed from these special sets of generators; the formation of each G being effected by taking one number (which may be unity) from each set and forming their product.

In order to obtain the most general set of generators, G, we have now only to obtain the most general method of producing each of these subsidiary sets of generators. We may then combine them as products (one from each set) in any manner we please.

(26.) Suppose that $\Gamma_1, \Gamma_2, \ldots$ independent generators, generate completely the p-power-exponent numbers.

The exponent of each number, Γ, must be some power of p.

Let them be $p^{n_1}, p^{n_2}, \ldots$ respectively.

One condition that the numbers, Γ, must satisfy is that the number of numbers they generate, which belong to any power of p, p^t as exponent, should agree with the number already found. (Proposition 23.)

Consider any number generated

$$a \equiv \Gamma_1^{i_1}\Gamma_2^{i_1} \ldots \pmod{m}.$$

Since $\Gamma_1, \Gamma_2, \ldots$ are *independent* generators, therefore the exponent of a is the greatest of the separate exponents of $\Gamma_1^{i_1}, \Gamma_2^{i_2}, \ldots$ (Proposition 8. Corollary).

Now Γ_1 has exponent p^{n_1}, therefore of the numbers of the form $\Gamma_1^{i_1}$, there are

$$\begin{array}{llll} p^{n_1} \text{ with exponent a power of } p & & & \ngtr p^{n_1}, \\ p^{n_1-1} \quad ,, & ,, & ,, & \ngtr p^{n_1-1}, \\ \quad \text{\&c.,} & & & \end{array}$$

and similarly for each of the others, $\Gamma_2, \Gamma_3. \ldots$

Hence the number of numbers of the form a which have, as exponent, a power of $p \ngtr p^t$, is $p^{(\Sigma n)_t}$ (using the notation of Proposition 23).

Hence the number of numbers, with exponent p^s generated by the numbers Γ, is

$$p^{(\Sigma n)_s} - p^{(\Sigma n)_{s-1}},$$

and, therefore, we must have

$$p^{(\Sigma n)_s} - p^{(\Sigma n)_{s-1}} = p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}},$$

for all values of s from 1 up to the greatest of the values of the numbers l.

Hence we get

$$\begin{aligned}(\Sigma n)_1 &= (\Sigma l)_1,\\ (\Sigma n)_2 &= (\Sigma l)_2,\\ (\Sigma n)_3 &= (\Sigma l)_3,\\ &\text{\&c.}\end{aligned}$$

The first of these equations shows that the number of numbers l is the same as that of the numbers n.

The second then shows that, of each set, the number of numbers which exceed 1 is the same.

The third then shows that, of each set, the number of numbers which exceed 2 is the same.

And so on.

Hence the two sets of numbers $l_1, l_2, \ldots$ and $n_1, n_2, \ldots$ *are identical* (in some order) *term for term.*

We have, therefore, shown that the most general set of p-power-exponent generators must have as exponents the powers of p, p^{h_1}, p^{h_2}, . . . , which occur, one each, as a principal factor of $\phi(\mathrm{P}_1{}^{\lambda_1})$, $\phi(\mathrm{P}_2{}^{\lambda_2})$ . . .

When $p = 2$ we have the special case of the 2-power-exponent generators.

First suppose $\kappa' \overset{=}{>} 1$, then

$$2^{(\Sigma\kappa)_\sigma + 1} - 2^{(\Sigma\kappa)_{\sigma-1} + 1} = 2^{(\Sigma n)_\sigma} - 2^{(\Sigma n)_{\sigma-1}},$$

and so

$$\begin{aligned}(\Sigma\kappa)_1 + 1 &= (\Sigma n)_1,\\ (\Sigma\kappa)_2 + 1 &= (\Sigma n)_2,\\ &\text{\&c.}\end{aligned}$$

Hence the set of numbers n are identical with the set κ *together with unity:* and so the exponents of the 2-power-exponent generators are

$$2.\ 2^{\kappa'}.\ 2^{\kappa_1}.\ 2^{\kappa_2}.\ \ldots$$

Secondly, suppose $\kappa = 2$, then $\kappa' = 0$, and the exponents of the generators are

$$2.\ 2^{\kappa_1}.\ 2^{\kappa_2}.\ \ldots$$

Lastly, when $\kappa = 0$ or 1, we have

$$2^{(\Sigma\kappa)_\sigma} - 2^{(\Sigma\kappa)_\sigma - 1} = 2^{(\Sigma n)_\sigma} - 2^{(\Sigma n)_\sigma - 1},$$

and therefore

$$\begin{gathered}(\Sigma\kappa)_1 = (\Sigma n)_1,\\ (\Sigma\kappa)_2 = (\Sigma n)_2,\\ \&c.,\end{gathered}$$

and so the numbers n are identical with the numbers κ.

Hence the exponents of the 2-power-exponent generators are

$$2^{\kappa_1}, 2^{\kappa_2}, 2^{\kappa_3} \ldots$$

We can now see the least possible number of numbers G that can generate the complete set of $\phi(m)$ numbers.

Since each number G contains not more than one generator from each of the subsidiary sets of generators as a factor, it follows that there cannot be less generators G than the number of generators in that subsidiary set which contains most.

Now, since $\phi(\mathrm{P}_1^{\lambda_1}) = \mathrm{P}^{\lambda_1 - 1}(\mathrm{P}_1 - 1)$, and P_1 is odd, therefore $\mathrm{P}_1 - 1$ is even. Hence $\kappa_1 = 1$ at least. Therefore of the generators of the 2-power-exponent numbers there are at least as many as there are odd principal factors $\mathrm{P}_1^{\lambda_1}, \mathrm{P}_2^{\lambda_2}, \ldots,$ in m: and so the number of 2-power-exponent generators is never less than the number of generators in any other subsidiary set.

Therefore when

$$\left.\begin{aligned} m &= \mathrm{P}_1^{\lambda_1}, \ldots \\ \text{or} \qquad & 2.\mathrm{P}_1^{\lambda_1}, \ldots \end{aligned}\right\}$$

the least number of generators G is the number of primes P_1, when

$$m = 2^2.\mathrm{P}_1^{\lambda_1}, \ldots$$

the least number of generators G is the number of primes P_1, + 1, and when

$$m = 2^{\kappa' + 2}\mathrm{P}_1^{\lambda_1}, \ldots$$

the least number of generators G is the number of primes P_1, + 2.

(27.) We shall next form a set of p-power-exponent generators of a particular kind, similar to the complete generators of Proposition (24).

Let a be any p-power-exponent number

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots \pmod{m}.$$

Then we must have

$\alpha_0 \equiv 1 \pmod{2^\kappa}$
α_1 with exp a power of p (or unity) mod $P_1^{\lambda_1}$
α_2 ,, ,, p ( ) mod $P_2^{\lambda_2}$
&c.

Let

γ_1 be a number with exp p^{l_1} mod $P_1^{\lambda_1}$
γ_2 ,, ,, p^{l_2} mod $P_2^{\lambda_2}$,
&c.,

(if in any case p is not a factor of $\phi(P^\lambda)$ γ is $\equiv 1$).

Then we may put

$\alpha_1 \equiv \gamma_1^{i_1} \pmod{P_1^{\lambda_1}}$
$\alpha_2 \equiv \gamma_2^{i_2} \pmod{P_2^{\lambda_2}}$,
&c.,

and therefore

$$a \equiv \xi_0 + \gamma_1^{i_1}\xi_1 + \gamma_2^{i_2}\xi_2 + \ldots \pmod{m}$$
$$a \equiv (\xi_0 + \xi_1 + \ldots)(\xi_0 + \gamma_1\xi_1 + \ldots)^{i_1}(\xi_0 + \xi_1 + \gamma_2\xi_2 + \ldots)^{i_2} \ldots \pmod{m}$$
$$\equiv g_1^{i_1} g_2^{i_2} \ldots \pmod{m}.$$

The first factor and each of the following in which the γ is $= 1$ is $\equiv 1 \pmod{m}$.

The number of factors remaining is the number of the principal factors P^λ which have a power of p as a factor of $\phi(P^\lambda)$.

These factors generate the p-power-exponent numbers: corresponding to each set of indices i is one of the numbers a and *vice versâ*.

These generators are only a very special kind of p-power-exponent generators inasmuch as each is congruent to unity for each but one of the principal factors of m as modulus. They will be called unitary generators and will be useful for the discussion of the more general type.

In the case when $p = 2$, let

f be $\equiv 2^\kappa - 1$ or $2^{\kappa-1} - 1 \pmod{2^\kappa}$
γ_0 have exp $2^{\kappa-2} \pmod{2^\kappa}$

(if $\kappa = 2$, f does not occur; if $\kappa = 0$ or 1, neither g nor f),

γ_1 have exp $2^{\kappa_1} \pmod{P_1^{\lambda_1}}$
γ_2 ,, $2^{\kappa_2} \pmod{P_2^{\lambda_2}}$,
&c.,

and thus

$$a \equiv (f\xi_0 + \xi_1 + \ldots)^{\gamma} (\gamma_0\xi_0 + \xi_1 + \ldots)^{i_0} (\xi_0 + \gamma_1\xi_1 + \ldots)^{i_1} \ldots \pmod{m},$$

and all the factors in this product (omitting those whose γ is $= 1$) form a complete set of 2-power-exponent generators.

Thus, in either case, when p is an odd prime or is equal to 2 we can form a set of p-power-exponent generators (having the exponents found to be necessary in Proposition 26), such that each is congruent to unity for all but one of the principal factors of m.

Example.—Let

$$m = 308 = 2^2 .\ 7 .\ 11.$$

$$\phi(m) = (2).\ (2.\ 3)\ (2.\ 5).$$

The highest exponent is 30.

There are

$$\left.\begin{array}{ll} \text{} & 7 \text{ numbers with exp } 2 \\ \text{and} & 1 \quad \text{,,} \quad \text{,,} \quad 1 \end{array}\right\} \text{(Example Proposition 26.)}$$

We will form unitary generators of these eight numbers. Since 2 enters only in the *square* into m, f is not needed.

We must take

$$\gamma_0 = 3 \text{ (with exp 2 mod } 2^2\text{)},$$
$$\gamma_1 = 6 \text{ (with exp 2 mod 7)},$$
$$\gamma_2 = 10 \text{ (with exp 2 mod 11)}.$$

$$\xi_0 = 77, \qquad \xi_1 = 176, \qquad \xi_2 = 56.$$

Thus

$$\left.\begin{array}{l} g_0 \equiv (\ 3 - 1)\ \ 77 + 1 = 155 \pmod{308} \\ g_1 \equiv (\ 6 - 1)\ 176 + 1 = 881 \equiv 265 \pmod{308} \\ g_2 = (10 - 1)\ \ 56 + 1 = 505 \equiv 197 \pmod{308} \end{array}\right\}$$

are a (in this case *the*) set of 2-power-exponent unitary generators.

The numbers with exponent 2 are given as products of powers of these by the following table of indices.

Numbers.	197.	265.	155.	111.	43.	153.	307.
Index of 155 . . .	0	0	1	1	1	0	1
Index of 265 . . .	0	1	0	1	0	1	1
Index of 197 . . .	1	0	0	0	1	1	1

(28.) In what follows we shall need the following lemma.

Consider

$$\begin{aligned} a_1x + b_1y + c_1z + \ldots &\equiv k_1 \pmod{p^a} \\ a_2x + b_2y + c_2z + \ldots &\equiv k_2 \pmod{p^a} \\ a_3x + b_3y + c_3z + \ldots &\equiv k_3 \pmod{p^a} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

as many congruences as unknown quantities.

What is necessary in order that the congruences may have one solution, for any assigned set of values of the k's, and *one only?*

Multiply the equations in order by the minors of the elements of the 1st column of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & . \\ a_2 & b_2 & c_2 & . \\ a_3 & b_3 & c_3 & . \\ . & . & . & . \end{vmatrix} \text{ and add.}$$

We get

$$x \begin{vmatrix} a_1 & b_1 & c_1 & . \\ a_2 & b_2 & c_2 & . \\ a_3 & b_3 & c_3 & . \\ . & . & . & . \end{vmatrix} \equiv \begin{vmatrix} k_1 & b_1 & c_1 & . \\ k_2 & b_2 & c_2 & . \\ k_3 & b_3 & c_3 & . \\ . & . & . & . \end{vmatrix} \pmod{p^a}.$$

In order that this may have one solution, and one only, the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & . \\ a_2 & b_2 & c_2 & . \\ a_3 & b_3 & c_3 & . \\ . & . & . & . \end{vmatrix}$$

must be prime to p^a, and, therefore, prime to p.

This being so, then to each set of values of the k's corresponds a single set of values of $x, y, z. \ldots$

(29.) We have now to find the most general type of a set of p-power-exponent generators. (The work is in no respect different in the case when $p = 2$.)

Let the exponents necessary for a set of generators be $p^{l_1}, p^{l_2}, \ldots p^{l_\mu}$ as determined by Proposition (26).

Let

$$\left.\begin{array}{llll} g_1 & \text{with exp.} & & p^{l_1} \\ g_2 & ,, & ,, & p^{l_2} \\ \vdots & & & \\ g_\mu & ,, & ,, & p^{l_\mu} \end{array}\right\} \text{be a set of \textit{unitary} generators. (Proposition 27.)}$$

Suppose that $\Gamma_1, \Gamma_2, \dots \Gamma_\mu$ are the most general set of p-power-exponent generators. We know already (Proposition 27) that their exponents are $p^{l_1}, p^{l_2}, \dots$ and we may suppose them to be so in this order.

Since each number Γ is itself a p-power-exponent number, each is expressible as a product of powers of the g's.

Let

$$\left.\begin{aligned} \Gamma_1 &\equiv g_1^{i_{11}} g_2^{i_{21}} g_3^{i_{31}} \dots g_\mu^{i_{\mu 1}} \pmod{m} \\ \Gamma_2 &\equiv g_1^{i_{12}} g_2^{i_{22}} g_3^{i_{32}} \dots g_\mu^{i_{\mu 2}} \pmod{m} \\ &\vdots \\ \Gamma_\mu &\equiv g_1^{i_{1\mu}} g_2^{i_{2\mu}} g_3^{i_{3\mu}} \dots g_\mu^{i_{\mu\mu}} \pmod{m} \end{aligned}\right\}$$

(*Note.*—$i_{r,s}$ is the index of g_r in the value of Γ_s.)

Since the exponent of Γ_1 is p^{l_1}, it follows that p^{l_1} is the least multiplier that makes

$$i_{21} p^{l_1} \equiv 0 \pmod{p^{l_2}},$$
$$i_{31} p^{l_1} \equiv 0 \pmod{p^{l_3}},$$
&c.,

(Proposition 8, Corollary), and similarly for p^{l_2}, &c.

We may insert here a lemma, which will be useful presently.

Lemma.—If $l_r > l_s$, then i_{rs} is divisible by p.

$$i_{rs} \text{ is the index of } g_r \text{ in the product } \Gamma_s,$$
$$\Gamma_s \equiv \dots g_r^{i_{rs}} \dots \pmod{m}.$$

Hence (Proposition 8, Corollary) the exponent of Γ_s is $\nless$ exponent of $g_r^{i_{rs}}$.

Suppose, if possible, that i_{rs} can be prime to p.

Then the exponent of $g_r^{i_{rs}}$ is p^{l_r} (Proposition 4).

Now the exponent of Γ_s is p^{l_s},

therefore
$$p^{l_s} \nless p^{l_r},$$
therefore
$$l_s \nless l_r,$$

which is contrary to the supposition $l_s < l_r$, and therefore i_{rs} cannot be prime to p.

Hence, if $l_r > l_s$, then i_{rs} is divisible by p.

Take any one whatever of the p-power-exponent numbers, $g_1^{I_1} g_2^{I_2} \dots g_\mu^{I_\mu}$ (modulus m). Then, since the numbers Γ are also generators of these numbers (the p-power-exponent numbers), the numbers $g_1^{I_1} g_2^{I_2} \dots g_\mu^{I_\mu}$ (modulus m) must be expressible in the form $\Gamma_1^{x_1} \Gamma_2^{x_2} \dots \Gamma_\mu^{x_\mu}$ (modulus m) in one way, and one way only.

Therefore
$$g_1^{I_1} g_2^{I_2} \dots g_\mu^{I_\mu} \equiv \Gamma_1^{x_1} \Gamma_2^{x_2} \dots \Gamma_\mu^{x_\mu} \pmod{m}$$

must lead to

$$\begin{array}{l} \text{one value of } x_1 \pmod{p^{l_1}}, \\ \quad ,, \qquad ,, \quad x_2 \pmod{p^{l_2}}, \\ \text{\&c.} \end{array}$$

Substituting the values of the Γ's in terms of the g's,

$$(g_1^{i_{11}} g_2^{i_{21}} \ldots g_\mu^{i_{\mu 1}})^{x_1} (g_1^{i_{12}} g_2^{i_{22}} \ldots g_\mu^{i_{\mu 2}})^{x_2} \ldots (g_1^{i_{1\mu}} g_2^{i_{2\mu}} \ldots g_\mu^{i_{\mu\mu}})^{x_\mu} \equiv g_1^{I_1} g_2^{I_2} \ldots g_\mu^{I_\mu} \pmod{m},$$
$$g_1^{i_{11}x_1 + i_{12}x_2 + \ldots + i_{1\mu}x_\mu} . g_2^{i_{21}x_1 + i_{22}x_2 + \ldots i_{2\mu}x_\mu} \ldots g_\mu^{i_{\mu 1}x_1 + i_{\mu 2}x_2 + \ldots i_{\mu\mu}x_\mu} \equiv g_1^{I_1} g_2^{I_2} \ldots g_\mu^{I_\mu} \pmod{m}.$$

Therefore (Proposition 7) it follows that

$$\begin{array}{l} i_{11}x_1 + i_{12}x_2 + \ldots + i_{1\mu}x_\mu \equiv I_1 \pmod{p^{l_1}}, \\ i_{21}x_1 + i_{22}x_2 + \ldots + i_{2\mu}x_\mu \equiv I_2 \pmod{p^{l_2}}, \\ \vdots \\ i_{\mu 1}x_1 + i_{\mu 2}x_2 + \ldots + i_{\mu\mu}x_\mu \equiv I_\mu \pmod{p^{l_\mu}}, \end{array}$$

and we need that these shall have *one* set of values of x_1 (modulus p^{l_1}), x_2 (modulus p^{l_2}), &c., . . . and only one.

This being so the numbers Γ will generate completely the p-power-exponent numbers.

We may suppose that the moduli $p^{l_1}, p^{l_2} \ldots$ are in ascending order of magnitude, so that $l_s \overset{=}{<} l_{s+1}$.

Suppose that

$$l_1 = l_2 = l_3 = \ldots = l_a < l_{a+1} = l_{a+2} = \ldots = l_b < l_{b+1} = l_{b+2} = \ldots = l_c < \text{\&c.} \ldots = l_\mu.$$

Any one of the unknowns x_s is to be found with regard to mod p^{l_s}.

We can write x_s in the form

$$x_s \equiv \xi_s + \xi_{sa} p^{l_a} + \xi_{sb} p^{l_b} + \xi_{sc} p^{l_c} + \ldots \pmod{p^{l_s}},$$

where

$$\left.\begin{array}{l} \xi_s \ < p^{l_a} \\ \xi_{sa} < p^{l_b - l_a} \\ \xi_{sb} < p^{l_c - l_b} \end{array}\right\}$$
$$\text{\&c.}$$

This substitution gives in particular,

$$\begin{array}{ll} x_1 & \equiv \xi_1 \pmod{p^{l_a}}. \\ x_2 & \equiv \xi_2 \pmod{p^{l_a}}. \\ \vdots & \quad \vdots \quad\; \vdots \\ x_a & \equiv \xi_a \pmod{p^{l_a}}. \end{array}$$

$$\begin{aligned}
x_{a+1} &\equiv \xi_{a+1} + \xi_{a+1.a}\, p^{l} \pmod{p^{h}}.\\
x_{a+2} &\equiv \xi_{a+2} + \xi_{a+2.a}\, p^{l} \pmod{p^{h}}.\\
&\vdots\\
x_{b} &\equiv \xi_{b} + \xi_{ba}\, p^{l} \pmod{p^{h}}.\\
\\
x_{b+1} &\equiv \xi_{b+1} + \xi_{b+1.a}\, p^{l} + \xi_{b+1.b}\, p^{h} \pmod{p^{k}}.\\
x_{b+2} &\equiv \xi_{b+2} + \xi_{b+2.a}\, p^{l} + \xi_{b+2.b}\, p^{h} \pmod{p^{k}}.\\
&\vdots\\
x_{c} &\equiv \xi_{c} + \xi_{ca}\, p^{l} + \xi_{cb}\, p^{h} \pmod{p^{k}}.\\
&\qquad \&\text{c.} \qquad\qquad \&\text{c.}
\end{aligned}$$

First replace each modulus which exceeds p^{l} by p^{l}, and substitute the assumed values of the x's.

We get the μ congruences

$$\begin{aligned}
i_{11}\xi_1 + i_{12}\xi_2 + \ldots + i_{1\mu}\xi_\mu &\equiv \mathrm{I}_1 \pmod{p^{l}},\\
i_{21}\xi_1 + i_{22}\xi_2 + \ldots + i_{2\mu}\xi_\mu &\equiv \mathrm{I}_2 \pmod{p^{l}},\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
i_{\mu1}\xi_1 + i_{\mu2}\xi_2 + \ldots + i_{\mu\mu}\xi_\mu &\equiv \mathrm{I}_\mu \pmod{p^{l}},
\end{aligned}$$

and we need that these shall determine a single set of values for $\xi_1, \xi_2, \ldots \xi_\mu$ (mod p^{l}).

The necessary and sufficient condition is that the determinant

$$(i_{11}, i_{22}, i_{33}, \ldots i_{\mu\mu})$$

should be prime to p (Proposition 28).

Suppose that this is so, and that the numbers $\xi_1, \xi_2, \ldots \xi_\mu$ are determined.

When these values are substituted above, suppose that the values of the left-hand sides are $\mathrm{I}_1 - p^{l}\mathrm{I}'_1$, $\mathrm{I}_2 - p^{l}\mathrm{I}'_2$, &c. . . . (where the negative signs are written for convenience in what follows, and $\mathrm{I}'_1, \mathrm{I}'_2, \ldots$ are therefore negative).

Next replace each modulus which exceeds p^{h} by p^{h}, and substitute the assumed values of the x's.

The first a congruences are already satisfied.

The rest give

$$\begin{aligned}
\mathrm{I}_{a+1} - p^{l}\mathrm{I}'_{a+1} + p^{l}\,(i_{a+1.a+1}\,\xi_{a+1.a} + i_{a+1.a+2}\,\xi_{a+2.a} + \ldots + i_{a+1.\mu}\,\xi_{\mu a}) &\equiv \mathrm{I}_{a+1} \pmod{p^{h}}.\\
\mathrm{I}_{a+2} - p^{l}\mathrm{I}'_{a+2} + p^{l}\,(i_{a+2.a+1}\,\xi_{a+1.a} + i_{a+2.a+2}\,\xi_{a+2.a} + \ldots + i_{a+2.\mu}\,\xi_{\mu a}) &\equiv \mathrm{I}_{a+2} \pmod{p^{h}}.\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\mathrm{I}_{\mu} - p^{l}\mathrm{I}'_{\mu} + p^{l}\,(i_{\mu.a+1}\,\xi_{a+1.a} + i_{\mu.a+2}\,\xi_{a+2.a} + \ldots i_{\mu\mu}\,\xi_{\mu a}) &\equiv \mathrm{I}_{\mu} \pmod{p^{h}}.
\end{aligned}$$

Therefore

$$\begin{aligned} &i_{a+1.a+1}\,\xi_{a+1.a} + i_{a+1.a+2}\,\xi_{a+2.a} + \ldots + i_{a+1.\mu}\,\xi_{\mu a} \equiv \mathrm{I}'_{a+1} \pmod{p^{l_b-l_a}} \\ &i_{a+2.a+1}\cdot\xi_{a+1.a} + i_{a+2.a+2}\,\xi_{a+2.a} + \ldots + i_{a+2.\mu}\,\xi_{\mu a} \equiv \mathrm{I}'_{a+2} \pmod{p^{l_b-l_a}} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &i_{\mu.a+1}\,\xi_{a+1.a} + i_{\mu.a+2}\cdot\xi_{a+2.a} + \ldots\quad i_{\mu\mu}\,\xi_{\mu a} \equiv \mathrm{I}'_{\mu} \pmod{p^{l_b-l_a}}, \end{aligned}$$

and we need that these shall determine a single set of values of

$$\xi_{a+1.a},\ \xi_{a+2.a},\ \ldots\ \xi_{\mu a} \pmod{p^{l_b-l_a}}.$$

The necessary and sufficient condition is that the determinant

$$(i_{a+1,a+1},\ i_{a+2,a+2},\ \ldots\ i_{\mu\mu})$$

should be prime to p.

Suppose that this is so, and the numbers $\xi_{a+1,a}$, $\xi_{a+2.a}$, … $\xi_{\mu a}$ are determined: and suppose that when substituted above the values of the left-hand sides become $\mathrm{I}'_{a+1} - p^{l_b-l_a}\mathrm{I}''_{a+1}$, &c.

Next replace each modulus that exceeds p^{l_b} by p^{l_b}, and substitute again the values of x.

The first b congruences are now satisfied.

The rest give

$$\begin{aligned} &(\mathrm{I}_{b+1} - \mathrm{I}'_{b+1}p^{l_a}) + (\mathrm{I}'_{b+1} - \mathrm{I}''_{b+1}p^{l_b-l_a})\,p^{l_a} + p^{l_b}(i_{b+1.b+1}\,\xi_{b+1.b} + \ldots\ i_{b+1.\mu}\,\xi_{\mu b}) \equiv \mathrm{I}_{b+1} \pmod{p^{l_b}} \\ &(\mathrm{I}_{b+2} - \mathrm{I}'_{b+2}p^{l_a}) + (\mathrm{I}'_{b+2} - \mathrm{I}''_{b+2}p^{l_b-l_a})\,p^{l_a} + p^{l_b}(i_{b+2.b+1}\,\xi_{b+1.b} + \ldots\ i_{b+2.\mu}\,\xi_{\mu b}) \equiv \mathrm{I}_{b+2} \pmod{p^{l_b}} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &(\mathrm{I}_{\mu} - \mathrm{I}'_{\mu}p^{l_a}) \quad + (\mathrm{I}'_{\mu} - \mathrm{I}''_{\mu}p^{l_b-l_a}) \quad + p^{l_b}(i_{\mu.b+1}\,\xi_{b+1.b} \quad + \ldots\ i_{\mu\mu}\,\xi_{\mu b}) \equiv \mathrm{I}_{\mu} \pmod{p^{l_b}}. \end{aligned}$$

Therefore

$$\begin{aligned} &i_{b+1.b+1}\,\xi_{b+1.b} + i_{b+1.b+2}\,\xi_{b+2.b} + \ldots + i_{b+1.\mu}\,\xi_{\mu b} \equiv \mathrm{I}''_{b+1} \pmod{p^{l_c-l_b}} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &i_{\mu.b+1}\,\xi_{b+1.b} + i_{\mu.b+2}\,\xi_{b+2.b} + \ldots\ i_{\mu\mu}\,\xi_{\mu b} \equiv \mathrm{I}''_{\mu} \pmod{p^{l_c-l_b}}, \end{aligned}$$

and we need that these shall determine a single set of values of

$$\xi_{b+1.b},\ \xi_{b+2,b},\ \ldots\ \xi_{\mu b} \pmod{p^{l_c-l_b}}.$$

The necessary and sufficient condition for this is that the determinant

$$(i_{b+1.b+1},\ i_{b+2,b+2},\ \ldots\ i_{\mu\mu})$$

should be prime to p.

These successive steps are to be continued until all the quantities ξ are determined. They, and therefore the numbers x will be determined uniquely, provided that the conditions that

$$\left.\begin{array}{l} (i_{11} \ldots i_{\mu\mu}) \\ (i_{a+1}\, i_{a+1}, \ldots i_{\mu\mu}) \\ (i_{b+1,\, b+1}, \ldots i_{\mu\mu}) \\ \quad \&c., \end{array}\right\} \text{shall all be prime to } p \text{, are satisfied.}$$

When these conditions are satisfied the generators Γ generate the complete set of p-power-exponent numbers.

(30.) We have not yet seen whether the conditions just found are independent or not. We shall find that the first condition $(i_{11}, i_{22}, \ldots i_{\mu\mu})$ includes all the others; it may, however, be replaced by others of a similar kind, but practically simpler.

Let us write down the complete determinant and divide it into squares and rectangles thus :—

$$\begin{array}{cccc|cc|c}
i_{11} & i_{12} & \ldots\ i_{1a-1} & i_{1a} & i_{1a+1} & i_{1b} & \\
i_{21} & i_{22} & \ldots\ i_{2.\,a-1} & i_{2a} & \vdots & & \\
\vdots & & & & \vdots & & \&c. \\
i_{a-1.\,1} & i_{a-1.\,2} & \ldots\ i_{a-1.\,a-1} & i_{a-1.\,a} & \vdots & & \\
i_{a1} & i_{a2} & \ldots\ i_{a.\,a-1} & i_{aa} & i_{aa+1} & i_{ab} & \\
\hline
i_{a+1.\,1} & & \ldots & i_{a+1.\,a} & i_{a+1.\,a+1} & i_{a+1.\,b} & \\
\vdots & & & & \vdots & & \\
i_{b1} & & \ldots & i_{ba} & i_{ba+1} & i_{bb} & \\
\hline
 & & \&c. & & & & \\
\end{array}$$

For reference, we may name these squares and rectangles, thus :—

$$\begin{array}{c|c|cc}
(aa) & (ab) & (ac) & \\
\hline
(ba) & (bb) & (bc) & \&c. \\
\hline
(ca) & (cb) & (cc) & \\
\end{array}$$

$$\&c.$$

The lemma proved in the course of the last proposition (that if $l_r > l_s$, then i_{rs} is divisible by p), shows that every element i, which is found to the left and below the determinants (aa), (bb), (cc) . . . is divisible by p, *i.e.*, every element in

$$\begin{array}{c|c|c|c} & & & \\ \hline (ba) & & & \\ \hline (ca) & (cb) & & \\ \hline (da) & (db) & (dc) & \end{array}$$

is divisible by p. &c.,

The complete determinant $(i_{11}, \ldots i_{\mu\mu})$ can be expressed as the sum of products of pairs of complementary determinants, one of these in each such product being the determinant formed by any a rows taken from (aa), (ba), (ca) . . .

Now, any row taken from (ba), (ca), . . . has every element divisible by p. Hence, the only determinant formed by a rows taken from (aa), (ba), (ca), . . . which is not immediately divisible by p, is the determinant $(aa) = (i_{11}, \ldots i_{aa})$.

Hence

$$(i_{11}, \ldots i_{\mu\mu}) \equiv (i_{11}, \ldots i_{aa})\,(i_{a+1.\,a+1} \ldots i_{\mu\mu}) \pmod p.$$

Therefore, if $(i_{11} \ldots i_{\mu\mu})$ is prime to p, then $(i_{11}, \ldots i_{aa})$ and $(i_{a+1.\,a+1}, \ldots i_{\mu\mu})$ are both prime to p (and *vice versâ*).

Again, take the determinant $(i_{a+1.\,a+1}, \ldots i_{\mu\mu})$,

$$\begin{array}{c|c|c} (bb) & (bc) & \\ \hline (cb) & (cc) & \quad \&c. \\ \hline & & \end{array}$$

&c.

This can be expressed as a sum of products of determinants in the same way: one determinant in each product being formed by $(b - a)$ rows taken from (bb), (cb) . . . Of these, only (bb) is not immediately divisible by p.

Therefore

$$(i_{a+1.a+1}, \dots i_{\mu\mu}) \equiv (i_{a+1.a+1}, \dots i_{bb})\,(i_{b+1.b+1}, \dots i_{\mu\mu}) \pmod{p},$$

and, since

$$(i_{a+1.a+1}, \dots i_{\mu\mu}) \text{ is prime to } p,$$

therefore

$$(i_{a+1.a+1}, \dots i_{bb}) \text{ and } (i_{b+1.b+1}, \dots i_{\mu\mu}) \text{ are both prime to } p,$$

and so on.

Hence, as stated above, the single condition $(i_{11}, \dots i_{\mu\mu})$ prime to p, includes all the other conditions found necessary in Proposition (29), and is therefore a necessary and sufficient condition that the numbers Γ (having the proper set of exponents) may completely generate the p-power-exponent numbers.

Also, this single condition may be replaced (with much advantage practically) by the equivalent set of conditions—

$$\left.\begin{array}{c} (i_{11}, \dots i_{aa}) \\ (i_{a+1}, i_{a+1}, \dots i_{bb}) \\ (i_{b+1}, i_{b+1}, \dots i_{cc}) \\ \&\text{c.} \end{array}\right\} \text{all prime to } p.$$

(31.) We have now a complete series of methods for ascertaining whether a given set of numbers G can act as generators of the complete system of $\phi(m)$ numbers (modulus m).

(1.) Find the exponent to which each belongs. (Proposition 17.)

(2.) The principal factors of these exponents must be the numbers found in Proposition (26).

(3.) Express each number G as a product of numbers with principal factors of exponent of G as exponents. (Proposition 10.)

(4.) Express each of these last as a product of powers of unitary generators, and thus get for each a set of indices.

(5.) Apply the series of conditions of Proposition (30).

If (2) and (5) are both satisfied then the numbers G are complete generators.

Conversely to form any set of generators—

(1.) For each prime p in $\phi(m)$ form a set of unitary generators.

(2.) Form from these any set of p-power-exponent generators satisfying the conditions of Proposition 30.

(3.) Multiply together one generator from each of the sets just formed.

Examples.

Let us form a set of generators for mod $308 = 2^2.\ 7.\ 11$.

$$m = 2^2.\ 7.\ 11.$$
$$\phi(7) = 2.\ 3.$$
$$\phi(11) = 2.\ 5.$$

The principal factors of the exponents of the generators are

2. 2. 2. 3. 5.

We form first the numbers with exponent 2 to generate the 2-exponent numbers.

These numbers expressed in terms of the unitary generators, 155, 265, 197, are given by

No.	Ind. of 155.	Ind. of 265.	Ind. of 197.
197.	0.	0.	1.
265.	0.	1.	0.
155.	1.	0.	0.
111.	1.	1.	0.
43.	1.	0.	1.
153.	0.	1.	1.
307.	1.	1.	1.

Of these (by Proposition 30) we can take any three such that the determinant formed by their indices is prime to 2, *i.e.*, odd.

For example,

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1,$$

and so 43, 153, 307 may be taken as generators.

Secondly we need a number with exponent 3.

Of these there are two, 177 and 221. Let us take 177.

Lastly we need a number with exponent 5.

Of these there are four, 113, 141, 169, 225. Let us take 113.

We have now 43, 153, 307, independent generators with exponent 2,

177 with exp 3,
113 „ „ 5.

These we may combine in any manner we please as products, taking only one from each set in each product.

Let us take 43. 177. $113 \equiv 107 \pmod{308}$ with exponent 30.

We have thus obtained three generators

$$\left.\begin{array}{l} 107 \text{ with exp } 30 \\ 153 \quad „ \quad „ \quad 2 \\ 307 \quad „ \quad „ \quad 2 \end{array}\right\}$$

which generate the complete set of $\phi(308)$ numbers.

Conversely, we can verify that 107. 153. 307 are complete generators.

First we find that

$$\left.\begin{array}{lll} 107 & \text{has exp} & 30 \\ 153 & \text{,,} \quad \text{,,} & 2 \\ 307 & \text{,,} \quad \text{,,} & 2 \end{array}\right\}$$

and so the exponents have the necessary principal factors.

$$\begin{aligned} 107 &\equiv 107^{15}.\ 107^{10}.\ 107^{6} \pmod{308}, \\ &\equiv 43.\ 177.\ 113 \pmod{308}, \end{aligned}$$

where

$$\left.\begin{array}{lll} 43 & \text{has exp} & 2 \\ 177 & \text{,,} \quad \text{,,} & 3 \\ 113 & \text{,,} \quad \text{,,} & 5 \end{array}\right\}$$

We only need to show that 43, 153, 307 are independent; and since the indices of these when referred to the unitary generators are

$$\begin{array}{rccc} 43, & 1 & 0 & 1 \\ 153, & 0 & 1 & 1 \\ 307, & 1 & 1 & 1 \end{array}$$

and the determinant is prime to 2, we see that the generators 43. 153. 307 are independent.

Let us form generators for modulus $999 = 3^3.\ 37$.

$$\begin{aligned} m &= 3^3.\ 37, \\ \phi(m) &= (3^2.\ 2)(3^2.\ 2^2). \end{aligned}$$

The principal factors of the exponents of the generators are

$$2.\ 2^2.\ 3^2.\ 3^2.$$

If

$$\begin{aligned} a &\equiv \alpha_1 \pmod{3^3}, \\ &\equiv \alpha_2 \pmod{37}, \\ a &\equiv \alpha_1\xi_1 + \alpha_2\xi_2 \pmod{3^3.\ 37}. \end{aligned}$$

$$\begin{aligned} 37x_1 &\equiv 1 \pmod{3^3} \\ 10x_1 &\equiv 1 \pmod{27} \\ x_1 &\equiv 19 \pmod{3^3} \\ \xi_1 &\equiv 37.\ 19 \equiv 703 \pmod{999} \end{aligned}$$

$$\begin{aligned} 27x_2 &\equiv 1 \pmod{37} \\ x_2 &\equiv 11 \pmod{37} \\ \xi_2 &\equiv 297 \pmod{999} \end{aligned}$$

and, therefore,

$$a \equiv 703\alpha_1 + 297\alpha_2 \pmod{999}.$$

We want first any number with exponent 2.

Suppose we take $\alpha_1 = 26$ and $\alpha_2 = 36$ each with exponent 2 (for moduli 3^3 and 37 respectively).

Then we get

$$\begin{aligned} a &\equiv 703.\,26 + 297.\,36 \pmod{999} \\ &\equiv 998 \pmod{999} \text{ with exp } 2. \end{aligned}$$

We want next a number with exponent 4.

Suppose we take

$$\alpha_1 = 1$$

and

$$\alpha_2 = 6 \text{ (with exp 4, mod 37).}$$

Then we get

$$\begin{aligned} a &\equiv 703 + 297.\,6 \pmod{999} \\ &\equiv 487 \pmod{999} \text{ with exp } 4. \end{aligned}$$

Lastly, we want two independent generators with exponent 3^2.

We first form unitary generators with exponents 3^2

$$\alpha_1 = 4 \quad \alpha_2 = 1 \text{ gives 112 with exp } 3^2,$$

and

$$\alpha_1 = 1 \quad \alpha_2 = 12 \text{ gives 271 with exp } 3^2.$$

We may take for our 3-power-exponent generators any two numbers of the form $112^{i_1}\,271^{i_2}$, $112^{i'_1}\,271^{i'_2}$, provided that

$$\begin{vmatrix} i_1 & i_2 \\ i'_1 & i'_2 \end{vmatrix} \text{ is prime to 3.}$$

Suppose we take it

$$\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

and then the generators are

$$112.\,271 \equiv 382 \pmod{999}$$

and

$$112^2.\,271 \equiv 826 \pmod{999}.$$

Thus we have

998 with exp 2,
487 with exp 4,

and

$$382 \text{ and } 826 \text{ with exp } 3^2.$$

Hence we may form, say,

$$998.\ 826 \equiv 173 \pmod{999} \text{ with exp } 2.\ 3^3.$$

and

$$487.\ 382 \equiv 220 \pmod{999} \text{ with exp } 2^2.\ 3^2.$$

Thus

$$\left.\begin{array}{l} 173 \text{ with exp } 18 \\ 220 \text{ with exp } 36 \end{array}\right\}$$

generate completely the $2^3.\ 3^4$ numbers prime to 999.

INDICES AND TABLES OF INDICES.

(32.) Let $G_1\, G_2 \ldots G_\kappa$ be a complete set of independent generators for modulus m. Let their exponents be $t_1, t_2, \ldots t_\kappa$.

Then any number a prime to m is expressible in the form

$$a \equiv G_1^{i_1}\, G_2^{i_2} \ldots G_\kappa^{i_\kappa} \pmod{m},$$

where

$$i_1 < t_1,\ i_2 < t_2, \text{ \&c.}$$

We may thus make a table giving the set of indices $i_1, i_2, \ldots i_\kappa$ which correspond to any number a, and conversely the number a which corresponds to any set of indices.

We may conveniently write

$$a \equiv (i_1, i_2, \ldots i_\kappa),$$

and then if

$$a' \equiv (i'_1, i'_2, \ldots i'_\kappa)$$

we shall have

$$aa' \equiv (i_1 + i'_1, i_2 + i'_2, \ldots i_\kappa + i'_\kappa)$$

and

$$a^s \equiv (i_1 s, i_2 s, \ldots i_\kappa s).$$

With tables of this kind for every modulus we can at once solve any congruence of the form

$$ax^n \equiv b \pmod{m}$$

whatever m may be, if a and b are both prime to m. For suppose that

$$a \equiv (i_1\, i_2 \ldots)$$
$$b \equiv (i'_1\, i'_2 \ldots)$$
$$x \equiv (I_1, I_2, \ldots)$$

Then we must have

$$i_1 + nI_1 \equiv i'_1 \pmod{t_1}$$
$$i_1 + nI_2 \equiv i'_2 \pmod{t_2},$$
&c.

The G.C.M. of n and t_1 must divide $i'_1 - i_1$, &c., in order that the congruences may be soluble.

These being so we find $I_1, I_2, \ldots$ by reference to tables with moduli $t_1, t_2, \ldots$ which are (or may be) composite numbers.

If

ν_1 is the G.C.M. of n and t_1, I_1 will have ν_1 values mod t_1,
ν_2 ,, ,, n and t_2, I_2 ,, ν_2 ,, t_2,
&c.

Hence, by reference again to the table for modulus m, we obtain at once the $\nu_1, \nu_2, \nu_3 \ldots$ numbers which satisfy the congruence $ax^n \equiv b \pmod{m}$.

Let us now solve the same congruence with the help only of tables of indices for powers of primes as moduli.

If

α_0 is a solution of the congruence $ax^n \equiv b \pmod{2^\kappa}$,
α_1 ,, ,, ,, $ax^n \equiv b \pmod{P_1^{\lambda_1}}$,
&c.,

then

$$x \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 \ldots \pmod{m}$$

is a solution of the congruence

$$ax^n \equiv b \pmod{m}.$$

Hence the solution is conducted as follows:—

By means of the tables of indices find

all the values α_0 which satisfy $ax^n \equiv b \pmod{2^\kappa}$,
,, ,, α_1 ,, ,, $\pmod{P_1^{\lambda_1}}$,
&c.

Next find the values of the ξ's, Proposition (19), each being found by the tables.

Lastly the values of x which satisfy $ax^n \equiv b \pmod{m}$ must each be calculated separately by giving to each of $\alpha_0, \alpha_1, \ldots$ one of its possible values.

In the case of a multiple solution the labour involved in this last step may be very great; and consequently the saving of labour effected by the use of tables of indices for composite moduli proportionately so.

As an example of the use of tables of indices for composite moduli let us use the table given in the example to Proposition (24).

Solve

$$9x^5 \equiv 53 \pmod{112}.$$
$$53 \equiv (4.\ 1.\ 1.)$$

(written in this order the indices are referred to moduli 6. 4. 2),

$$9 \equiv (2.\ 2.\ 0),$$

and, therefore,

$$x^5 \equiv (2.\ 3.\ 1).$$

If, then,

$$x \equiv (a.\ b.\ c)$$
$$x^5 \equiv (5a.\ 5b.\ 5c),$$

and, therefore,

$$\begin{aligned} 5a &\equiv 2 \pmod 6\text{), and, therefore, } a = 4,\\ 5b &\equiv 3 \pmod 4\text{), and, therefore, } b = 3,\\ 5c &\equiv 1 \pmod 2\text{), and, therefore, } c = 1,\end{aligned}$$

therefore

$$x \equiv (4.\ 3.\ 1),$$
$$x \equiv 109 \pmod{112}.$$

Solve

$$47x^2 \equiv 55 \pmod{112}.$$
$$55 \equiv (3.\ 0.\ 1),$$
$$47 \equiv (5.\ 2.\ 1),$$

therefore

$$x^2 \equiv (4.\ 2.\ 0),$$

therefore, if

$$x \equiv (a.\ b.\ c)$$

$$\begin{array}{lll} 2a \equiv 4 \pmod 6, & 2b \equiv 2 \pmod 4, & 2c \equiv 0 \pmod 2,\\ a \equiv 2 \pmod 3, & b \equiv 1 \pmod 2, & c = 0 \text{ or } 1\\ a = 2 \text{ or } 5. & b = 1 \text{ or } 3. & \end{array}$$

Hence there are eight solutions

$$\begin{array}{lll} (2\ \ 1\ \ 0) \equiv 51 & \qquad & (5\ \ 1\ \ 0) \equiv 19\\ (2\ \ 1\ \ 1) \equiv 37 & & (5\ \ 1\ \ 1) \equiv 5\\ (2\ \ 3\ \ 0) \equiv 107 & & (5\ \ 3\ \ 0) \equiv 75\\ (2\ \ 3\ \ 1) \equiv 93 & & (5\ \ 3\ \ 1) \equiv 61. \end{array}$$

PART II.—ON THE RESIDUES OF POWERS OF NUMBERS FOR ANY MODULUS, COMPOSITE AND COMPLEX.

In order to maintain as far as possible the parallelism of Parts I. and II. a number of facts and relations peculiar to complex numbers are noted (most without proof) in the following Preface to Part II.

Preface to Part II.

Complex primes.—i.e., numbers either real or complex which have no real or complex factors.

These are of two kinds

(i.) Real primes of the form $4k + 3$; *e.g.* 3, 7, 11, 19, . . .

(ii.) The factors of real primes of the form $4k + 1$, which are expressible as the sum of two squares, *e.g.*, $1 + 2i$, $2 + i$, $3 + 2i$, $2 + 3i$ . . . Among the last we include the factors of the real prime 2, viz., $1 + i$.

These two kinds of prime we shall call respectively *pure* and *mixed*, speaking of either as a *complex* prime. By a real prime we shall mean the primes ordinarily so called, real numbers which have no real factors.

To express any number as a product of its prime factors.

Let $a + bi$ be the number. Let d be the G.C.M. of a and b, and suppose that

$$a + bi = d(\alpha + \beta i).$$

d is a real number and can be expressed as a product of real primes.

Of these, those that are not complex primes can be separated each into its two factors.

All the factors of $\alpha^2 + \beta^2$ are themselves expressible as the sum of two squares.

Say

$$\alpha^2 + \beta^2 = (\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2) \ldots$$

Then the prime factors of $\alpha + \beta i$ are

$$\alpha_1 \pm i\beta_1,$$
$$\alpha_2 \pm i\beta_2, \ldots$$

where, in each pair, the sign must be so chosen that

$$\alpha + \beta i = (\alpha_1 \pm i\beta_1)(\alpha_2 \pm i\beta_2) \ldots$$

Example.

$$60 + 105i = 15(4 + 7i)$$
$$15 = 3.\,5 = 3(2 + i)(2 - i)$$
$$4^2 + 7^2 = 65 = 5.\,13 = (2^2 + 1^2)(3^2 + 2^2)$$

Thus

$$p^{n\rho - r} p'^{n\rho' - r'} \ldots \text{ is prime to } m/\kappa,$$

and, therefore, since a and b/κ are also prime to m/κ, the congruence (ii.) is soluble (if possible) by means of the tables giving the indices of numbers prime to the modulus.

Each solution ξ of the congruence (ii.) gives a single solution $x \equiv \xi p^{\rho} p'^{\rho'} \ldots$ (mod m) of the congruence $ax^n \equiv b \pmod{m}$.

Example.—Solve

$$22x^5 \equiv 54 \pmod{672}.$$

The congruence is

$$2.\ 11.\ x^5 \equiv 2.\ 3^3 \pmod{2^5.\ 3.\ 7} \quad \ldots\ldots\ldots\ldots \quad \text{(i.)}$$

Dividing by 2 we have

$$11.\ x^5 \equiv 3^3 \pmod{2^4.\ 3.\ 7} \quad \ldots\ldots\ldots\ldots \quad \text{(ii.)}$$

and any solution x of this gives the two solutions x, $x + 336$ of the congruence (i.).

From (ii.) we get, dividing by 3,

$$11 \frac{x^5}{3} \equiv 9 \pmod{2^4.\ 7}.$$

In congruence (ii.) let $x = 3\xi$.

Then

$$11.\ 3^4.\ \xi^5 \equiv 9 \pmod{2^4.\ 7}$$

$$11.\ 3^2.\ \xi^5 \equiv 1 \pmod{2^4.\ 7}.$$

Now (see table, p. 222)

$$11 \equiv (4.\ 3.\ 0),$$

$$9 \equiv (2.\ 2.\ 0),$$

therefore

$$9.\ 11 \equiv (0.\ 1.\ 0),$$

therefore

$$\xi^5 \equiv (0.\ 3.\ 0),$$

therefore

$$\xi \equiv (0.\ 3.\ 0)$$

$$\xi \equiv 43 \pmod{2^4.\ 7}.$$

Hence (ii.) has the solution

$$x \equiv 129 \pmod{336},$$

and therefore the solutions of (i.) are

$$x \equiv 129,\ 465 \pmod{672}.$$

Then

$$X = x + d\,(\alpha\xi - \beta\eta)$$
$$Y = y + d\,(\beta\xi + \alpha\eta).$$

The second equation gives $y \equiv Y \pmod{d}$, which determines y.
Then

$$\beta\xi + \alpha\eta = \frac{Y-y}{d},$$

α and β are co-prime, and therefore we can find α' and β' so that $\alpha'\beta - \alpha\beta' = 1$, and then

$$\left.\begin{aligned} \xi &= \alpha'\frac{Y-y}{d} + \lambda\alpha \\ \eta &= -\beta'\frac{Y-y}{d} - \lambda\beta \end{aligned}\right\}\text{where } \lambda \text{ is some integer.}$$

Therefore,

$$\begin{aligned} X &= x + d\left[\frac{Y-y}{d}(\alpha\alpha' + \beta\beta') + \lambda(\alpha^2 + \beta^2)\right] \\ &= x + (Y-y)(\alpha\alpha' + \beta\beta') + \lambda\,.\,d\,(\alpha^2 + \beta^2), \end{aligned}$$

and therefore

$$x \equiv X - (Y-y)(\alpha\alpha' + \beta\beta') \pmod{d\,.\,\overline{\alpha^2 + \beta^2}},$$

which determines x.

So x and y are given by

$$\begin{aligned} &y \equiv Y \pmod{d} \\ &x \equiv X - (Y-y)(\alpha\alpha' + \beta\beta') \pmod{d\,.\,\alpha^2 + \beta^2} \end{aligned}$$

where $\alpha\alpha' + \beta\beta'$ is a constant depending on the modulus.

Example.

Mod 3 (3 + 2i)

$$\left.\begin{aligned} d = 3.\qquad \alpha &= 3 \qquad & \beta &= 2 \\ \alpha' &= -1 \qquad & \beta' &= -1 \end{aligned}\right\} \alpha\alpha' + \beta\beta' = -5.$$

The reducing formulæ are

$$\begin{aligned} &y \equiv Y \pmod{3} \\ &x \equiv X + 5\,(Y-y) \pmod{39}. \end{aligned}$$

Thus to find the residues of the successive powers of $1 + i$:—

$$\begin{aligned}
1 + {} & i \\
& 2i \\
-2 + {} & 2i \equiv 37 + 2i \\
35 + {} & 39i \equiv 35 \\
35 + {} & 35i \equiv 5 + 2i \\
3 + {} & 7i \equiv 33 + i \\
32 + {} & 34i \equiv 2 + i \\
1 + {} & 3i \equiv 16 \\
16 + {} & 16i \equiv 13 + i \\
12 + {} & 4i \equiv 33 + 2i \\
31 + {} & 35i \equiv 1 + 2i \\
-1 + {} & 3i \equiv 14 \\
14 + {} & 14i \equiv 35 + 2i \qquad (\text{mod } 9 + 6i) \\
33 + {} & 37i \equiv 18 + i \\
17 + {} & 19i \equiv 29 + i \\
28 + {} & 30i \equiv 22 \\
22 + {} & 22i \equiv 10 + i \\
9 + {} & 11i \equiv 15 + 2i \\
13 + {} & 17i \equiv 10 + 2i \\
8 + {} & 12i \equiv 29 \\
29 + {} & 29i \equiv 8 + 2i \\
6 + {} & 10i \equiv 12 + i \\
11 + {} & 13i \equiv 32 + i \\
31 + {} & 33i \equiv 1
\end{aligned}$$

In the two special cases

(i.) $m = \alpha + \beta i$

$$X + iY \equiv x \ (\text{mod } \alpha + \beta i) \quad \text{and} \quad x$$

is determined by

$$x \equiv X - Y\,(\alpha\alpha' + \beta\beta')\ (\text{mod } \alpha^2 + \beta^2).$$

(ii.) $m = d$

$$X + iY \equiv x + iy \ (\text{mod } d) \quad \text{and} \quad x \quad \text{and} \quad y$$

are given by

$$x \equiv X \ (\text{mod } d),$$

$$y \equiv Y \ (\text{mod } d).$$

Number of residues prime to any modulus.

Let the modulus be expressed as a product of powers of its prime factors, $m = p^\alpha q^\beta \ldots$ then of the N (m) residues,

$$\Phi (m) = [\mathrm{N} (p^\alpha) - \mathrm{N} (p^{\alpha-1})] [\mathrm{N} (q^\beta) - \mathrm{N} (q^{\beta-1})] \ldots$$

are prime to m.

In particular for a pure prime p,

$$\Phi (p) = p^2 - 1 \quad \text{and} \quad \Phi (p^\alpha) = p^{2\alpha} - p^{2\alpha-2};$$

and for a mixed prime, $a + bi$,

$$\Phi (a + bi) = a^2 + b^2 - 1 \quad \text{and} \quad \Phi (a + bi)^\alpha = (a^2 + b^2)^\alpha - (a^2 + b^2)^{\alpha-1}.$$

This value Φ, when we are dealing with complex numbers, must be distinguished from the value ϕ when we are concerned only with real numbers.

To a mixed prime ϕ is inapplicable, and to a real it has the relations $\phi (p) = p - 1$, $\phi (p^\alpha) = p^\alpha - p^{\alpha-1}$.

Any modulus may be multiplied by -1, i, *or* $-i$, for if the modulus is a factor of any number it remains a factor on being multiplied by -1, i, or $-i$. This enables us to change any modulus into one entirely positive :—

For

$$\left.\begin{aligned} -\alpha - \beta i &= -1 (\alpha + \beta i) \\ -\beta + \alpha i &= i (\alpha + \beta i) \\ \beta - \alpha i &= -i (\alpha + \beta i) \end{aligned}\right\}.$$

PART II.—RESIDUES OF POWERS OF NUMBERS FOR ANY MODULUS, COMPOSITE AND COMPLEX.

(i.) a being a number prime to m each term of the series of residues

$$a, a^2, a^3, \ldots (\text{mod } m)$$

is one of the Φ (m) residues which are prime to m. Hence, as in (Proposition 1), we see that, if t is the smallest integer for which $a^t \equiv 1$ (mod m), the infinite series of residues consists of a repetition of t terms beginning from the first term. The number t is called the exponent of a for the modulus m.

Example.—The series of residues of the first twenty-four powers of the number $2 + i$ for the modulus 9 are

$2 + i$
$3 + 4i$
$2 + 2i$
$2 + 6i$
$7 + 5i$
$8i$
$1 + 7i$
$4 + 6i$
$2 + 7i$
$6 + 7i$
$5 + 2i$
8
$7 + 8i$
$6 + 5i$
$7 + 7i$
$7 + 3i$
$2 + 4i$
i
$8 + 2i$
$5 + 3i$
$7 + 2i$
$3 + 2i$
$4 + 7i$
1

after which the set of twenty-four terms constantly repeats itself.

(ii.) The proof is identical with 2, that if a has exponent t mod m and $a^s \equiv 1$, then t divides s.

(iii.) The proof of FERMAT'S theorem $a^{\Phi(m)} \equiv 1$ (mod m) is identical with that in (3) if for "the $\phi(m)$ numbers less than m and prime to it," we substitute "the $\Phi(m)$ residues of m that are prime to it."

Hence the corollary, that the exponent of any number, modulus m, is a divisor of $\Phi(m)$.

Example.—Mod $9 = 3^2$. $\Phi(3^2) = 3^4 - 3^2 = 72$. In Proposition (i.) we saw that the exponent of $2 + i$, mod 9, is 24, a divisor of 72.

(iv.) The proof is identical with that of (4). If a has exponent t, mod m, then a^s has exponent τ: where $t = \kappa\tau$ and κ is the G.C.M. of s and t.

Example.—Taking again the successive powers of $2 + i$, mod 9, we find the exponent of each of the twenty-four numbers.

No.	s.	τ = exp.
$2 + i$	1	24
$3 + 4i$	2	12
$2 + 2i$	3	8
$2 + 6i$	4	6
$7 + 5i$	5	24
$8i$	6	4
$1 + 7i$	7	24
$4 + 6i$	8	3
$2 + 7i$	9	8
$6 + 7i$	10	12
$5 + 2i$	11	24
8	12	2
$7 + 8i$	13	24
$6 + 5i$	14	12
$7 + 7i$	15	8
$7 + 3i$	16	3
$2 + 4i$	17	24
i	18	4
$8 + 2i$	19	24
$5 + 3i$	20	6
$7 + 2i$	21	8
$3 + 2i$	22	12
$4 + 7i$	23	24
1	24	1

(v.) The proof is the same as that of (5).

If the exponent of a is t, and of a' is t', and t and t' are co-prime, then the exponent of aa' is tt'.

And hence the corollary, that the same is true for any number of numbers: if $a, a', a'', \ldots$ have exponents $t, t', t'', \ldots$ co-prime, then $aa'a'' \ldots$ has exponent $tt't'' \ldots$

Example.—The exponent of 3 for mod $1 + 5i$ is 3.

Its successive powers have residues 3. 9. 1.

The exponent of 5 for mod $1 + 5i$ is 4.

Its successive powers have residues 5. 25. 21. 1.

Hence the exponent of $5 \times 3 = 15$ is $4 \times 3 = 12$.

The successive powers of 15 have residues 15. 17. 21. 3. 19. 25. 11. 9. 5. 23. 7. 1.

Example.—The exponent of 2, mod 7, is 3. The residues are 2. 4. 1.

The exponent of $5 + 2i$ is 8. The residues are $5 + 2i$, $6i$, $2 + 2i$, 6, $2 + 5i$, i, $5 + 5i$, 1.

Hence the exponent of $10 + 4i \equiv 3 + 4i$ is $3 \times 8 = 24$.

The successive residues are

$3 + 4i$	$3i$	$2 + 2i$	5	$1 + 6i$	i.
$3 + 3i$	4	$5 + 2i$	$5i$	$1 + i$	6.
$4 + 3i$	$4i$	$5 + 5i$	2	$6 + i$	$6i$.
$4 + 4i$	3	$2 + 5i$	$2i$	$6 + 6i$	1.

(vi.) The proof is the same as for (6).

Suppose that a has exponent t, and a' has exponent t', and that t and t' are not coprime. Then, if t and t' contain no prime factor raised to the same power in both, the exponent of aa' is the L.C.M. of t and t'.

Example.—The exponent of 35 for mod $9 + 6i$ is 6.

The residues of its powers are 35. 16. 14. 22. 29. 1.

The exponent of $3 + 7i$ for the same modulus is 4.

The residues of its powers are $3 + 7i$. 14. $15 + 2i$. 1.

Hence the exponent of $35\,(3 + 7i)$ is the L.C.M. of 6 and $4 = 12$.

$$35\,(3 + 7i) = 105 + 245i \equiv 33 + 2i \pmod{9 + 6i}.$$

The residues of its powers are

$33 + 2i$	29	$33 + i$	22	$2i$	14.
$12 + i$	16	$15 + 2i$	35	$18 + i$	1.

(vii.) The proof is the same as for (7).

If a has exponent t, a' has exponent t', a'' has exponent t'', &c. for modulus m, and if of the $tt't'' \ldots$ numbers $a^r a'^{r'} a''^{r''} \ldots$ (modulus m), formed by giving to r all values modulus t, to r' all values modulus t', &c., no two are congruent; then if

$$a^s a'^{s'} \ldots \equiv 1 \pmod{m},$$

we must have

$$a^s \equiv 1, \;\; a'^{s'} \equiv 1, \ldots$$

in other words, if a product of powers of numbers that are independent generators be congruent to unity, then each of these powers is itself congruent to unity.

(viii.) Proof identical with (8).

The exponent of the product of a number of independent generators is the L.C.M. of their exponents.

In particular if the separate exponents are all powers of the same prime, then the exponent is equal to the greatest of them.

(ix.) Proof the same as for (9).

If the exponent of a for modulus m is t, and for modulus n is t', and m and n are co-prime, then the exponent of a for modulus mn is the L.C.M. of t and t'.

Also the corollary, if the exponents of a for moduli $m, m', m'', \ldots$ are respectively $t, t', t'', \ldots$ $m, m', m'' \ldots$ being co-prime, then the exponent of a for modulus $mm'm'' \ldots$ is the L.C.M. of the numbers $t, t', t' \ldots$

Example.

$$1 + i \equiv 4 \pmod{2 + i},$$

and the residues of its powers are 4. 1. Hence the exponent of $1 + i$ for modulus $2 + i$ is 2.

$$1 + 1 \equiv 6 \pmod{3 + 2i},$$

and the residues of its powers are 6. 10. 8. 9. 2. 12. 7. 3. 5. 4. 11. 1. Hence the exponent of $1 + i$ for modulus $3 + 2i$ is 12.

The moduli $2 + i$, $3 + 2i$ are co-prime, therefore the exponent of $1 + i$ for modulus $(2 + i)(3 + 2i)$ is 12.

$$1 + i \equiv 19 \pmod{4 + 7i},$$

and the residues of its powers are

19. 36. 34. 61. 54. 51. 59. 16. 44. 56. 24. 1.

Example.

$1 + 2i$ has exp 8, mod 3 ;

the residues of its powers are

$1 + 2i$. i. $1 + i$. 2. $2 + i$. $2i$. $2 + 2i$. 1.

$1 + 2i \equiv 11 \pmod{3 + 2i}$, and has exp 12 ;

the residues of its powers are

11. 4. 5. 3. 7. 12. 2. 9. 8. 10. 6. 1.

$1 + 2i \equiv 26 \pmod{6 + 1}$, and has exp 3,

the residues of its powers are

26. 10. 1.

Hence, since 3, $3 + 2i$, $6 + i$ are co-prime, $1 + 2i$ has for modulus $3(3 + 2i)(6 + i)$ exponent = L.C.M. of 8, 12, 3, *i.e.*, $1 + 2i$ has exponent 24 for modulus $48 + 45i$. The residues of its powers are

$$1 + 2i.\ \ 90 + i.\ \ 1339 + i.\ \ 692.\ \ 275 + i.\ \ 1419 + 2i.\ \ 1370 + 2i.$$
$$1231.\ \ 1012 + 2i.\ \ 291 + i.\ \ 1015 + i.\ \ 482.$$

$$32 + i.\ \ 540 + 2i.\ \ 827 + 2i.\ \ 211.\ \ 244 + 2i.\ \ 969 + i.\ \ 439 + i.$$
$$269.\ \ 1043 + i.\ \ 741 + 2i.\ \ 503 + 2i.\ \ \ \ 1.$$

(x.) Proof identical with (10).

If the exponent of a is t, and $t = pqr \ldots$ where $p, q, r \ldots$ are co-prime factors of t, then a can be expressed as a product of numbers whose exponents are $p, q, r \ldots$

Example.—$2 + i$ has exponent 24 for modulus 9. (See example Proposition iv.)

$$24 = 3.\ 8 \qquad \left.\begin{aligned}16 &\equiv 1 \ (\mathrm{mod}\ 3)\\ &\equiv 0 \ (\mathrm{mod}\ 8)\end{aligned}\right\} \quad \begin{aligned}9 &\equiv 1 \ (\mathrm{mod}\ 8).\\ &\equiv 0 \ (\mathrm{mod}\ 3).\end{aligned}$$

Hence

$$\begin{aligned}2 + i &\equiv (2 + i)^{16}.\ (2 + i)^{9} \ (\mathrm{mod}\ 9),\\ &\equiv (7 + 3i)\ (2 + 7i) \ (\mathrm{mod}\ 9),\end{aligned}$$

where

and

$$\left.\begin{aligned}&7 + 3i \text{ has exp } 3\\ &2 + 7i \text{ has exp } 8\end{aligned}\right\} \text{ (See example Proposition iv.)}$$

Example.—$33 + 2i$ has exponent 12 for modulus $9 + 6i$. (See example Proposition vi.)

$$12 = 3.\ 4 \qquad \begin{aligned}4 &\equiv 1 \ (\mathrm{mod}\ 3)\\ &\equiv 0 \ (\mathrm{mod}\ 4)\end{aligned} \qquad \begin{aligned}9 &\equiv 1 \ (\mathrm{mod}\ 4)\\ &\equiv 0 \ (\mathrm{mod}\ 3).\end{aligned}$$

Therefore

$$\begin{aligned}33 + 2i &\equiv (33 + 2i)^{4}\ (33 + 2i)^{9} \ (\mathrm{mod}\ 9 + 6i)\\ &\equiv 22\ (15 + 2i) \ (\mathrm{mod}\ 9 + 6i),\end{aligned}$$

where 22 has exponent 3, and $15 + 2i$ has exponent 4.

(xi.) The number of numbers with exponent t, when the modulus is a prime (pure or mixed), is $\phi(t)$.

Any exponent t is a divisor of $\Phi(p)$, p being the prime modulus (Corollary, Proposition iii.). Exactly as in (11) we see that if there be *one* number with exponent t there are $\phi(t)$ and no more.

Now $t_1, t_2, \ldots$ being all the divisors of any real number $\Phi(p)$,

$$\phi(t_1) + \phi(t_2) + \ldots = \Phi(p).$$

Corresponding to each value t there are $\phi(t)$ numbers, or none with t as exponent.

The number of residues altogether (prime to the modulus p) is $\Phi(p)$. Hence, in no case can there fail to be $\phi(t)$ numbers with exponent t, t being any divisor of $\Phi(p)$.

Corollary.—In particular, any prime modulus p has numbers with exponents $\Phi(p)$, *i.e., has primitive roots.*

The number of these primitive roots is $\phi[\Phi(p)]$.

If p is a pure prime this is $\phi(p^2 - 1)$.

If p is a mixed prime, $= \alpha + \beta i$, the number is $\phi(\alpha^2 + \beta^2 - 1)$.

Example.—Modulus $5 + 2i$. 10 is a primitive root with exponent

$$\Phi(5 + 2i) = (5^2 + 2^2 - 1) = 28.$$

The residues of its powers are given in the table.

Number	10	13	14	24	8	22	17	25	18	6	2	20	26	28
Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Number	19	16	15	5	21	7	12	4	11	23	27	9	3	1
Index	15	16	17	18	19	20	21	22	23	24	25	26	27	28 or 0

The $\phi(28) = 12$ primitive roots are 10 14 8 18 2 26 19 15 21 11 27 3.

Example.—Modulus 7. Primitive roots have exponent $\Phi(7) = 7^2 - 1 = 48$. $2 + i$ is a primitive root, and the residues of its powers are given in the table.

Number	$2 + i$	$3 + 4i$	$2 + 4i$	$3i$	$4 + 6i$	$2 + 2i$	$2 + 6i$	5
Index	1	2	3	4	5	6	7	8
Number	$3 + 5i$	$1 + 6i$	$3 + 6i$	i	$6 + 2i$	$3 + 3i$	$3 + 2i$	4
Index	9	10	11	12	13	14	15	16
Number	$1 + 4i$	$5 + 2i$	$1 + 2i$	$5i$	$2 + 3i$	$1 + i$	$1 + 3i$	6
Index	17	18	19	20	21	22	23	24
Number	$5 + 6i$	$4 + 3i$	$5 + 3i$	$4i$	$3 + i$	$5 + 5i$	$5 + i$	2
Index	25	26	27	28	29	30	31	32
Number	$4 + 2i$	$6 + i$	$4 + i$	$6i$	$1 + 5i$	$4 + 4i$	$4 + 5i$	3
Index	33	34	35	36	37	38	39	40
Number	$6 + 3i$	$2 + 5i$	$6 + 5i$	$2i$	$5 + 4i$	$6 + 6i$	$6 + 4i$	1
Index	41	42	43	44	45	46	47	0

The $\phi(48) = 16$ primitive roots are

$$\begin{array}{cccc} 2+i & 4+6i & 2+6i & 3+6i \\ 6+2i & 1+4i & 1+2i & 1+3i \\ 5+6i & 3+i & 5+i & 4+i \\ 1+5i & 6+3i & 6+5i & 6+4i \end{array}$$

(xii.) The exponents to which a number a belongs for successive powers of a prime as moduli.

If we make one slight alteration the proof of (12) holds good throughout in the case of complex numbers for powers of a *pure* prime p as moduli. [In place of "$x < p$" read "$x =$ one of the $p^2 - 1$ residues, mod p."]

The exponent of a for mod p^λ is

$$t \text{ if } \lambda \leqq s,$$

$$tp^{\lambda - s} \text{ if } \lambda > s,$$

where t is the exponent of a for mod p, and p^s is the greatest power of p that divides $a^t - 1$.

Corollary.—The greatest value of t is $p^2 - 1$. (Proposition xi.)

The greatest value that $p^{\lambda - s}$ can have is got by making $s = 1$, *i.e.*, by taking a so that $a^{p^2-1} - 1$ (though necessarily divisible by p) is not divisible by p^2.

Hence the greatest exponent that a number can have for mod p^λ is

$$(p^2 - 1)p^{\lambda - 1}.$$

Now $\Phi(p^\lambda) = p^{2\lambda} - p^{2(\lambda - 1)} = (p^2 - 1)p^{2(\lambda - 1)}$, which is $p^{\lambda - 1}$ times the highest exponent.

Hence for a *power* of a pure prime as modulus primitive roots do not exist.

Consider next the case of a mixed prime (not $1 + i$).

Suppose $\alpha + \beta i$ is the prime.

Then, if $(\alpha + \beta i)^\lambda = \mathrm{A} + \mathrm{B}i$, A and B must be co-prime, for otherwise $\mathrm{A} + \mathrm{B}i$ would be divisible by some number other than $\alpha + \beta i$.

Hence the $\mathrm{N}(\alpha + \beta i)^\lambda = (\alpha^2 + \beta^2)^\lambda$ residues can be taken to be the pure numbers

$$0, 1, 2, \ldots (\alpha^2 + \beta^2)^\lambda - 1.$$

Suppose that a is any one of these numbers.

Then, if we have any congruence of the form

$$a^r \equiv b\,[\text{mod}\,(\alpha + \beta i)^\lambda],$$

$$a^r - b \equiv 0\,[\text{mod}\,(\alpha + \beta i)^\lambda],$$

it follows, since $a^r - b$ is a pure number, that

$$a^r - b \equiv 0 \,[\text{mod}\,(\alpha - \beta i)^\lambda],$$

therefore

$$a^r - b \equiv 0 \,[\text{mod}\,(\alpha^2 + \beta^2)^\lambda],$$
$$a^r \equiv b \,[\text{mod}\,(\alpha^2 + \beta^2)^\lambda].$$

Hence for the modulus $(\alpha + \beta i)^\lambda$ the residue of any power of a number is the same as for the modulus $(\alpha^2 + \beta^2)^\lambda$.

Now $\alpha^2 + \beta^2$ is a *real prime.*

Therefore the exponent to which a belongs for modulus $(\alpha + \beta i)^\lambda$ is

$$t \text{ if } \lambda \leqq s,$$
$$t\,(\alpha^2 + \beta^2)^{\lambda - s} \text{ if } \lambda > s,$$

where t is the exponent of a for mod $\alpha^2 + \beta^2$, and $(\alpha^2 + \beta^2)^s$ is the highest power of $\alpha^2 + \beta^2$ that divides $a^t - 1$.

The greatest exponent is

$$(\alpha^2 + \beta^2 - 1)(\alpha^2 + \beta^2)^{\lambda - 1},$$

and

$$\Phi\,(\alpha + \beta i)^\lambda = (\alpha^2 + \beta^2)^\lambda - (\alpha^2 + \beta^2)^{\lambda - 1} = (\alpha^2 + \beta^2 - 1)(\alpha^2 + \beta^2)^{\lambda - 1}.$$

Thus for powers of a *mixed prime* primitive roots *do* exist.

(xii*a*.) We see from the last proposition that for a power of a mixed prime as modulus primitive roots exist, and any one of them generates by its powers all the residues prime to the modulus. But in the case of a power of a pure prime p^λ, the highest exponent is $p^{\lambda-1}(p^2 - 1)$, whereas $\Phi\,(p^\lambda) = p^{2(\lambda-1)}(p^2 - 1)$.

We wish now to find how all the $p^{2(\lambda-1)}(p^2 - 1)$ numbers can be generated.

$$\left.\begin{array}{l} \text{Take a number } g \text{ with exp } p^{\lambda-1}(p^2 - 1) \\ \text{and ,, \quad ,, \quad } f \text{ \quad ,, \quad ,, \quad } p^{\lambda-1} \end{array}\right\}$$

g can be expressed as a product

$$g \equiv f'h \;(\text{mod } p^\lambda),$$

where

$$\left.\begin{array}{l} f' \text{ has exp } p^{\lambda-1} \\ h \text{ \quad ,, \quad ,, \quad } p^{2-1} \end{array}\right\}, \text{ Proposition x.}$$

$$\left.\begin{array}{r} f \text{ and } f' \text{ are each} \equiv 1 \;(\text{mod } p) \\ \not\equiv 1 \;(\text{mod } p^2) \end{array}\right\}, \text{ Proposition xii.}$$

Suppose

$$f \equiv 1 + \alpha p \pmod{p^2} \qquad \alpha \not\equiv 0 \pmod{p},$$
$$f' \equiv 1 + \alpha' p \pmod{p^2} \qquad \alpha' \not\equiv 0 \pmod{p}.$$

The $p^{\lambda-1} \times p^{\lambda-1}(p^2 - 1)$ numbers generated by products of powers of f and g will be the $p^{2(\lambda-1)}(p^2 - 1)$ residues prime to the modulus p^λ, provided that no two of them are congruent. We shall now show that no two can be congruent provided that k is a *complex* and not a *real* number, k being determined by the congruence $k\alpha' \equiv \alpha \pmod{p}$.

Suppose that two of the numbers generated are congruent, say

$$g^i f^j \equiv g^{i'} f^{j'} \pmod{p^\lambda},$$

where

$$i \not\equiv i' \pmod{p^{\lambda-1} . p^2 - 1},$$
$$j \not\equiv j' \pmod{p^{\lambda-1}}.$$

This is equivalent to

$$g^{i''} \equiv f^{j''} \pmod{p^\lambda},$$

where

$$i'' \equiv i - i' \not\equiv 0 \pmod{p^{\lambda-1} . p^2 - 1},$$
$$j'' \equiv j - j' \not\equiv 0 \pmod{p^{\lambda-1}},$$

j'' may be divisible by a power of p.

Suppose

$$j'' = \beta p^{\lambda-s},$$

where β is prime to p, and s may be any one of the numbers 2, 3. . . . λ.

Then

$$f^{j''} \text{ has exp } p^{s-1}. \quad \text{(Proposition iv.)},$$

therefore

$$g^{i''} \text{ has exp } p^{s-1},$$

therefore

$$i'' = \beta'(p^2 - 1)p^{\lambda-s},$$

where β' is prime to p. (Proposition iv.) Therefore

$$g^{\beta'(p^2-1)p^{\lambda-s}} \equiv f^{\beta p^{\lambda-s}} \pmod{p^\lambda},$$

and, therefore, since

$$h^{p^{s-1}} \equiv 1 \pmod{p^\lambda},$$
$$f'^{\beta'(p^2-1)p^{\lambda-s}} \equiv f^{\beta p^{\lambda-s}} \pmod{p^\lambda}.$$

Raise both sides to the power p^{s-2}, then

$$f'^{\beta'(p^2-1)p^{\lambda-2}} \equiv f^{\beta p^{\lambda-2}} \pmod{p^\lambda}.$$

Let

$$\beta\xi \equiv 1 \pmod{p}$$

determine ξ, and suppose

$$\beta'(p^2 - 1)\,\xi \equiv \gamma \pmod{p}.$$

Raise both sides to the power ξ, then

$$f'^{\gamma p^{\lambda-2}} \equiv f^{p^{\lambda-2}} \pmod{p^\lambda}.$$

Now

$$f \equiv 1 + \alpha p \pmod{p^2},$$

therefore

$$f^{p^{\lambda-2}} \equiv 1 + \alpha p^{\lambda-1} \pmod{p^\lambda},$$

and

$$f' \equiv 1 + \alpha' p \pmod{p^2},$$

therefore

$$f'^{\gamma p^{\lambda-2}} \equiv 1 + \alpha'\gamma p^{\lambda-1} \pmod{p^\lambda},$$

therefore

$$1 + \alpha'\gamma p^{\lambda-1} \equiv 1 + \alpha p^{\lambda-1} \pmod{p^\lambda},$$

therefore

$$\gamma\alpha' \equiv \alpha \pmod{p},$$

therefore

$$\gamma \equiv k \pmod{p}.$$

Now i'', j'', β, β', and, therefore, ξ, and finally γ, are *real* numbers, whereas, by supposition, k is complex, therefore on this supposition g and f are complete generators.

Example.—Modulus 3^2.

$$\Phi(3^2) = 3^4 - 3^2 = 72.$$

The highest exponent is

$$3(3^2 - 1) = 24.$$

We shall find that $2 + i$ and $4 + 3i$ can be taken for generators.

$$\left.\begin{aligned} g &= 2 + i \text{ has exp } 24 \bmod 3^2 \\ f &= 4 + 3i \text{ has exp } 3 \bmod 3^2 \end{aligned}\right\}$$

$$\begin{aligned} 2 + i &\equiv (2 + i)^{16}\,(2 + i)^9 \pmod{9} \\ &\equiv (7 + 3i)\,(2 + 7i) \pmod{9}, \end{aligned}$$

and

$$f' = 7 + 3i \text{ has exp } 3.$$

Now

$$f \equiv 1 + (1 + i)\,.\,3 \pmod{3^2}$$

$$f' \equiv 1 + (2 + i)\,.\,3 \pmod{3^2}$$

and

$$k(2+i)\equiv(1+i) \pmod{3}$$

gives

$$k\equiv 2i,$$

which is not a real number.

The following table gives the indices of the 72 numbers.

	0	1	2
1	$2+i$	$5+i$	$8+i$
2	$3+4i$	$7i$	$6+i$
3	$2+2i$	$2+5i$	$2+8i$
4	$2+6i$	$8+3i$	5
5	$7+5i$	$4+5i$	$1+5i$
6	$8i$	$3+5i$	$6+2i$
7	$1+7i$	$1+4i$	$1+i$
8	$4+6i$	7	$1+3i$
9	$2+7i$	$5+7i$	$8+7i$
10	$6+7i$	$3+i$	$4i$
11	$5+2i$	$5+5i$	$5+8i$
12	8	$5+6i$	$2+3i$
13	$7+8i$	$4+8i$	$1+8i$
14	$6+5i$	$2i$	$3+8i$
15	$7+7i$	$7+4i$	$7+i$
16	$7+3i$	$1+6i$	4
17	$2+4i$	$5+4i$	$8+4i$
18	i	$6+4i$	$3+7i$
19	$8+2i$	$8+5i$	$8+8i$
20	$5+3i$	2	$8+6i$
21	$7+2i$	$4+2i$	$1+2i$
22	$3+2i$	$6+8i$	$5i$
23	$4+7i$	$4+4i$	$4+i$
0	1	$4+3i$	$7+6i$

[The pair of indices for any number are found at the end of the row and at the head of the column in which the number is situated.

Thus

$$1+8i\equiv(2+i)^{13}(4+3i)^{2} \pmod{9}.]$$

(xiii.–xvi.) Propositions (13) to (16) are concerned with the moduli 2^λ. In Propositions xiii.–xvi. we shall investigate the moduli analogous to these in the case of complex numbers, viz. $(1+i)^\lambda$. [Propositions xiii.–xvi. will not be made to correspond individually to Propositions (13)–(16): they are intended to cover the same ground.]

(xiii.) In the case of the smaller values of λ, from 1 to 7, we shall find results, some of which, though they are really particular cases of the general results for any value of λ, are not conveniently included under them.

We shall start by a separate treatment of each of the first seven moduli, finding—

(i.) the numbers which belong to each exponent;

(ii.) what numbers must be taken in order to generate the complete set of residues.

Mod $1+i$. There is only one residue, viz., 1, whose exponent is 1.

Mod $(1+i)^2$. $\Phi(1+i)^2 = 2$. There are two numbers, viz.,

and $$\left.\begin{array}{r} i \text{ with exp } 2 \\ 1 \text{ with exp } 1 \end{array}\right\}.$$

Thus, for this modulus, i is a primitive root.

Mod $(1+i)^3$. $\Phi(1+i)^3 = 4$. The four residues are—

i, $2+i$ with exp 4,
3 with exp 2,
1 with exp 1.

i and $2+i$ are both primitive roots.

Mod $(1+i)^4$. $\Phi(1+i)^4 = 8$. The 8 numbers with their exponents are

Exp 4.	i,	$3i$,	$2+i$,	$2+3i$.
Exp 2.	3,	$1+2i$,	$3+2i$.	
Exp 1.	1.			

The square of each of the four numbers with exponent 4 is congruent to the same number with exponent 2, viz. 3.

Hence, to generate the 8 numbers, we must take any one of the four numbers with exponent 4, and either of the two numbers with exponent 2, viz. $1+2i$, $3+2i$.

Mod $(1+i)^5$. $\Phi(1+i)^5 = 16$. The 16 numbers, with their exponents are

Exp 4.	i	$3i$	$2+i$	$2+3i$	$4+i$	$6+i$	$4+3i$	$6+3i$.
Exp 2.	3	5	7	$1+2i$	$3+2i$	$5+2i$	$7+2i$	
Exp 1.	1.							

The square of each of the 8 numbers with exponent 4 gives the same number with exponent 2, viz. 7.

Also $7 \equiv 3 . 5 \equiv (1 + 2i)(3 + 2i) \equiv (5 + 2i)(7 + 2i) . [\text{mod } (1 + i)^5]$.

Hence, firstly, we must take as generators

$$\text{and} \qquad \left.\begin{array}{l}\text{one number with exponent 4}\\[1em] \text{two numbers with exponent 2}\end{array}\right\};$$

and, secondly, the two numbers with exponent 2 must neither of them be 7, and they must not be a pair of the numbers as arranged above whose product is congruent to 7.

Mod $(1 + i)^6$. $\Phi (1 + i)^6 = 32$. The 32 numbers with their exponents are arranged below.

Exp 1.	Exp 2.	Exp 4.							
1	$5+4i$	$1+2i$	$1+6i$	$3+2i$	$3+6i$	$5+2i$	$5+6i$	$7+2i$	$7+6i$
	$3+4i$	$2+i$	$2+3i$	$2+5i$	$2+7i$	$6+i$	$6+3i$	$6+5i$	$6+7i$
	7	$4+i$	$4+3i$	$4+5i$	$4+7i$	i	$3i$	$5i$	$7i$
	3								
	5								
	$1+4i$								
	$7+4i$								

The 24 numbers with exponent 4 are written in three rows, and the number with exponent 2, to which the square of any one of them is congruent, is to be found written at the end of the row.

Moreover, the three numbers, with exponent 2 (viz. $5 + 4i$, $3 + 4i$ and 7), which thus appear as the squares of numbers with exponent 4, are such that the product of any two is congruent to the third; for

$$7 (5 + 4i)(3 + 4i) \equiv 1 \; [\text{mod } (1 + i)^6].$$

Hence, to generate the 32 numbers, we must take

two numbers, with exponent 4, not in the same row

and

one number, with exponent 2, selected from 3, 5, $1 + 4i$, $7 + 4i$.

Mod $(1+i)^7$. $\Phi(1+i)^7 = 64$. The 64 numbers, with their exponents, are arranged below.

Exp 1.	*Exp* 2.	*Exp* 4.							
1	7	$4+i$	$12+i$	$3i$	$8+3i$	$5i$	$8+5i$	$4+7i$	$12+7i$
	9	3	5	11	13	$1+4i$	$7+4i$	$9+4i$	$15+4i$
	15	i	$8+i$	$4+3i$	$12+3i$	$4+5i$	$12+5i$	$7i$	$8+7i$
	$3+4i$	$2+i$	$10+i$	$2+3i$	$10+3i$	$6+5i$	$14+5i$	$6+7i$	$14+7i$
	$5+4i$	$5+2i$	$7+2i$	$13+2i$	$15+2i$	$1+6i$	$3+6i$	$9+6i$	$11+6i$
	$11+4i$	$6+i$	$14+i$	$6+3i$	$14+3i$	$2+5i$	$10+5i$	$2+7i$	$10+7i$
	$13+4i$	$1+2i$	$3+2i$	$9+2i$	$11+2i$	$5+6i$	$7+6i$	$13+6i$	$15+6i$

The 56 numbers with exponent 4 are arranged in rows collinearly with the numbers, with exponent 2, to which their squares are congruent.

The 7 numbers with exponent 2 can be arranged in 7 sets of 3, so that of each 3 the product of any two is congruent to the third (or the product of the three congruent to unity). Thus:

$$\left.\begin{aligned} 7.\ 9.\ 15 &\equiv 1 \\ 7\,(13+4i)\,(3+4i) &\equiv 1 \\ 7\,(5+4i)\,(11+4i) &\equiv 1 \\ 9\,(3+4i)\,(11+4i) &\equiv 1 \\ 9\,(5+4i)\,(13+4i) &\equiv 1 \\ 15\,(3+4i)\,(5+4i) &\equiv 1 \\ 15\,(11+4i)\,(13+4i) &\equiv 1 \end{aligned}\right\} \pmod{1+i)^7.$$

To generate the 64 numbers, we must take three numbers each with exponent 4. The product of powers of three such numbers can only be congruent to unity without each being separately congruent to unity, if the product of their squares be so.

Hence, the three generators must be chosen, one each from three of the above rows, and the numbers, with exponent 2 found in these rows, must not have their product congruent to unity.

(xiv.) *The numbers which have as exponents* 2, 4, *and* 8 *for mod* $(1+i)^\lambda$.

We shall use the results of the three following Lemmas:

Lemma (1).—If

$$a^2 \equiv 1 \ [\text{mod } (1+i)^\kappa]$$

$$(a-1)(a+1) \equiv 0 \ [\text{mod } (1+i)^\kappa].$$

Now, if

$$a \equiv 1 \ [\text{mod } (1+i)^3]$$

$$a+1 \equiv 2 \ [\text{mod } (1+i)^3]$$

$$\equiv 0 \ [\text{mod } (1+i)^2],$$

i.e., if $a - 1$ is divisible by $(1 + i)^3$ or any higher power, then $a + 1$ is divisible by $(1 + i)^2$ as the highest power.

Similarly, we can show that if a higher power than $(1 + i)^2$ divides $a + 1$, then no higher power than $(1 + i)^2$ divides $a - 1$.

Therefore, if $\kappa > 4$, either

$$a \equiv 1 \, [\text{mod } (1 + i)^{\kappa - 2}]$$

or

$$a \equiv - 1 \, [\text{mod } (1 + i)^{\kappa - 2}],$$

therefore, κ being > 4, the solutions of

$$a^2 \equiv 1 \, [\text{mod } (1 + i)^{\kappa}]$$

are given by

$$a \equiv \pm 1 \, [\text{mod } (1 + i)^{\kappa - 2}].$$

Lemma (2).—If

$$a^2 \equiv - 1 \, [\text{mod } (1 + i)^{\kappa}]$$

$$(a + i) \, (a - i) \equiv 0 \, [\text{mod } (1 + i)^{\kappa}]$$

Now, if

$$a \equiv i \, [\text{mod } (1 + i)^3]$$

$$a + i \equiv 2i \, [\text{mod } (1 + i)^3]$$

$$\equiv 0 \, [\text{mod } (1 + i)^2].$$

Thus as above, if either of $a + i$, $a - i$ is divisible by a power of $(1 + i)$ above the second, then $(1 + i)^2$, is the highest power that divides the other.

Therefore, if $\kappa > 4$ the solutions of

$$a^2 \equiv - 1 \, [\text{mod } (1 + i)^{\kappa}]$$

are given by

$$a \equiv \pm i \, [\text{mod } (1 + i)^{\kappa - 2}].$$

Lemma (3).

$$a^2 \equiv \pm i \, [\text{mod} (1 + i)^{\kappa}]$$

is impossible for any value of $\kappa > 1$.

For if so,

$$a^2 \equiv \pm i \, [\text{mod } (1 + i)^2],$$

or

$$a^2 \equiv i \, [\text{mod } (1 + i)^2],$$

since

$$- i \equiv i \, [\text{mod } (1 + i)^2],$$

which is impossible, for

$$1^2 \not\equiv i \, [\text{mod } (1 + i)^2],$$

and

$$i^2 \not\equiv i \, [\text{mod } (1 + i)^2].$$

Numbers with exponent 2 *mod* $(1+i)^\lambda$. $(\lambda > 4.)$

All these numbers, together with 1, which has exponent 1, satisfy the congruence

$$\alpha^2 \equiv 1\,[\text{mod }(1+i)^\lambda];$$

therefore

$$\alpha \equiv \pm 1\,[\text{mod }(1+i)^{\lambda-2}]$$

gives (together with unity) the numbers which have exponent 2, mod $(1+i)^\lambda$.

We get thus 7 numbers with exponent 2, mod $(1+i)^\lambda$. They are congruent to

$$\left.\begin{array}{l} -1 \\ \pm 1 + (1+i)^{\lambda-2} \\ \pm 1 + (1+i)^{\lambda-1} \\ \pm 1 + (1+\iota)^{\lambda-2} + (1+i)^{\lambda-1} \end{array}\right\}[\text{mod }(1+i)^\lambda].$$

The product of any two of these is congruent to a third. If we name the 7 numbers thus—

$$\begin{aligned} \alpha &\equiv -1, \\ \beta &\equiv +1 + (1+i)^{\lambda-2}, \\ \gamma &\equiv -1 + (1+i)^{\lambda-2}, \\ \delta &\equiv +1 + (1+i)^{\lambda-1}, \\ \epsilon &\equiv -1 + (1+i)^{\lambda-1}, \\ \eta &\equiv +1 + (1+i)^{\lambda-2} + (1+i)^{\lambda-1}, \\ \theta &\equiv -1 + (1+i)^{\lambda-2} + (1+i)^{\lambda-1}, \end{aligned}$$

then these relations may be written thus—

$$\begin{aligned} \alpha\beta\gamma &\equiv 1, \\ \alpha\delta\epsilon &\equiv 1, \\ \alpha\eta\theta &\equiv 1, \\ \beta\delta\eta &\equiv 1 \\ \beta\epsilon\theta &\equiv 1, \\ \gamma\delta\theta &\equiv 1, \\ \gamma\epsilon\eta &\equiv 1, \end{aligned}$$

[which includes $\alpha\beta \equiv \gamma$, $\alpha\gamma \equiv \beta$, $\beta\gamma \equiv \alpha$, because $\alpha^2 \equiv 1$, $\beta^2 \equiv 1$, $\gamma^2 \equiv 1$].

Numbers with exponent 4, *mod* $(1+i)^\lambda$. $(\lambda > 6.)$

The numbers which satisfy

$$\alpha^4 \equiv 1\,[\text{mod}\,(1+i)^\lambda],$$

and which do not satisfy

$$a^2 \equiv 1 \,[\mathrm{mod}\ (1+i)^\lambda],$$

are the numbers with exponent 4.

Hence Lemma (1) gives us

$$a^2 \equiv \pm 1 \,[\mathrm{mod}\ (1+i)^{\lambda-2}],$$

and then Lemma (2) gives

$$a \equiv \pm 1,\ \pm i \,[\mathrm{mod}\ (1+i)^{\lambda-4}].$$

If of these numbers we exclude

$$a \equiv \pm 1 \,[\mathrm{mod}\ (1+i)^{\lambda-2}],$$

we have the numbers with exponent 4.

Hence the numbers with exponent 4 are

$$\equiv \pm i \,[\mathrm{mod}\ (1+i)^{\lambda-4}],$$

and

$$\left.\begin{array}{l} \equiv \pm 1 + (1+i)^{\lambda-4} \\ \quad \pm 1 + (1+i)^{\lambda-3} \\ \quad \pm 1 + (1+i)^{\lambda-4} + (1+i)^{\lambda-3} \end{array}\right\} \mathrm{mod}\ (1+i)^{\lambda-2}.$$

The number of these is

$$\begin{aligned} &2\mathrm{N}\,[(1+i)^4] + 6\mathrm{N}\,[(1+i)^2] \\ &= 2\,.\,2^4 + 2\,.\,3\,.\,2^2 \\ &= 2^5 + 3\,.\,2^3 \\ &= 2^5 + 2^4 + 2^3 = 56. \end{aligned}$$

Numbers with exponent 8 *mod* $(1+i)^\lambda$. $\lambda > 8$.

The numbers are those that satisfy

$$a^8 \equiv 1 \,[\mathrm{mod}\ (1+i)^\lambda],$$

excluding those that satisfy

$$a^4 \equiv 1 \,[\mathrm{mod}\ (1+i)^\lambda].$$

Now

$$a^8 \equiv 1 \,[\mathrm{mod}\ (1+i)^\lambda]$$

gives

$a^4 \equiv \pm 1 \,[\mathrm{mod}\ (1+i)^{\lambda-2}]$. Lemma (1).

$a^2 \equiv \pm 1 \,[\mathrm{mod}\ (1+i)^{\lambda-4}]$. Lemmas (1), (2), and (3)

$a \equiv \pm 1,\ \pm i \,[\mathrm{mod}\ (1+i)^{\lambda-6}]$. Lemmas (1) and (2)

Of these we have to exclude

$$a \equiv \pm 1\ \ \pm i, \,[\mathrm{mod}\ (1+i)^{\lambda-4}].$$

Hence the numbers with exponent 8 are

$$\left.\begin{array}{l}\pm 1, \pm i, +(1+i)^{\lambda-6} \\ \pm 1, \pm i, +(1+i)^{\lambda-5} \\ \pm 1, \pm i, +(1+i)^{\lambda-6}+(1+i)^{\lambda-5}\end{array}\right\} \bmod (1+i)^{\lambda-4}.$$

In number they are

$$\begin{aligned} & 4 \times 3 \times \mathrm{N}(1+i)^4 \\ = & \ 2^2 . 3 . 2^4 \\ = & \ 2^6+2^7. \end{aligned}$$

We have not specially examined the modulus $(1+i)^8$. The general results already obtained give us the numbers with exponents 2 and 4. We will find the numbers with exponent 8.

Of the solutions of

$$a^8 \equiv 1\,[\bmod\,(1+i)^8]$$

we must exclude those of

$$a^4 \equiv 1\,[\bmod\,(1+i)^8].$$

$$a^8 \equiv 1\,[\bmod\,(1+i)^8]$$

gives

$$a^4 \equiv \pm 1\,[\bmod\,(1+i)^6]$$

and

$$a^2 \equiv \pm 1\,[\bmod\,(1+i)^4],$$

and therefore

$$a \equiv i,\ 3i,\ 2+i,\ 2+3i,\ 3,\ 1+2i,\ 3+2i,\ 1\,[\bmod\,(1+i)^4]. \quad [\text{See mod } (1+i)^4.]$$

The numbers to be excluded are

$$\begin{aligned} a^4 & \equiv 1\,[\bmod\,(1+i)^8] \\ a^2 & \equiv \pm 1\,[\bmod\,(1+i)^6] \\ a & \equiv \pm 1, \pm i\,[\bmod\,(1+i)^4], \end{aligned}$$

i.e.,

$$a \equiv 1,\ 3,\ i,\ 3i\,[\bmod\,(1+i)^4].$$

Hence the numbers $\equiv 2+i,\ 1+2i,\ 2+3i,\ 3+2i\,[\bmod\,(1+i)^4]$ have exponent 8. The number of them is $4 \times 2^4 = 2^6$.

(xv.) *The numbers which have exponent* 2^s *for mod* $(1+i)^\lambda$. [$\lambda > 8$ and $s > 3$.] The numbers are those which satisfy

$$a^{2^s} \equiv 1 \, [\text{mod } (1+i)^\lambda],$$

and which do not satisfy

$$a^{2^{s-1}} \equiv 1 \, [\text{mod } (1+i)^\lambda].$$

The first gives

$$a \equiv \pm 1, \ \pm i \ [\text{mod } (1+i)^{\lambda-2s}] \text{ if } \lambda - 2s > 2,$$

and the second

$$a \equiv \pm 1, \ \pm i, \ [\text{mod } (1+i)^{\lambda-2s+2}].$$

Hence the numbers with exponent 2^s for mod $(1+i)^\lambda$, $(\lambda - 2s > 2)$, are

$$\left.\begin{array}{l} \pm 1, \ \pm i, \ + (1+i)^{\lambda-2s} \\ \pm 1, \ \pm i, \ + (1+i)^{\lambda-2s+1} \\ \pm 1, \ \pm i, \ + (1+i)^{\lambda-2s} + (1+i)^{\lambda-2s+1} \end{array}\right\} \text{mod } (1+i)^{\lambda-2s+2}.$$

The number of them is

$$\begin{aligned} & 12 \times \text{N} \, (1+i)^{2s-2} \\ & = 2^2 . \, 3 . \, 2^{2s-2} \\ & = 2^{2s} + 2^{2s+1}. \end{aligned}$$

If λ is odd, $= 2\mu + 1$ the greatest value of s is $\mu - 1$, and then $\lambda - 2s > 2$. So there are $2^{2\mu-2} + 2^{2\mu-1}$ numbers with exponent $2^{\mu-1}$ for mod $(1+i)^{2\mu+1}$, viz.,

$$\left.\begin{array}{l} \pm 1, \ \pm i, + (1+i)^3 \\ \pm 1, \ \pm i, + (1+i)^4 \\ \pm 1, \ \pm i, + (1+i)^3 + (1+i)^4 \end{array}\right\} \text{mod } (1+i)^5,$$

$$\equiv \left.\begin{array}{cccc} 7+2i, & 5+2i, & 6+3i, & 6+i \\ 5 & 3 & 4+i & 3i \\ 3+2i, & 1+2i, & 2+3i, & 2+i \end{array}\right\} \text{mod } (1+i)^5.$$

So for mod $(1+i)^{2\mu+1}$ there are

$$\left.\begin{array}{lllll} 2^{2\mu-1} + 2^{2\mu-2} & \text{numbers} & \text{with} & \text{exp} & 2^{\mu-1} \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot & \cdot \ \cdot \ \cdot & \cdot \ \cdot \ \cdot & \cdot \ \cdot & \cdot \\ 2^{2s+1} + 2^{2s} & \text{,,} & \text{,,} & \text{,,} & 2^s \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot & \cdot \ \cdot \ \cdot & \cdot \ \cdot \ \cdot & \cdot \ \cdot & \cdot \\ 2^7 + 2^6 & \text{,,} & \text{,,} & \text{,,} & 2^3 \\ 2^5 + 2^4 + 2^3 & \text{,,} & \text{,,} & \text{,,} & 2^2 \\ 2^2 + 2 + 1 & \text{,,} & \text{,,} & \text{,,} & 2 \\ 1 & \text{,,} & \text{,,} & \text{,,} & 1 \end{array}\right\}$$

making, in all, $2^{2\mu} = \Phi \, (1+i)^{2\mu+1}$ numbers, as it should.

If λ is even and $= 2\mu$, then we may have $s = \mu + 1$, and in this case $\lambda - 2s = 2$. We wish, therefore, to find the numbers with exponent $2^{\mu-1}$ for mod $(1 + i)^{2\mu}$. We have to take the solutions of

$$a^{2^{\mu-1}} \equiv 1 \; [\text{mod } (1 + i)^{2\mu}]$$

and exclude those of

$$a^{2^{\mu-2}} \equiv 1 \; [\text{mod } (1 + i)^{2\mu}].$$

The first leads to

$$a^2 \equiv \pm 1 \; [\text{mod } (1 + i)^4],$$

and the second to

$$a \equiv \pm 1, \; \pm i \; [\text{mod } (1 + i)^4],$$

therefore

$$a \equiv 1 + 2i, \; 2 + i, \; 3 + 2i, \; 2 + 3i \; [\text{mod } (1 + i)^4].$$

Hence the numbers with exponent $2^{\mu-1}$ for mod $(1 + i)^{2\mu}$ are

$$\equiv 1 + 2i, \; 2 + i, \; 3 + 2i, \; 2 + 3i \; [\text{mod } (1 + i)^4].$$

The number of them is $4 \times \text{N} \, (1 + i)^{2\mu-4}$

$$= 2^2 . \, 2^{2\mu-4}$$
$$= 2^{2\mu-2}.$$

For mod $(1 + i)^{2\mu}$ there are

$$\left.\begin{array}{llllll}
2^{2\mu-2} & \text{numbers with exp} & & & 2^{\mu-1} \\
2^{2\mu-3} + 2^{2\mu-4} & \text{,,} & \text{,,} & \text{,,} & 2^{\mu-2} \\
\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot \\
2^{2s+1} + 2^{2s} & \text{,,} & \text{,,} & \text{,,} & 2^{s} \\
\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot & \cdot \\
2^7 + 2^6 & \text{,,} & \text{,,} & \text{,,} & 2^3 \\
2^5 + 2^4 + 2^3 & \text{,,} & \text{,,} & \text{,,} & 2^2 \\
2^2 + 2 + 1 & \text{,,} & \text{,,} & \text{,,} & 2 \\
1 & \text{,,} & \text{,,} & \text{,,} & 1
\end{array}\right\}$$

making, in all, $2^{2\mu-1} = \Phi \, (1 + i)^{2\mu}$ numbers as it should.

(xvi.) *Generators for the modulus* $(1 + i)^{\lambda}$.

If $\lambda = 2\mu + 1$, it is easy to see that in order to generate $2^{2\mu}$ numbers, of which for each exponent there shall be the right number of numbers (as just found), we must take three generators :—

$$\left.\begin{array}{l}
\text{two generators with exp } 2^{\mu-1} \\
\text{and} \qquad \text{one generator with exp } 2^2
\end{array}\right\}.$$

If $\lambda = 2\mu$, we can similarly see that we must take three generators

$$\left.\begin{array}{lll} \text{one generator with exp} & & 2^{\mu-1} \\ \quad ,, & ,, & 2^{\mu-2} \\ \quad ,, & ,, & 2^2 \end{array}\right\} .$$

We have now to show how these may be selected, so that all the numbers generated may be incongruent.

Any number, with exponent > 2 modulus $(1+i)^\lambda$, has one of its powers, and one only, which has exponent 2.

Suppose all the numbers modulus $(1+i)^\lambda$, arranged in 7 groups, so that all the numbers in a group may have the same number, with exponent 2, as a power.

We shall now prove four Lemmas with regard to these groups.

Lemma (i.). A power of a number belonging to any group belongs to the same group.

Let a have exponent 2^s, and belong to the group α, *i.e.*,

$$a^{2^{s-1}} \equiv \alpha \, [\text{mod } (1+i)^\lambda].$$

Take any power of a, say $a^{2^\sigma i}$ where i is odd.

Its exponent is $2^{s-\sigma}$. (Proposition iv.)

Therefore

$$(a^{2^\sigma i})^{2^{s-\sigma-1}} \text{ has exp } 2,$$

i.e.,

$$a^{2^{s-1} i} \text{ has exp } 2,$$

and it is $\equiv \alpha^i$, and, therefore, $\equiv \alpha$, because i is odd.

Therefore, any power of a belongs to the same group.

Lemma (ii.). If a and a' have exponent 2^s, and belong to the same group α, then

$$(aa')^{2^{s-1}} \equiv 1 \, [\text{mod } (1+i)^\lambda].$$

For

$$a^{2^{s-1}} \equiv \alpha.$$

$$a'^{2^{s-1}} \equiv \alpha.$$

Therefore

$$(aa')^{2^{s-1}} \equiv \alpha^2 \equiv 1 \, [\text{mod } (1+i)^\lambda].$$

Lemma (iii). If

a has exp 2^s and belongs to group α,

and

b ,, 2^s ,, ,, β,

then

ab ,, 2^s ,, ,, γ,

where

$$\alpha\beta\gamma \equiv 1.$$

For

$$a^{2^s-1} \equiv \alpha,$$
$$b^{2^s-1} \equiv \beta.$$

Therefore

$$(ab)^{2^s-1} \equiv \alpha\beta \equiv \gamma \ [\text{mod } (1+i)^\lambda],$$

and, therefore,

ab has exp 2^s, and belongs to group γ.

Lemma (iv.). If

a has exp $2^{s+\sigma}$ and belongs to group α,

and

b ,, 2^s ,, ,, β,

then

ab ,, $2^{s+\sigma}$,, ,, α.

For

$$a^{2^{s+\sigma}-1} \equiv \alpha,$$
$$b^{2^s-1} \equiv \beta,$$

and, therefore,

$$b^{2^{s+\sigma}-1} \equiv 1.$$

Therefore

$$(ab)^{2^{s+\sigma}-1} \equiv \alpha.$$

Therefore

ab has exp $2^{s+\sigma}$, and belongs to group α.

Now, suppose that we take these generators with the necessary exponents. They must be chosen so that the numbers they generate are all incongruent.

Let g_1, g_2, g_3 be the generators.

Then, $g_1^{i_1}, g_2^{i_2}, g_3^{i_3} \equiv g_1^{i_1'}, g_2^{i_2'}, g_3^{i_3'}$ [mod $(1+i)^\lambda$] must be impossible unless

$$i_1 \equiv i_1' \ (\text{mod exp } g_1),$$
$$i_2 \equiv i_2' \ (\text{mod exp } g_2),$$
$$i_3 \equiv i_3' \ (\text{mod exp } g_3),$$

i.e., $g_1^{j_1}, g_2^{j_2}, g_3^{j_3} \equiv 1$ [mod $(1+i)^\lambda$] must be possible only when

$$j_1 \equiv 0 \ (\text{mod exp } g_1), \quad j_2 \equiv 0 \ (\text{mod exp } g_2), \quad j_3 \equiv 0 \ (\text{mod exp } g_3).$$

We shall now show that to render this so, the three generators must belong to three different groups, and that those three groups must not be any one of the seven sets of three, such as α, β, γ, for which $\alpha\beta\gamma \equiv 1$.

Let us denote the exponents of g_1, g_2, g_3 by 2^{s_1}, 2^{s_2}, 2^{s_3}. [When λ is odd these are $2^{\mu-1}$, $2^{\mu-1}$, 2^2, and when λ is even, $2^{\mu-1}$, $2^{\mu-2}$, 2^2.]

First suppose that any two of the generators belong to the *same* group.

Say g_1 and g_2 both belong to the same group; s_1 and s_2 may be equal or unequal.

If $s_1 = s_2$, Lemma (ii.) shows that the exponent of $g_1 g_2$ divides 2^{s_1-1}, therefore

$$(g_1 g_2)^{2^{s_1-1}} \equiv 1 \; [\bmod\, (1+i)^{\lambda}],$$
$$g_1^{2^{s_1-1}} g_2^{2^{s_2-1}} \equiv 1 \; [\bmod\, (1+i)^{\lambda}],$$

where

$$2^{s_1-1} \not\equiv 0 \; (\bmod\, 2^{s_1}),$$
$$2^{s_2-1} \not\equiv 0 \; (\bmod\, 2^{s_2}),$$

therefore g_1, g_2, cannot belong to the same group.

If $s_1 = s_2 + \sigma$, then $g_2^{2^{\sigma}}$ has exponent s_1, and Lemma (ii.) shows that

$$g_1^{2^{s_1-1}} g_2^{2^{\sigma+s_1-1}} \equiv 1 \; [\bmod\, (1+i)^{\lambda}],$$

where

$$2^{s_1-1} \equiv 0 \; [\bmod\, (1+i)^{\lambda}],$$

therefore g_1 and g_2 cannot belong to the same group.

Hence the three generators must belong to three different groups.

Next suppose that the three generators belong to three groups, such as α, β, γ.

Then

$$g_1^{2^{s_1-1}} g_2^{2^{s_2-1}} g_3^{2^{s_3-1}} \equiv \alpha\beta\gamma \equiv 1 \; [\bmod\, (1+i)^{\lambda}],$$

where

$$2^{s_1-1} \not\equiv 0 \; (\bmod\, 2^{s_1}),$$
$$2^{s_2-1} \not\equiv 0 \; (\bmod\, 2^{s_2}),$$
$$2^{s_3-1} \not\equiv 0 \; (\bmod\, 2^{s_3}),$$

and therefore the three generators must not belong to three such groups.

Finally, we can see that if the three generators are taken from three different groups, excluding the seven sets of three groups, then

$$g_1^{j_1} g_2^{j_2} g_3^{j_3} \equiv 1 \; [\bmod\, (1+i)^{\lambda}]$$

is not possible unless

$$j_1 \equiv 0 \; (\bmod\, 2^{s_1}),$$
$$j_2 \equiv 0 \; (\bmod\, 2^{s_2}),$$
$$j_3 \equiv 0 \; (\bmod\, 2^{s_3}).$$

Lemma (i.) shows that $g_1^{j_1}$, $g_2^{j_2}$, $g_3^{j_3}$ belong to the same groups as do g_1, g_2, g_3. Suppose that these are α, β, and δ.

The Lemmas (iii.) and (iv.) show that the group to which $g_1^{j_1} g_2^{j_2} g_3^{j}$ belongs is one of

$$\alpha, \ \beta, \ \delta, \ \alpha\beta, \ \alpha\delta, \ \beta\delta, \ \text{or} \ \alpha\beta\delta,$$

i.e.,

$$\alpha, \ \beta, \ \delta, \ \gamma, \ \epsilon, \ \eta, \ \text{or} \ \theta,$$

and in no case is the product congruent to unity, save when each factor is so separately.

The result, therefore, is that for modulus $(1 + i)^\lambda$ we have to take three generators with the exponents determined in the last proposition, and such that the product of the three numbers with exponent 2 that they separately produce shall not be congruent to unity.

Example.—Mod $(1 + i)^8$. $\Phi (1 + i)^8 = 2^7$.

The exponents of the generators are

$$2^3, 2^2, 2^2.$$

The 2^6 numbers with exp 2^3 are

$$\left.\begin{array}{ll} 1 + 2i, & 3 + 2i \\ 2 + i, & 2 + 3i \end{array}\right\} (\text{mod } 4).$$

The $2^5 + 2^4 + 2^3$ numbers with exp 2^2 are

$$\left.\begin{array}{r} i \\ 3i \end{array}\right\} (\text{mod } 4) \text{ and } 3, \ 5, \ 3 + 4i, \ 5 + 4i, \ 1 + 4i, \ 7 + 4i \ (\text{mod } 8).$$

The $2^2 + 2 + 1$ numbers with exp 2 are

$$15, \ 9, \ 7, \ 1 + 8i, \ 15 + 8i, \ 9 + 8i, \ 7 + 8i.$$

If of these we take

$$2 + i \text{ with exp } 8,$$
$$3 \text{ with exp } 4,$$

and

$$i \text{ with exp } 4,$$

we have

$$(2 + i)^4 \equiv 9 + 8i \ [\text{mod } (1 + i)^8],$$
$$3^2 \equiv 9 \qquad [\text{mod } (1 + i)^8],$$

and

$$i^2 \equiv 15 \qquad [\text{mod } (1 + i)^8],$$

and

$$9 + 8i \,.\, 9 \,.\, 15 \equiv 15 + 8i \not\equiv 1 \ [\text{mod } (1 + i)^8],$$

therefore, $2 + i$, 3, and i generate the 2^7 numbers.

Example.—Mod $(1+i)^9$. $\Phi(1+i)^9 = 2^8$. The three generators have exponents $2^3, 2^3, 2^2$.

The $2^7 + 2^6$ numbers with exp 2^3 are (see Proposition xv.)—

$$\equiv 7+2i,\ 5+2i,\ 6+3i,\ 6+i,\ 5,\ 3,\ 4+i,\ 3i,\ 3+2i,\ 1+2i,\ 2+3i,\ 2+i\,[\text{mod}\,(1+i)^5].$$

The $2^5 + 2^4 + 2^3$ numbers with exp 2^2 are

$$\equiv \pm i\,[\text{mod}\,(1+i)^5] \text{ and } 7,\ 9,\ 3+4i,\ 5+4i,\ 11+4i,\ 13+4i\,[\text{mod}\,(1+i)^7].$$

The $2^2 + 2 + 1$ numbers with exp 2, are

$$15,\ 17,\ 31,\ 7+8i,\ 23+8i,\ 25+8i,\ 9+8i.$$

If of these we take

$$2+i \text{ with exp } 8,$$
$$3 \text{ with exp } 8,$$

and

$$i \text{ with exp } 4,$$

we have

$$\left.\begin{aligned} (2+i)^4 &\equiv 9+8i \\ 3^4 &\equiv 17 \\ i^2 &\equiv 15 \end{aligned}\right\} \text{ and } 15\,.\,17\,.\,(9+8i) \equiv 7+8i \not\equiv 1.$$

Therefore $2+i$, 3, and i generate the 2^8 numbers.

(xvii.) From Propositions ix., xii., xiii., and xiv. the exponent of a number for any modulus (to which it is prime) is readily determined.

Let the modulus be expressed as a product of powers of its prime factors,

$$m = (1+i)^\kappa \mathrm{P}_1^{\lambda_1}\mathrm{P}_2^{\lambda_2}\ldots \mathrm{Q}_1^{\mu_1}\mathrm{Q}_2^{\mu_2}\ldots,$$

where $\mathrm{P}_1, \mathrm{P}_2, \ldots$ are pure primes, and $\mathrm{Q}_1, \mathrm{Q}_2, \ldots$ are mixed primes, and equal to $\alpha_1 + \beta_1 i,\ \alpha_2 + \beta_2 i \ldots$

Then, by Proposition ix., the exponent of any number a is the L.C.M. of its separate exponents for the moduli

$$(1+i)^\kappa,\ \mathrm{P}_1^{\lambda_1},\ \mathrm{P}_2^{\lambda_2}, \ldots \mathrm{Q}_1^{\mu_1},\ \mathrm{Q}_2^{\mu_2}\ldots$$

The greatest exponent possible.—The greatest exponent for mod $(1+i)^\kappa$ is

$$1 \text{ if } \kappa = 1,$$
$$2 \text{ if } \kappa = 2,$$
$$4 \text{ if } 2 < \kappa < 8,$$
$$\cdots\cdots\cdots$$
$$2^{\kappa'-1} \text{ if } 8 \leqq \kappa = 2\kappa' \text{ or } 2\kappa' + 1.$$

The greatest exponent for mod $P_1^{\lambda_1}$ is

$$P_1^2 - 1 \text{ if } \lambda_1 = 1,$$
$$P_1^{\lambda_1 - 1}(P_1^2 - 1) \text{ if } \lambda_1 > 2.$$

The greatest exponent for mod $Q_1^{\mu_1}$ is

$$\alpha_1^2 + \beta_1^2 - 1 \text{ if } \mu_1 = 1,$$
$$(\alpha_1^2 + \beta_1^2)^{\mu_1 - 1}(\alpha_1^2 + \beta_1^2 - 1) \text{ if } \mu_1 > 1,$$

and the greatest exponent for mod m is the L.C.M. of these separate greatest exponents.

Primitive roots exist when the greatest exponent is equal to $\Phi(m)$, and

$$\Phi(m) = \Phi(1+i)^{\kappa} \,.\, \Phi(P_1^{\lambda_1}) \,.\, \Phi(P_2^{\lambda_2}) \ldots \Phi(Q_1^{\mu_1})\,\Phi(Q_2^{\mu_2}) \ldots$$

and

$$\begin{aligned} \Phi(1+i)^{\kappa} &= 1 \text{ if } \kappa = 1 \\ &= 2^{\kappa - 1} \text{ if } \kappa > 1, \\ \Phi(P_1^{\lambda_1}) &= P_1^2 - 1 \text{ if } \lambda_1 = 1 \\ &= P_1^{2(\lambda_1 - 1)}(P_1^2 - 1) \text{ if } \lambda_1 > 2, \\ \Phi(Q_1^{\mu_1}) &= \alpha_1^2 + \beta_1^2 - 1 \text{ if } \mu_1 = 1 \\ &= (\alpha_1^2 + \beta_1^2)^{\mu_1 - 1}(\alpha_1^2 + \beta_1^2 - 1). \end{aligned}$$

The only cases in which the greatest exponent can be equal to $\Phi(m)$ are those in which the separate exponents are each equal to the Φ of the modulus they refer to and are also co-prime.

Hence we have the following moduli possessed of primitive roots :—

(1) The moduli $1 + i$, $(1 + i)^2$, $(1 + i)^3$;

(2) Any pure prime and $(1 + i) \times$ any pure prime;

(3) A power of a mixed prime and $(1 + i) \times$ a power of a mixed prime.

Example.—To find the exponent of $3 + 2i$ for mod 1000.

$$1000 = 2^3 \,.\, 5^3 = -(1+i)^6 (2+i)^3 (1+2i)^3.$$

The exp of $3 + 2i$ for mod $(1 + i)^6$ is 4.

$$3 + 2i \equiv 1 \pmod{2 + i},$$

and therefore has exponent 1.

$$(3 + 2i)^1 \not\equiv 1 \,[\text{mod } (2 + i)^2]$$

therefore

$$3 + 2i \text{ has exp } 5^2 \text{ for mod } (2+i)^3.$$

$$3 + 2i \equiv 2 \ (\text{mod } 1 + 2i),$$

and therefore has exp 4.

$$(3 + 2i)^4 \equiv 21 \ [\text{mod } (1 + 2i)^2]$$

$$\not\equiv 1,$$

therefore

$$3 + 2i \text{ has exp } 4 \,.\, 5^2 \text{ for mod } (1 + 2i)^3.$$

The exponent required is the L.C.M. of 4, 5^2 and $4 \,.\, 5^2 = 100$.

Therefore the exponent of $3 + 2i$ for mod 1000 is 100.

(xviii.) Proof identical with (18).

If a be not prime to m and $m = p\mathrm{P}$ where p consists of powers of those primes which occur in a, and P is prime to a, then the series of residues

$$a, a^2, a^3, \ldots a^r, \ldots a^{r+t}, \ldots (\text{mod } m)$$

consists of periods of t terms, the first period beginning at the r^{th} term: where t is the exponent of a for mod P, and r is the least number that makes a^r divisible by p.

Corollary.—When $a \equiv 1$ (mod P) $t = 1$ and the period consists of one term.

When a is divisible by p the first period starts from the first term.

When both these conditions hold good then every power of a is $\equiv a$ (mod m).

Example.—Residues of powers of $2 + i$ for mod 10.

$$10 = -(1+i)^2(2+i)(1+2i),$$

$$p = 2 + i,$$

$$\mathrm{P} = (1+i)^2(1+2i),$$

therefore $r = 1$, and t is the exponent of $2 + i$ for mod $(1+i)^2(1+2i)$.

Now

$$\left.\begin{array}{ll} 2 + i \equiv i \,[\text{mod } (1+i)^2] \text{ and therefore has exp} & 2 \\ \text{and} & \\ 2 + i \equiv 4 \ (\text{mod } 1 + 2i) \quad ,, \quad ,, \quad ,, & 2 \end{array}\right\}$$

therefore

$$t = 2.$$

The period consists of two terms and the first period begins at the first term

$$2 + i,$$

$$(2+i)^2 \equiv 3 + 4i,$$

$$(2+i)^3 \equiv 2 + i,$$

&c.

Example.—Residues of powers of $(1 + i)$ for mod $100 = (1 + i)^4 (2 + i)^2 (1 + 2i)^2$,

$$p = (1 + i)^4,$$

and therefore

$$r = 4,$$
$$\mathrm{P} = (2 + i)^2 (1 + 2i)^2.$$

$$1 + i \text{ has exp } 4 \text{ mod } 2 + i \text{ and } (1 + i)^4 \not\equiv 1 \; [\text{mod } (2 + 1)^2],$$

therefore

$$1 + i \text{ has exp } 20 \text{ mod } (2 + i)^2.$$

Similarly

$$1 + i \text{ has exp } 4 \text{ mod } 1 + 2i \text{ and } (1 + i)^4 \not\equiv 1 \; [\text{mod } (1 + 2i)^2],$$

therefore

$$1 + i \text{ has exp } 20 \text{ mod } (1 + 2i)^2,$$

and therefore

$$t = 20.$$

The residues are

$1 + i$	$2i$	$98 + 2i$	96	$96 + 96i$	$92i$	$8 + 92i$	16	$16 + 16i$	$32i$
		$68 + 32i$	36	$36 + 36i$	$72i$	$28 + 72i$	56	$56 + 56i$	$12i$
		$88 + 12i$	76	$76 + 76i$	$52i$	$48 + 52i.$			

Example.—The residues of the powers of $688 + 784i$ for mod 1000.

$$1000 = -(1 + i)^6 (2 + i)^3 (1 + 2i)^3.$$
$$688 + 784i \equiv 0 \; [\text{mod } (1 + i)^6].$$
$$688 + 784i \equiv 688 - 57\,(784) \; [\text{mod } (2 + i)^3]. \quad \text{(See preface.)}$$
$$\equiv 0 \qquad [\text{mod } (2 + i)^3].$$

and

$$688 + 784i \equiv 688 + 57\,(784) \; [\text{mod } (1 + 2i)^3].$$
$$\equiv 1 \qquad [\text{mod } (1 + 2i)^3].$$

Hence, $r = 1$ and $t = 1$, and so all powers of the number $688 + 784i$ are congruent to itself for mod 1000.

$$\begin{array}{lll}
688 & 688 & 784 \\
\underline{688} & \underline{784} & \underline{784} \\
504 & 752 & 136 \\
04 & 04 & 72 \\
\underline{8} & \underline{6} & \underline{8} \\
344 & 392 & 656 \\
 & \underline{2} & \\
 & 784 &
\end{array}$$

therefore

$$(688 + 784i)^2 \equiv (344 - 656 + 784i) \pmod{1000}$$
$$\equiv 688 + 784i \pmod{1000}.$$

(xix. and xx.) The proofs and results are the same as in (19) and (20).

Example.—Mod $10 = 2.\ 5 = -(1+i)^2(2+i)(1+2i)$.

To find a so that

$$a \equiv a_1 \,[\text{mod}\,(1+i)^2]$$
$$\equiv a_2 \,[\text{mod}\,(2+i)]$$
$$\equiv a_3 \,[\text{mod}\,(1+2i)].$$

We shall have

$$a \equiv a_1\xi_1 + a_2\xi_2 + a_3\xi_3 \pmod{10},$$

where ξ_1, ξ_2, ξ_3 are found thus :—

$$\xi_1 = (2+i)(1+2i)\,x_1 \equiv 1 \,[\text{mod}\,(1+i)^2],$$
$$5i\,x_1 \equiv 1 \,[\text{mod}\,(1+i)^2],$$
$$i\,x_1 \equiv 1 \,[\text{mod}\,(1+i)^2],$$
$$x_1 \equiv i,$$

and therefore

$$\xi_1 \equiv 5 \pmod{10}.$$
$$\xi_2 = (1+i)^2(1+2i)\,x_2 \equiv 1 \,(\text{mod}\ 2+i),$$
$$(2i-4)\,x_2 \equiv 1 \,(\text{mod}\ 2+i),$$
$$-8x_2 \equiv 1 \,(\text{mod}\ 2+i),$$
$$x_2 \equiv 3 \,(\text{mod}\ 2+i),$$

and therefore

$$\xi_2 \equiv 8 + 6i \pmod{10}.$$
$$\xi_3 = (1+i)^2(2+i)\,x_3 \equiv 1 \,(\text{mod}\ 1+2i),$$
$$(4i-2)\,x_3 \equiv 1 \,(\text{mod}\ 1+2i),$$
$$-4x_3 \equiv 1 \,(\text{mod}\ 1+2i),$$
$$x_3 \equiv 1 \,(\text{mod}\ 1+2i),$$

and therefore

$$\xi_3 \equiv 8 + 4i \pmod{10}.$$

Hence

$$a \equiv 5a_1 + (8+6i)\,a_2 + (8+4i)\,a_3 \pmod{10},$$

e.g., if

$$\left.\begin{aligned} a_1 &\equiv 1 \,[\text{mod}\,(1+i)^2] \\ a_2 &\equiv 4 \,(\text{mod}\ 2+i) \\ a_3 &\equiv 3 \,(\text{mod}\ 1+2i) \end{aligned}\right\}$$

then

$$\begin{aligned} a &\equiv 5 + (8 + 6i)\,4 + (8 + 4i)\,.\,3\,.\,(\text{mod } 10),\\ &\equiv 5 + 2 + 4i + 4 + 2i \ (\text{mod } 10),\\ &\equiv 1 + 6i \ (\text{mod } 10). \end{aligned}$$

(xxi.) The number of numbers which belong to a given exponent when the modulus is a power of a prime.

We have three cases to consider

(1) When the modulus is a power of a pure prime.
(2) When the modulus is a power of a mixed prime.
(3) When the modulus is a power of $1 + i$.

(1.) We saw in Proposition xii*a*, that for the modulus p^λ we can generate the $\Phi(p^\lambda) = p^{2(\lambda-1)}(p^2 - 1)$ residues prime to the modulus by three generators

and
$$\left.\begin{array}{l} f \text{ and } f' \text{ each with exp } p^{\lambda-1} \\ h \text{ with exp } p^2 - 1. \end{array}\right\}$$

How many of the residues have exponent $p^s t$, where $s \not> \lambda - 1$ and t divides $p^2 - 1$?

Of the $p^2 - 1$ powers of h, $\phi(t)$ have exponent t.

Of the $p^{\lambda-1}$ powers of f, p^s have exponent a power of $p \not> p^s$.

Of the $p^{\lambda-1}$ powers of f', p^s have exponent a power of $p \not> p^s$.

Therefore f and f' generate p^{2s} numbers with exponent $\not> p^s$, and similarly p^{2s-2} numbers with exponent $\not> p^{s-1}$.

Therefore f and f' generate $p^{2s} - p^{2s-2}$ numbers with exponent p^s.

Therefore f, f', and g generate $(p^{2s} - p^{2s-2})\,\phi(t)$ numbers with exponent $p^s t$, *i.e.*, the number of numbers with exponent $p^s t$ mod p^λ is $(p^{2s} - p^{2s-2})\,\phi(t)$.

In particular the number of numbers with exponent p^s is $p^{2s} - p^{2s-2}$, and the number of numbers with exponent t is $\phi(t)$ if t is prime to p.

(2.) Any power of a mixed prime, $(\alpha + \beta i)^\lambda$, has primitive roots, and hence, as in (21), the number of numbers with exponent t [any divisor of $(\alpha^2 + \beta^2)^\lambda - (\alpha^2 + \beta^2)^{\lambda-1}$] is $\phi(t)$.

(3.) The number of numbers with exponent a given power of 2 for modulus $(1 + i)^\nu$ was found completely in Propositions xiv. and xv.

(xxi*a*.) It will be convenient for the succeeding propositions to express the number of numbers which have a given exponent for modulus $(1 + i)^\lambda$, in terms of the exponents of the generators. By so doing we shall avoid the detailed discussion of the cases arising from different values of λ, which was necessary in Part I.

Suppose 2^κ, $2^{\kappa'}$, $2^{\kappa''}$ are the exponents of the generators.

The exponent of the product of any powers of the generators is equal to the highest exponent of the three (Proposition viii.).

The generator with exponent 2^κ generates

$$2^s \text{ numbers with exp} \ngtr 2^s \text{ if } \kappa > s,$$
$$2^\kappa \text{ numbers with exp} \ngtr 2^s \text{ if } \kappa \overset{=}{<} s.$$

Suppose by $(\kappa)_s$ we denote that κ is to be replaced by s if κ exceeds s. Then in either case the generator with exponent 2^κ generates

$$2^{(\kappa)_s} \text{ numbers with exp} \ngtr 2^s.$$

Similarly the two other generators generate respectively

$$2^{(\kappa')_s} \text{ numbers with exp} \ngtr 2^s,$$

and

$$2^{(\kappa'')_s} \text{ numbers with exp} \ngtr 2^s.$$

Therefore the three generators generate

$$2^{(\kappa + \kappa' + \kappa'')_s} \text{ numbers with exp} \ngtr 2^s,$$

i.e., there are, for modulus $(1 + i)^\lambda$,

$$2^{(\kappa + \kappa' + \kappa'')_s} \text{ numbers with exp} \ngtr 2^s.$$

Similarly there are

$$2^{(\kappa + \kappa' + \kappa'')_{s-1}} \text{ numbers with exp} \ngtr 2^{s-1}.$$

Hence there are

$$2^{(\kappa + \kappa' + \kappa'')_s} - 2^{(\kappa + \kappa' + \kappa'')_{s-1}}$$

numbers with exponent 2^s for modulus $(1 + i)^\lambda$.

This result clearly holds good when one or two of the κ's is absent, as is the case when λ is less than 5.

(xxii.) The number of numbers, for modulus m, each of which has as exponent some power of a prime p, p being a divisor of $\Phi(m)$.

Let

$$m = (1 + i)^\kappa P_1^{\lambda_1} P_2^{\lambda_2} \dots Q_1^{\mu_1} Q_2^{\mu_2} \dots$$

where P_1, P_2, ... are pure primes and Q_1, Q_2, ... are mixed primes.

$$\Phi(m) = \Phi(1 + i)^\kappa . \Phi(P_1^{\lambda_1}) . \Phi(P_2^{\lambda_2}) \dots \Phi(Q_1^{\mu_1}) \Phi(Q_2^{\mu_2}) \dots$$
$$\Phi(1 + i)^\kappa = 2^{\kappa_0} 2^{\kappa_0'} 2^{\kappa_0''}$$

where 2^{κ_0}, $2^{\kappa_0'}$, $2^{\kappa_0''}$, are the exponents of the generators for mod $(1 + i)^\kappa$, and suppose that

$$\Phi(P_1^{\lambda_1}) = P_1^{2(\lambda_1 - 1)} (P_1^2 - 1) = 2^{\kappa_1} p^{l_1} q^{m_1} \dots$$
$$\Phi(P_2^{\lambda_2}) = P_2^{2(\lambda_2 - 1)} (P_2^2 - 1) = 2^{\kappa_2} p^{l_2} q^{m_2} \dots$$

&c. &c.

With regard to these we shall follow a convention which will be useful in simplifying the next proposition.

Firstly, we write $2^{\kappa_0}2^{\kappa_0'}2^{\kappa_0''}$, and not $2^{\overline{\kappa_0+\kappa_0'+\kappa_0''}}$. Thus the value of $(\Sigma\kappa)_s$ will be $(\kappa_0+\kappa_0'+\kappa_0''+\kappa_1+\kappa_2+\ldots)_s$, and not $\overline{(\kappa_0+\kappa_0'+\kappa_0''}+\kappa_1+\kappa_2+\ldots)_s$.

Secondly, in $2^{\kappa_1}p^{l_1}q^{m_1}\ldots$ occurs the prime P_1 raised to the power $2(\lambda_1-1)$.

We shall suppose it written $P_1^{\lambda_1-1}\,.\,P_1^{\lambda_1-1}$, and not $P_1^{2(\lambda_1-1)}$.

For the rest of the principal factors no such arrangement is necessary.

Let

$$\begin{array}{c}\Phi(Q_1^{\mu_1}) = 2^{\kappa_1'}p_1^{l_1'}q_1^{m_1'}\ldots\\ \Phi(Q_2^{\mu_2}) = 2^{\kappa_2'}p_1^{l_2'}q_1^{m_2'}\ldots\\ \text{\&c.} \qquad\qquad \text{\&c.}\end{array}$$

As in (22) the number of numbers which have some power of p as exponent is the product of the number of such numbers for each separate modulus $(1+i)^{\kappa}$, $P_1^{\lambda_1}$, &c.

$$\left.\begin{array}{llllll}\text{Now the number of numbers for mod} & P_1^{\lambda_1} & \text{is the power of } p \text{ in} & \Phi(P_1^{\lambda_1})\\ \quad ,, \qquad ,, \qquad ,, & P_2^{\lambda_2} & \quad ,, \qquad ,, & \Phi(P_2^{\lambda_2})\\ \quad ,, \qquad ,, \qquad ,, & Q_1^{\mu_1} & \quad ,, \qquad ,, & \Phi(Q_1^{\mu_1})\\ & \text{\&c.} & & \end{array}\right\}\text{Prop. (xxi.).}$$

Hence the number of numbers with exponent a power of p is $p^{\Sigma(l)}$.

In particular the number of numbers with exponent a power of 2 is $2^{\Sigma(\kappa)}$.

(xxiii.) The number of numbers having a given exponent p^s for modulus m, p^s being a divisor of the greatest exponent.

Any such number has a power of p or unity for its exponent for each of the moduli $(1+i)^{\kappa}$, $P_1^{\lambda_1}$, . . . &c., and the greatest power of p among these exponents must be p^s.

As in (23) the number of numbers, modulus $P_1^{\lambda_1}$, which have as exponent a power of p not greater than p^s is (if P_1 is not p) $p^{(l_1)_s}$.

In the particular case when P_1 is the prime p, if

$$\left.\begin{array}{l}\quad\lambda_1 - 1 \text{ is } \geqq s \text{ there are } p^{2s} \text{ numbers, mod } P_1^{\lambda_1} \text{ with exp a power of } p \ngtr p^s\\ \text{and if}\\ \quad\lambda_1 - 1 \text{ is } < s \text{ there are } p^{2(\lambda_1-1)} \quad ,, \qquad ,, \qquad ,, \qquad ,,\end{array}\right\}$$

In either case the number is $p^{(\overline{\lambda_1-1}+\overline{\lambda_1-1})_s}$.

Now we have arranged that in $\Phi(P_1^{\lambda_1})$ the power of P_1 (p in this case) shall be written $p^{l_1}p^{l_1}$, and not p^{2l_1}: thereby we easily express the number of numbers with exponent a power of $p \ngtr p^s$ for modulus $P_1^{\lambda_1}$, which is $p^{(l_1)_s}p^{(l_1)_s}$ or $p^{(l_1+l_1)_s}$, but which is *not* $p^{(2l_1)_s}$.

For modulus $Q_1^{\mu_1}$ the number of numbers with exponent a power of $p \ngtr p^s$ is $p^{(l_1')_s}$.

Multiplying these numbers we get $p^{(\Sigma l)_s}$ as the number of numbers with exponent a power of $p \ngtr p^s$.

Hence the number of numbers with exponent p^s modulus m is

$$p^{(\Sigma l)_s} - p^{(\Sigma l)_{s-1}}.$$

Example.—Mod $3^3(3 + 2i)$. The greatest exponent is the L.C.M. of $3^2(3^2 - 1)$ and 12, $= 72$.

We will find how many numbers have exponent 3^2, and how many numbers have exponent 3.

$$\Phi(3^3) = 2^3 . 3^2 . 3^2.$$
$$\Phi(3 + 2i) = 2^2 . 3.$$

The number of numbers with exponent 3^2 is

$$3^{(2+2+1)_2} - 3^{(2+2+1)_1}$$
$$= 3^5 - 3^3.$$

The number of numbers with exponent 3 is

$$3^{(2+2+1)_1} - 1$$
$$= 3^3 - 1.$$

Hence

$$\begin{array}{lll} & 3^5 - 3^3 & \text{numbers have exp } 3^2.\\ & 3^3 - 1 & \text{numbers have exp } 3.\\ \text{and} & & \\ & 1 & \text{number has exp } 1.\end{array}\left.\vphantom{\begin{array}{l}a\\a\\a\\a\end{array}}\right\}$$

(xxiii*a*.) By writing $\Phi(1 + i)^\kappa$ in the form $2^{\kappa_0}2^{\kappa_0'}2^{\kappa_0''}$ we are enabled to apply exactly the same method to this case as we have to the case of any odd prime p. The result we obtain is that the number of numbers with exponent 2^s for mod m is

$$2^{(\Sigma\kappa)_s} - 2^{(\Sigma\kappa)_{s-1}}.$$

Example.—Mod $12 + 4i = i(1 + i)^5(1 + 2i)$. Highest exponent $= 2^2$. $\Phi(m) = 2^6$.

$$\Phi(1 + i)^5 = 2^2 . 2 . 2.$$
$$\Phi(1 + 2i) = 2^2.$$

The number of numbers with exponent 2^2

$$= 2^{(2+1+1+2)_2 - (2+1+1+2)_1}$$
$$= 2^6 - 2^4.$$

The number of numbers with exponent 2

$$\begin{aligned} &= 2^{(2+1+1+2)_1} - 1 \\ &= 2^4 - 1. \end{aligned}$$

Thus there are

$$\left.\begin{array}{llll} 2^6 - 2^4 & \text{numbers with exp} & & 4. \\ 2^4 - 1 & ,, & ,, & 2. \\ 1 & ,, & ,, & 1. \end{array}\right\}$$

Corollary to xxiii. and xxiii*a*.—The number of numbers that belong to any exponent is simply the product of the number of numbers that belong to each of its principal factors as exponents.

Thus, for mod m, the number of numbers with exp $t = p^s q^\sigma \ldots$ is

$$[p^{(\Sigma l)_s} - p^{(\Sigma l)_s - 1}] [q^{(\Sigma m)_\sigma} - q^{(\Sigma m)_\sigma - 1}] \ldots$$

Example.—Mod $(1+i)^6 . 3^2 (3+2i)(4+i) = -i . 72 (10 + 11i)$.

$$\begin{aligned} \Phi (1+i)^6 &= 2 .\ 2^2 .\ 2^2. \\ \Phi (3^2) &= 2^3 .\ 3 .\ 3. \\ \Phi (3+2i) &= 2^2 .\ 3. \\ \Phi (4+i) &= 2^4. \end{aligned}$$

The highest exponent is $2^4 .\ 3^3$.

$$\Phi (m) = 2^{14} .\ 3^3.$$

The numbers κ are 1. 2. 2. 3. 2. 4.

$$\begin{aligned} (\Sigma\kappa)_4 &= 14, \\ (\Sigma\kappa)_3 &= 13, \\ (\Sigma\kappa)_2 &= 11, \\ (\Sigma\kappa)_1 &= 6, \\ (\Sigma\kappa)_0 &= 0. \end{aligned}$$

Therefore

$$\begin{array}{lllll} 2^{14} - 2^{13} & \text{numbers} & \text{have} & \text{exp} & 2^4, \\ 2^{13} - 2^{11} & ,, & ,, & ,, & 2^3, \\ 2^{11} - 2^6 & ,, & ,, & ,, & 2^2, \\ 2^6 - 1 & ,, & ,, & ,, & 2. \end{array}$$

The numbers l for the prime 3 are 1. 1. 1.

$$(\Sigma l)_1 = 3,$$
$$(\Sigma l)_0 = 0,$$

therefore $3^3 - 1$ numbers have exp 3.

From these we deduce that

$$\begin{array}{llll}
(3^3 - 1)(2^{14} - 2^{13}) & \text{numbers have exp} & & 3.\ 2^4, \\
(3^3 - 1)(2^{13} - 2^{11}) & ,, \quad ,, & ,, & 3.\ 2^3, \\
(3^3 - 1)(2^{11} - 2^{6}) & ,, \quad ,, & ,, & 3.\ 2^2, \\
(3^3 - 1)(2^{6} - 1) & ,, \quad ,, & ,, & 3.\ 2.
\end{array}$$

Including 1 which has exponent 1, this makes the complete set of $2^{14}.\ 3^3$ numbers.

(xxiv.) A special set of generators which generate the $\Phi(m)$ numbers, modulus m.

$$m = (1 + i)^{\kappa}\, \mathrm{P}_1^{\lambda_1} \ldots \mathrm{Q}_1^{\mu_1} \ldots$$

Three numbers ϕ, ϕ', ϕ'', generate the residues for modulus $(1 + i)^{\kappa}$. (Propositions xiii. and xvi.) Two numbers g_1, f_1, generate the residues for modulus $\mathrm{P}_1^{\lambda_1}$. (Proposition xii*a*.), &c. One number g'_1 generates the residues for modulus $\mathrm{Q}_1^{\mu_1}$. (Proposition xii.), &c.

Suppose any number a modulus m is

$$\begin{aligned}
&\equiv \alpha_0\, [\text{mod}\ (1 + i^{\kappa})] \\
&\equiv \alpha_1\, (\text{mod}\ \mathrm{P}_1^{\lambda_1}), \\
&\qquad \&\text{c.} \\
&\equiv \alpha'_1\, (\text{mod}\ \mathrm{Q}_1^{\mu_1}), \\
&\qquad \&\text{c.}
\end{aligned}$$

Then

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \ldots + \alpha'_1\xi'_1 + \ldots\ (\text{mod}\ m). \quad \text{(Proposition xix.)}$$

If now

$$\begin{aligned}
\alpha_0 &\equiv \phi^{i_0}\phi'^{i'_0}\phi''^{i''_0}\, [\text{mod}\ (1 + i)^{\kappa}] \\
\alpha_1 &\equiv g_1^{i_1} f_1^{j_1}\, (\text{mod}\ \mathrm{P}_1^{\lambda_1}), \\
&\qquad \&\text{c.} \\
\alpha'_1 &\equiv g'^{\,i'_1}_1\, (\text{mod}\ \mathrm{Q}_1^{\mu_1}), \\
&\qquad \&\text{c.}
\end{aligned}$$

Thus

$$\begin{aligned}
a &\equiv \phi^{i_0}\phi'^{i'_0}\phi''^{i''_0}\xi_0 + g_1^{i_1} f_1^{j_1}\xi_1 + \ldots + g'^{\,i'_1}_1\xi'_1 + \ldots\ (\text{mod}\ m) \\
&\equiv [\phi\xi_0 + \xi_1 + \ldots + \xi'_1 + \ldots]^{i_0} [\phi'\xi_0 + \xi_1 + \ldots + \xi'_1 + \ldots]^{i'_0} \\
&\qquad [\phi''\xi_0 + \xi_1 + \ldots + \xi'_1 + \ldots]^{i''_0} [\xi_0 + g_1\xi_1 + \ldots + \xi'_1 + \ldots]^{i_1} \\
&\qquad\quad [\xi_0 + f_1\xi_1 + \ldots]^{j_1} \ldots [\xi_0 + \xi_1 + \ldots + g'_1\xi'_1 + \ldots]^{i'_1} \ldots\ (\text{mod}\ m) \\
&\equiv [(\phi - 1)\,\xi_0 + 1]^{i_0} [(\phi' - 1)\,\xi_0 + 1]^{i'_0} [(\phi'' - 1)\,\xi_0 + 1]^{i''_0} [(g_1 - 1)\,\xi_1 + 1]^{i_1} \\
&\qquad\qquad [(f_1 - 1) + 1]^{j_1} \ldots [(g'_1 - 1) + 1]^{i'_1} \ldots\ (\text{mod}\ m),
\end{aligned}$$

where the numbers in the square brackets are a set of generators whose exponents are, in order, $\kappa_0, \kappa'_0, \kappa''_0, P_1^{\lambda_1-1}(P_1^2-1), P_1^{\lambda_1-1}, \ldots \Phi(Q_1^{\mu_1}) \ldots$ whose product is equal to $\Phi(m)$.

Example.—Mod $10 = -(1+i)^2(2+i)(1+2i)$

$$\begin{array}{llllll} i & \text{with exp } 2 & \text{generates the } 2 & \text{numbers mod} & (1+i)^2 \\ 2 & \quad ,, \quad 4 & \quad ,, \quad 4 & \quad ,, & 2+i \\ 2 & \quad ,, \quad 4 & \quad ,, \quad 4 & \quad ,, & 1+2i. \end{array}$$

Hence the following will generate the $\Phi(10) = 32$ numbers, mod 10, viz.,

$$\left.\begin{array}{l} i\xi_1 + \xi_2 + \xi_3 \\ \xi_1 + 2\xi_2 + \xi_3 \\ \xi_1 + \xi_2 + 2\xi_3 \end{array}\right\} \text{mod } 10,$$

where

$$\xi_1 = 5 \quad \xi_2 = 8 + 6i \quad \xi_3 = 8 + 4i. \quad \text{(See Example, Proposition xix.)}$$

$$\left.\begin{array}{l} 5i + 8 + 6i + 8 + 4i \equiv 6 + 5i \\ 5 + 2(8 + 6i) + 8 + 4i \equiv 9 + 6i \\ 5 + 8 + 6i + 2(8 + 4i) \equiv 9 + 4i \end{array}\right\} \text{(mod 10)}.$$

Hence

$$\left.\begin{array}{lll} 9 + 4i & \text{with exp} & 4 \\ 9 + 6i & \quad ,, & 4 \\ 6 + 5i & \quad ,, & 2 \end{array}\right\} \text{generate the 32 residues.}$$

The indices corresponding to each of the 32 numbers are given in the following table.

Numbers.	Indices of			Numbers.	Indices of		
	$6+5i$.	$9+6i$.	$9+4i$.		$6+5i$.	$9+6i$.	$9+4i$.
1	0	0	0	$6+5i$	1	0	0
$9+4i$	0	0	1	$4+9i$	1	0	1
$5+2i$	0	0	2	$7i$	1	0	2
$7+8i$	0	0	3	$2+3i$	1	0	3
$9+6i$	0	1	0	$4+i$	1	1	0
7	0	1	1	$2+5i$	1	1	1
$3+8i$	0	1	2	$8+3i$	1	1	2
$5+4i$	0	1	3	$9i$	1	1	3
$5+8i$	0	2	0	$3i$	1	2	0
$3+2i$	0	2	1	$8+7i$	1	2	1
9	0	2	2	$4+5i$	1	2	2
$1+6i$	0	2	3	$6+i$	1	2	3
$7+2i$	0	3	0	$2+7i$	1	3	0
$5+6i$	0	3	1	i	1	3	1
$1+4i$	0	3	2	$6+9i$	1	3	2
3	0	3	3	$8+5i$	1	3	3

E.g., $6 + i \equiv (6 + 5i)(9 + 6i)^2(9 + 4i)^3$ (mod 10).

(xxv.) Proposition (25) is completely applicable to the case of complex numbers and moduli. The result arrived at we may re-state as follows:—The most general set of numbers which generate the $\Phi\,(m)$ residues for modulus m must always be constructed in this manner, viz., we must form

$$\left.\begin{array}{c}\text{a set of } p\text{-power-exponent generators}\\ \text{a set of } q\text{-power-exponent generators}\\ \text{\&c.} \qquad \text{\&c.}\end{array}\right\} p,\ q, \ldots \text{ being the prime factors of } \Phi\,(m),$$

each generator must then be formed by taking numbers from these sets, not more than one from each, and forming their product. Each number, moreover, in these subsidiary sets is to appear once and once only as a factor of one of the generators that are thus formed.

It remains to investigate the most general mode of formation of a set of p-power-exponent generators.

(xxvi.) The proof of Proposition (26) holds good for complex numbers and moduli. It shows that the exponents of any set of p-power-exponent generators must be the same set of powers of p as those which occur in the Φ's of the principal factors of m; *i.e.*, they are what have been denoted by p^{l_1}, p^{l_2}, . . . $p^{l'_1}$, $p^{l'_2}$, . . (see Proposition xxii., in which the convention stated must be strictly attended to). The convention of Proposition xxii. makes the treatment of the 2-power-exponent numbers uniform with that of the p-power-exponent numbers; hence the same result is true for the 2-power-exponent numbers, viz., that the exponents of any set of 2-power-exponent generators must be 2^{κ_0}, $2^{\kappa'_0}$, $2^{\kappa''_0}$, 2^{κ_1}, . . . $2^{\kappa'_1}$ . . .

The least number of generators for a given modulus m.

As in (26) we see that the least number of generators is equal to the number of terms of that row which contains most among the following:—

$$\begin{array}{rl}\kappa_0\ \kappa'_0\ \kappa''_0\ \kappa_1\ \kappa_2 \ldots & \kappa'_1\ \kappa'_2 \ldots\\ l_1\ l_2 \ldots & l'_1\ l'_2 \ldots\\ m_1\ m_2 \ldots & m'_1\ m'_2 \ldots\end{array}$$

Consider first the set of numbers, $\kappa_0\ \kappa'_0\ \kappa''_0\ \kappa_1 \ldots \kappa_1 \ldots$

Since P_1 is an odd prime, therefore

$$\Phi\,(P_1^{\lambda_1}) = P_1^{2(\lambda_1-1)}(P_1^2 - 1)$$

is even, and therefore κ_1 always occurs.

Similarly in

$$\Phi\,(Q_1^{\mu_1}) = (\alpha_1^2 + \beta_1^2)^{\mu_1-1}(\alpha_1^2 + \beta_1^2 - 1)$$

$\alpha_1^2 + \beta_1^2$ is a real odd prime number, and therefore

$$\Phi\ (Q_1^{\mu_1})$$

is even, and therefore κ'_1 always occurs.

For the numbers, $\kappa_0\ \kappa'_0\ \kappa''_0$, all three occur if

$$\kappa > 4 \qquad [m = (1+i)^\kappa P_1^{\lambda_1} \ldots Q_1^{\mu_1} \ldots],$$

two occur if

$$\kappa = 4,$$

one occurs if

$$\kappa = 3,\ 2,\ 1,$$

none occur if

$$\kappa = 0.$$

Hence, if we denote by n the number of different prime factors, $P_1\, P_2 \ldots Q_1\, Q_2 \ldots$ of m (excluding $1 + i$) then the number of terms in the set of numbers $\kappa_0\ \kappa'_0\ \kappa''_0\ \kappa_1 \ldots \kappa'_1 \ldots$ is

$$\begin{array}{lll} n + 3 & \text{if} & \kappa > 4, \\ n + 2 & \text{if} & \kappa = 4, \\ n + 1 & \text{if} & \kappa = 3,\ 2,\ 1, \\ n & \text{if} & \kappa = 0. \end{array}$$

These numbers give an inferior limit to the least number of generators. In all but one exceptional case the least number of generators coincides with these.

Consider next the set of numbers, $l_1\ l_2 \ldots l_1'\ l_2' \ldots$

l_1' occurs if p divides

$$\Phi\ (P_1^{\lambda_1}) = P_1^{2(\lambda_1 - 1)}(P_1^2 - 1).$$

If it divides

$$P_1^2 - 1$$

it occurs once. If it is identical with P_1 then it occurs twice ($l_1 = \lambda_1 - 1$), and the set of numbers would be written $l_1\ l_1\ l_2 \ldots l_1'\ l_2' \ldots$

l_2 occurs if p divides

$$\Phi\ (Q_1^{\mu_1}).$$

Therefore the number of terms in the set exceeds n (and is equal to $n + 1$) only when m is such that

$$\Phi\ (Q_1^{\mu_1}),\ \Phi\ (Q_2^{\mu_2}), \ldots\ \Phi\ (P_1^{\lambda_1}),\ \Phi\ (P_2^{\lambda_2}) \ldots$$

are all divisible by p and also one of the primes, P_1, P_2, ... is equal to p.

If this be the case when $\kappa = 0$ the least number of generators is $n + 1$.

The result may be stated thus :—

If n be the number of different prime factors of $m = (1 + i)^{\kappa} P_1^{\lambda_1} \ldots Q_1^{\mu_1}, \ldots$ excluding $(1 + i)$, then the least number of generators is

$$\begin{aligned} &n + 3 \text{ if } \kappa > 4, \\ &n + 2 \text{ if } \kappa = 4, \\ &n + 1 \text{ if } \kappa = 3, 2, 1, \\ &n \quad\;\;\; \text{ if } \kappa = 0, \end{aligned}$$

unless one of the primes P_1 divides $P_2^2 - 1$, $P_3^2 - 1, \ldots$ and $\Phi(Q_1^{\mu_1})$, $\Phi(Q_2^{\mu_2}), \ldots$ in which case it is $n + 1$.

(xxvii.) The formation of a set of *unitary* p-power-exponent generators for mod m.

Let a be any p-power-exponent number, and

$$a \equiv \alpha_0\xi_0 + \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots + \alpha'_1\xi'_1 + \alpha'_2\xi'_2 + \ldots \pmod{m}.$$

Then $\alpha_0, \alpha_1, \alpha_2, \ldots \alpha'_1, \ldots$ must each have as exponent, for its own modulus, either unity or a power of p.

Suppose first that p is not $= 2$, but an odd (real) prime.

Take

$$\begin{aligned} &\gamma_1 \text{ with exp } p^{l_1} \; (\text{mod } P_1^{\lambda_1}), \\ &\gamma_2 \quad \text{,,} \quad \text{,,} \quad p^{l_2} \; (\text{mod } P_2^{\lambda_2}), \\ &\quad \&\text{c.}, \qquad\qquad \&\text{c.}, \end{aligned}$$

if in any case p is not a factor of $\Phi(P^{\lambda})$ γ is $\equiv 1$.

If for one of these, say P_1, $p = P_1$, then take also β_1 with exponent p^{l_1}, modulus $P_1^{\lambda_1}$, as in Proposition xii., so that β_1 and γ_1, each with exponent $l_1 = \lambda_1 - 1$, generate all the p-power-exponent numbers, modulus $P_1^{\lambda_1}$.

Take also

$$\begin{aligned} &\gamma'_1 \text{ with exp } p^{l'_1}, \text{ mod } Q_1^{\mu_1}, \\ &\gamma'_2 \quad \text{,,} \quad \text{,,} \quad p^{l'_2}, \text{ ,, } \; Q_2^{\mu_2}, \\ &\quad \&\text{c.}, \qquad\qquad \&\text{c.} \end{aligned}$$

Then we have

$$\alpha_0 \equiv 1 \; [\text{mod } (1 + i)^{\kappa}],$$

and can put

$$\begin{aligned} &\alpha_1 \equiv \gamma_1^{i_1}\beta_1^{j_1} \; (\text{mod } P_1^{\lambda_1}), \\ &\alpha_2 \equiv \gamma_2^{i_2} \; (\text{mod } P_2^{\lambda_2}), \\ &\quad \&\text{c.}, \qquad \&\text{c.}, \\ &\alpha'_1 \equiv \gamma'^{\,l'_1}_1 \; (\text{mod } Q_1^{\mu_1}), \\ &\alpha'_2 \equiv \gamma'^{\,i'_2}_2 \; (\text{mod } Q_2^{\mu_2}), \\ &\quad \&\text{c.}, \end{aligned}$$

and so

$$a \equiv [(\gamma_1 - 1)\xi_1 + 1]^{i_1} [(\beta_1 - 1)\xi_1 + 1]^{j_1} [(\gamma_2 - 1)\xi_2 + 1]^{i_2} \ldots$$
$$[(\gamma'_1 - 1)\xi'_1 + 1]^{i'_1} [(\gamma'_2 - 1)\xi'_2 + 1]^{i'_2} \ldots \pmod{m}.$$

The quantities in the brackets we shall denote by $g_1 g_2 \ldots g_\mu$. Each is congruent to unity for all but one of the principal factors of m as moduli. Their exponents are the exponents of any set of p-power-exponent generators (Proposition xxvi.), viz., $p^{l_1}, p^{l_2}, \ldots$

Next let $p = 2$.

Then take ϕ, ϕ', ϕ'', generators for mod $(1 + i)^\kappa$

γ_1 a number with exp 2^{κ_1} (mod $P_1^{\lambda_1}$),

&c. &c.

γ'_1 a number with exp $2^{\kappa'_1}$ (mod $Q_1^{\mu_1}$),

&c. &c.

Then

$$\alpha_0 \equiv \phi^j \phi'^{j'} \phi''^{j''} \; [\text{mod } (1 + i)^\kappa]$$
$$\alpha_1 \equiv \gamma_1^{i_1} \pmod{P_1^{\lambda_1}},$$

&c.,

$$\alpha'_1 \equiv \gamma'^{\,i'_1}_1 \pmod{Q_1^{\mu_1}},$$

&c.

and, therefore,

$$a \equiv [(\phi - 1)\,\xi_0 + 1]^j [(\phi' - 1)\,\xi_0 + 1]^{j'} [(\phi'' - 1)\,\xi_0 + 1]^{j''} [(\gamma_1 - 1)\,\xi_1 + 1]^{i_1} \ldots$$
$$[(\gamma'_1 - 1)\,\xi'_1 + 1]^{i'_1} \ldots \pmod{m},$$

and the numbers in brackets are unitary generators of the numbers, modulus m, with exponent powers of 2.

Each is congruent to unity for all but one of the principal factors of m, and their exponents are $2^{\kappa_0}\, 2^{\kappa'_0}\, 2^{\kappa''_0}\, 2^{\kappa_1}\, 2^{\kappa_2} \ldots 2^{\kappa'_1}\, 2^{\kappa_2'} \ldots$

Thus, in either case, whether p is an odd prime or equal to 2, we can form a set of p-power-exponent generators (having the exponents found to be necessary in Proposition xxvi.), such that each is congruent to unity for all but one of the principal factors of m $[(1 + i), P_1^{\lambda_1}, \ldots Q_1^{\mu_1} \ldots]$ as moduli.

(xxviii–xxxi.) Propositions 28-30 are concerned with a discussion of the most general mode of formation of a set of p-power-exponent generators. They are throughout completely applicable to the case of complex numbers and moduli. The result we may state as follows. Let $g_1, g_2, \ldots g_\mu$ be a set of unitary p-power-exponent generators.

Then $\Gamma_1\, \Gamma_2 \ldots \Gamma_\mu$ (with the same set of powers of p as exponents) as given by

$$\left.\begin{array}{l} \Gamma_1 \equiv g_1^{i_{11}}\, g_2^{i_{21}}\, g_3^{i_{31}} \ldots g_\mu^{i_{\mu 1}} \\ \Gamma_2 \equiv g_1^{i_{12}}\, g_2^{i_{22}}\, g_3^{i_{32}} \ldots g_\mu^{i_{\mu 2}} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \Gamma_\mu \equiv g_1^{i_{1\mu}}\, g_2^{i_{2\mu}}\, g_3^{i_{3\mu}} \ldots g_\mu^{i_{\mu\mu}} \end{array}\right\} (\text{mod } m)$$

will be independent generators provided that the determinants formed by the indices i,

$$\left.\begin{array}{l} (i_{11}\, i_{22} \ldots i_{aa}) \\ (i_{a+1\, a+1} \ldots i_{bb}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (\ldots\ldots\ldots i_{\mu\mu}) \end{array}\right\} \text{are all prime to } p.$$

The indices i which occur in any one of these determinants, are those which occur as indices of generators g all with the same exponent; the generators Γ, in which they occur, having also this same exponent.

The summary of Proposition (31) is also true of generators for complex moduli.

(xxxii. and xxxiii.) With one modification, Propositions (22) and (23) hold good for complex numbers and primes.

If in

$$ax^n \equiv b \ (\text{mod } m)$$

a and m have G.C.M κ, then

$$\frac{a}{\kappa} x^n \equiv \frac{b}{\kappa} \left(\text{mod } \frac{m}{\kappa}\right).$$

Each solution, x, of the second congruence gives N (κ) solutions of the first, viz., all the numbers $x + s\frac{m}{\kappa}$, where s is any one of the N (κ) incongruent residues for modulus κ.

The solutions of the congruences which follow are intended as examples of these propositions and also as illustrations of the Tables placed in the Appendix.

Example 1.

$$7x^3 \equiv 3 \ (\text{mod } 4 + 2i).$$

From the tables

$$\begin{array}{rl} & {\scriptstyle (4)}\ \ {\scriptstyle (2)} \\ 3 & \equiv (3.\ 0) \\ 7 & \equiv (1.\ 0), \end{array}$$

therefore

$$x^3 \equiv (2.\ 0),$$

therefore

$$x \equiv (2.\ 0)$$

$$x \equiv 9 \ (\text{mod } 4 + 2i).$$

Example 2.

$$(2+i)\,x^3 \equiv 3+2i \pmod{9}.$$

From the tables

$$3+2i \equiv (\overset{(24)}{22}.\ \overset{(3)}{0})$$

$$2+i \equiv (\ 1.\ 0),$$

therefore

$$x^3 \equiv (21.\ 0),$$

therefore, if

$$x \equiv (a.\ b),$$

$$3a \equiv 21 \pmod{24},$$

therefore

$$a \equiv 7,\ 15,\ 23 \pmod{24}$$

$$3b \equiv 0 \pmod{3},$$

therefore

$$b \equiv 0,\ 1,\ 2 \pmod{3}.$$

There are thus nine solutions, viz. :

$$\begin{array}{lll}
(7.\ 0) \equiv 1+7i & (15.\ 0) \equiv 7+7i & (23.\ 0) \equiv 4+7i \\
(7.\ 1) \equiv 1+4i & (15.\ 1) \equiv 7+4i & (23.\ 1) \equiv 4+4i \\
(7.\ 2) \equiv 1+i & (15.\ 2) \equiv 7+i & (23.\ 2) \equiv 4+i
\end{array}$$

Example 3.

$$3x^3 \equiv 3+2i \pmod{10}.$$

From the tables

$$3+2i \equiv (\overset{(4)}{1}.\ \overset{(4)}{2}.\ \overset{(2)}{0})$$

$$3 \equiv (3.\ 3.\ 0),$$

therefore

$$x^3 \equiv (2.\ 3.\ 0),$$

and therefore

$$x \equiv (2.\ 1.\ 0)$$

$$x \equiv 3+8i.$$

Example 4.

$$3x^3 \equiv 3+2i \pmod{30+20i},$$

and so

$$3\,\frac{x^3}{3+2i} \equiv 1 \pmod{10}.$$

Let

$$x = (3+2i)\,\xi.$$

Then

$$3\,(3+2i)^2\,\xi^3 \equiv 1 \pmod{10}$$

Now

$$3+2i \equiv (1,\ 2,\ 6),$$

therefore

$$(3+2i)^2 \equiv (2,\ 6,\ 6);$$

also

$$3 \equiv (3,\ 3,\ 6),$$

therefore

$$3(3+2i)^2 \equiv (1,\ 3,\ 6)$$

therefore

[illegible]

therefore, if

[illegible]

[illegible]

and therefore

[illegible]

Example [illegible]

[illegible]

This is

[illegible]

therefore

[illegible]

Let

[illegible]

Then

[illegible]

From the latter

[illegible]

therefore

[illegible]

therefore

$$(2+i)^4 . 4 \equiv (10.\ 0),$$

therefore

$$\xi^5 \equiv (1.\ 2),$$

therefore, if

$$\begin{aligned} &\xi \equiv (a.\ b), \\ &5a \equiv 1 \,(\text{mod } 12), \quad a \equiv 5, \\ &5b \equiv 2 \,(\text{mod } 4), \quad b \equiv 2, \\ &\xi \equiv (5.\ 2), \\ &\quad \equiv 58 \,(\text{mod } 8+i). \end{aligned}$$

So

$$\begin{aligned} x &\equiv 58\,(2+i)\,(\text{mod } 15+10i) \\ &\equiv 1+3i. \end{aligned}$$

Example 6.

$$2x^2 \equiv 26 + 32i \,(\text{mod } 40),$$

and therefore

$$x^2 \equiv 13 + 16i \,(\text{mod } 20).$$

Each solution of the latter gives four of the former, viz.,

$$x, \quad x+20, \quad x+20i, \quad x+20+20i \,(\text{mod } 40).$$

The congruence is

$$x^2 \equiv -i\,(1+2i)^2\,(4+i)\,[\text{mod } (1+i)^4 .\ (2+i).\ (1+2i)],$$

therefore

$$\frac{x^2}{1+2i} \equiv -i\,(1+2i)\,(4+i)\,[\text{mod } (1+i)^4 .\ (2+i)].$$

Let

$$x \equiv (1+2i)\,\xi\,(\text{mod } 20).$$

Then

$$(1+2i)\,\xi^2 \equiv -i\,(1+2i)\,(4+i)\,[\text{mod } (1+i)^4 .\ (2+i)],$$

and therefore

$$\begin{aligned} \xi^2 &\equiv -i\,.\,(4+i)\,(\text{mod } 8+4i), \\ &\equiv 1-4i\,(\text{mod } 8+4i), \\ \xi^2 &\equiv 9\,(\text{mod } 8+4i). \end{aligned}$$

From the tables

$$9 \equiv (\overset{(4)}{2}.\ \overset{(4)}{0}.\ \overset{(2)}{0}),$$

and therefore

$$\xi^2 \equiv (2.\ 0.\ 0).$$

This gives eight solutions

$$\begin{aligned}\xi &\equiv (1.\ 0.\ 0) \equiv 17,\\ &\equiv (3.\ 0.\ 0) \equiv 13,\\ &\equiv (1.\ 2.\ 0) \equiv 7,\\ &\equiv (3.\ 2.\ 0) \equiv 3,\\ &\equiv (1.\ 0.\ 1) \equiv 1 + 2i,\\ &\equiv (3.\ 0.\ 1) \equiv 17 + 2i,\\ &\equiv (1.\ 2.\ 1) \equiv 11 + 2i,\\ &\equiv (3.\ 2.\ 1) \equiv 7 + 2i.\end{aligned}$$

Hence there are eight solutions of the congruence

$$x^2 \equiv 13 + 16i \pmod{20},$$

viz.,

$7 + 4i$	$17 + 4i$	$7 + 14i$	$17 + 14i$
$3 + 6i$	$13 + 6i$	$3 + 16i$	$13 + 16i$

and hence there are 32 solutions of the congruence

$$2x^2 \equiv 26 + 32i \pmod{40},$$

viz.,

$7 + 4i$	$17 + 4i$	$7 + 14i$	$17 + 14i$
$27 + 4i$	$37 + 4i$	$27 + 14i$	$37 + 14i$
$7 + 24i$	$17 + 24i$	$7 + 34i$	$17 + 34i$
$27 + 24i$	$37 + 24i$	$27 + 34i$	$37 + 34i$
$3 + 6i$	$13 + 6i$	$3 + 16i$	$13 + 16i$
$23 + 6i$	$33 + 6i$	$23 + 16i$	$33 + 16i$
$3 + 26i$	$13 + 26i$	$3 + 36i$	$13 + 36i$
$23 + 26i$	$33 + 26i$	$23 + 36i$	$33 + 36i$

APPENDIX.

TABLES OF INDICES FOR ALL MODULI WHOSE NORMS DO NOT EXCEED 100.

Description of Tables.

The following Appendix contains Tables of Indices for all the moduli whose norms do not exceed 100. For each modulus two tables are given: the first arranged so as to show readily the number that corresponds to given indices, the second to show what indices correspond to a given number. At the foot of any column, or the end of any row in which indices are tabulated, is placed in a bracket the exponent of the generator to which those indices refer. With each table are noted the formulæ necessary for finding to which of the numbers in the table any given number is congruent. There are also given for convenience the prime factors of the modulus, the norm, the highest exponent, the value of Φ expressed in factors which show the exponents of the generators, and the generators used in the table. All these will be found collected in the reference table next following. In this are also noted, for each modulus, the least possible number of generators and the values of the numbers ξ of Proposition xix. *E.g.*, for the modulus $5 + 5i$ we read thus :—

$$5 + 5i = -i(1 + i)(2 + i)(1 + 2i), \quad \mathrm{N}(5 + 5i) = 50, \quad \Phi(5 + 5i) = 16 = 2^2.2^2.$$

Highest exponent $= 4$. Least number of generators $= 2$, those used in the table being $4 + i$ and $9 + 4i$.

Also if

$$\left.\begin{aligned} a &\equiv \alpha_0 (\mathrm{mod}\ 1 + i) \\ &\equiv \alpha_1 (\mathrm{mod}\ 2 + i) \\ &\equiv \alpha_2 (\mathrm{mod}\ 1 + 2i) \end{aligned}\right\}$$

then

$$a \equiv 5\alpha_0 + (3 + i)\alpha_1 + (8 + 4i)\alpha_2 (\mathrm{mod}\ 5 + 5i).$$

The reducing formulæ (see preface, Part II.) are

$$\begin{aligned} y &\equiv \mathrm{Y} (\mathrm{mod}\ 5), \\ x &\equiv \mathrm{X} + \mathrm{Y} - y (\mathrm{mod}\ 10). \end{aligned}$$

The tables of indices for powers of $1 + i$ as moduli, up to $(1 + i)^8$, are placed at the end.

[The tables have been calculated with some care, but they have not been revised.]

m.	N (m).	$\Phi(m)$.	H.E.	No. of gens.	Gens. used.	The numbers ξ.	$d(\alpha^2+\beta^2)$.	d.	$-(\alpha x'+\beta\beta')$.
$1+i$	2	1	1	1	1	..	2	..	1
$2 \quad = -i(1+i)^2$	4	2	2	1	i	..	2	2	
$2+i$ $1+2i$	5	$4=2^2$	4	1	2	..	5	..	3 2
$2+2i = -i(1+i)^3$	8	$4=2^2$	4	1	i	..	4	2	1
3.	9	$8=2^3$	8	1	$1+i$	..	3	3	
$3+i \; = -i(1+i)(1+2i)$ $1+3i = \quad (1+i)(2+i)$	10	$4=2^2$	4	1	3	5, 6	10	..	7 3
$3+2i$ $2+3i$	13	$12=2^2.\,3$	12	1	2	..	13	..	5 8
$4 \quad = -\;(1+i)^4$	16	$8=2.\,2^2$	4	2	i $1+2i$	..	4	4	
$4+i$ $1+4i$	17	$16=2^4$	16	1	3	..	17	..	13 6
$3+3i = \quad (1+i).\,3$	18	$8=2^3$	8	1	$1+2i$	3, 4	6	3	1
$4+2i = -i(1+i)^2(2+i)$ $2+4i = -i(1+i)^2(1+2i)$	20	$8=2.\,2^2$	4	2	$7, 8+i$ $7, 4+i$	5, 6	10	2	3 7
$5 \quad = -i(2+i)(1+2i)$	25	$16=2^2.\,2^2$	4	2	$4+i, 4+4i$	$3+i, 3+4i$	5	5	
$4+3i = -i(1+2i)^2$ $3+4i = \quad (2+i)^2$	25	$20=2^2.\,5$	20	1	2	..	25	..	7 18
$5+i \; = -i(1+i)(2+3i)$ $1+5i = \quad (1+i)(3+2i)$	26	$12=2^2.\,3$	12	1	7	13, 14	26	..	21 5
$5+2i$ $2+5i$	29	$28=2^2.\,7$	28	1	2	..	29	..	12 17
$4+4i = -\;(1+i)^5$	32	$16=2.\,2.\,2^2$	4	3	i $1+2i$ 3	..	8	4	1
$5+3i = -i(1+i)(1+4i)$ $3+5i = \quad (1+i)(4+i)$	34	$16=2^4$	16	1	3	17, 18	34	..	21 13
$6 \quad = -i(1+i)^2.\,3$	36	$16=2.\,2^3$	8	2	$1+4i, 4+3i$	3, 4	6	6	
$6+i$. $1+6i$	37	$36=2^2.\,3^2$	36	1	2	..	37	..	31 6
$6+2i = -\;(1+i)^3(1+2i)$ $2+6i = -i(1+i)^3(2+i)$	40	$16=2^2.\,2^2$	4	2	$3, i$ $3, 14+i$	5, 16	20	2	7 13
$5+4i$ $4+5i$	41	$40=2^3.\,5$	40	1	6	..	41	..	9 32
$6+3i = \quad (2+i).\,3$ $3+6i = \quad (1+2i).\,3$	45	$32=2^2.\,2^3$	8	2	$1+i, i$ $10+2i, 9+2i$	6, 10	15	3	3 12
7.	49	$48=2^4.\,3$	48	1	$2+i$	..	7	7	
$7+i \; = -i(1+i)(2+i)^2$ $1+7i = -i(1+i)(1+2i)^2$	50	$20=2^2.\,5$	20	1	3	25, 26	50	..	43 7
$5+5i = -i(1+i)(2+i)(1+2i)$	50	$16=2^2.\,2^2$	4	2	$4+i, 9+4i$	$5, 3+i, 8+4i$	10	5	1

m.	N (m).	$\Phi(m)$.	H.E.	No. of gens.	Gens. used.	The numbers ξ.	$d(\alpha^2+\beta^2)$.	d.	$-(\alpha\alpha'+\beta\beta'.)$
$6+4i=-i(1+i)^2(3+2i)$. $4+6i=-i(1+i)^2(2+3i)$.	52	$24=2.\ 2^2.\ 3$	12	2	$15,\ 22+i$ $15,\ 6+i$	13, 14	26	2	5 21
$7+2i$ $2+7i$	53	$52=2^2.\ 13$	52	1	3	..	53	..	23 30
$7+3i=-i(1+i)(2+5i)$ $3+7i=(1+i)(5+2i)$	58	$28=2^2.\ 7$	28	1	3	29, 30	58	..	17 41
$6+5i$ $5+6i$	61	$60=2^2.\ 3.\ 5$	60	1	2	..	61	..	11 50
$8=i(1+i)^6$	64	$32=2.\ 2^2.\ 2^2$	4	3	$i,\ 1+2i,\ 3$	..	8	8	
$8+i=-i(1+2i)(3+2i)$ $1+8i=(2+i)(2+3i)$ $7+4i=-i(1+2i)(2+3i)$ $4+7i=(2+i)(3+2i)$	65	$48=2^2.\ 2^2.\ 3$	12	2	27, 2	26, 40	65	..	57 8 47 18
$8+2i=-i(1+i)^2(4+i)$ $2+8i=-i(1+i)^2(1+4i)$.	68	$32=2.\ 2^4$	16	2	$3,\ i$ $3,\ 26+i$	17, 18	34	2	13 21
$6+6i=-i(1+i)^3.\ 3$	72	$32=2^2.\ 2^3$	8	2	$2+i,\ i$	9, 4	12	6	1
$8+3i$ $3+8i$	73	$72=2^3.\ 3^2$	72	1	5	..	73	..	46 27
$7+5i=-i(1+i)(1+6i)$ $5+7i=(1+i)(6+i)$	74	$36=2^2.\ 3^2$	36	1	5	37, 38	74	..	43 31
$8+4i=-(1+i)^4(2+i)$ $4+8i=-(1+i)^4(1+2i)$.	80	$32=2.\ 2^2.\ 2^2$	4	3	$17,\ 8+i,\ 5+2i$ $17,\ 3i,\ 17+2i$	5, 16	20	4	8 17
$9=3^2$	81	$72=2^3.\ 3.\ 3$	24	2	$2+i,\ 4+3i$	..	9	9	
$9+i=-i(1+i)(4+5i)$ $1+9i=(1+i)(5+4i)$	82	$40=2^3.\ 5$	40	1	7	41, 42	82	..	73 9
$9+2i=-i(2+i)(1+4i)$ $2+9i=(1+2i)(4+i)$ $7+6i=(2+i)(4+i)$ $6+7i=-i(1+2i)(1+4i)$	85	$64=2^2.\ 2^4$	16	2	3, 52	51, 35	85	..	38 47 13 72
$8+5i$ $5+8i$	89	$88=2^3.\ 11$	88	1	3	..	89	..	34 55
$9+3i=-i(1+i)(1+2i).\ 3$ $3+9i=(1+i)(2+i).\ 3$	90	$32=2^2.\ 2^3$	8	2	$2+i,\ i$ $23+2i,\ 21+2i$	15, 6, 10	30	3	7 23
$9+4i$ $4+9i$	97	$96=2^5.\ 3$	96	1	5	..	97	..	22 75
$7+7i=(1+i).\ 7$.	98	$48=2^4.\ 3$	48	1	$2+i$	7. 8	14	7	1
$8+6i=-(1+i)^2(1+2i)^2$ $6+8i=-i(1+i)^2(2+i)^2$.	100	$40=2.\ 2^2.\ 5$	20	2	$3,\ 44+i$ $3,\ 8+i$	25, 26	50	2	7 43
$10=-i(1+i)^2(2+i)(1+2i)$	100	$32=2.\ 2^2.\ 2^2$	4	3	$9+4i,\ 9+6i,\ 6+5i$	$5,\ 8+6i,\ 8+4i$	10	10	

$m = 2 + i \quad N(m) = 5 \quad \Phi(m) = 4 = 2^2 \quad H.E = 4 \quad x \equiv X + 3Y \pmod{5}.$

$m = 1 + 2i \quad N(m) = 5 \quad \Phi(m) = 4 = 2^2 \quad H.E = 4 \quad x \equiv X + 2Y \pmod{5}.$

Generator 2.

N	2	4	3	1
I	1	2	3	0

(4)

$m = 3 \quad N(m) = 9 \quad \Phi(m) = 8 = 2^3 \quad H.E = 8 \quad x \equiv X \pmod{3} \quad y \equiv Y \pmod{3}.$

Generator $1 + i$.

0	1
1	1+*i*
2	2*i*
3	1+2*i*
4	2
5	2+2*i*
6	*i*
7	2+*i*

(8)

i	6
2*i*	2
1	0
1+*i*	1
1+2*i*	3
2	4
2+*i*	7
2+2*i*	5

(8)

$m = 3 + i = -i(1 + i)(1 + 2i) \quad N(m) = 10 \quad \Phi(m) = 4 = 2^2 \quad H.E = 4$
$x \equiv X + 7Y \pmod{10}.$

$m = 1 + 3i = (1 + i)(2 + i) \quad N(m) = 10 \quad \Phi(m) = 4 = 2^2 \quad H.E = 4$
$x \equiv X + 3Y \pmod{10}.$

Generator 3.

N	3	9	7	1
I	1	2	3	0

(4)

$m = 3 + 2i$ $\quad \mathrm{N}(m) = 13$ $\quad \Phi(m) = 12 = 2^2.3$ $\quad \mathrm{H.E} = 12$ $\quad x \equiv \mathrm{X} + 5\mathrm{Y} \pmod{13}$.

$m = 2 + 3i$ $\quad \mathrm{N}(m) = 13$ $\quad \Phi(m) = 12 = 2^2.3$ $\quad \mathrm{H.E} = 12$ $\quad x \equiv \mathrm{X} + 8\mathrm{Y} \pmod{13}$.

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	(12)
N	1	2	4	8	3	6	12	11	9	5	10	7	

N	1	2	3	4	5	6	7	8	9	10	11	12	
I	0	1	4	2	9	5	11	3	8	10	7	6	(12)

$m = 4 + i$ $\quad \mathrm{N}(m) = 17$ $\quad \Phi(m) = 16 = 2^4$ $\quad \mathrm{H.E} = 16$ $\quad x \equiv \mathrm{X} + 13\mathrm{Y} \pmod{17}$.

$m = 1 + 4i$ $\quad \mathrm{N}(m) = 17$ $\quad \Phi(m) = 16 = 2^4$ $\quad \mathrm{H.E} = 16$ $\quad x \equiv \mathrm{X} + 4\mathrm{Y} \pmod{17}$.

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(16)
N	1	3	9	10	13	5	15	11	16	14	8	7	4	12	2	6	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
I	0	14	1	12	5	15	11	10	2	3	7	13	4	9	6	8	(16)

$m = 3 + 3i = (1 + i)3$ $\quad \mathrm{N}(m) = 18$ $\quad \Phi(m) = 8 = 2^3$ $\quad \mathrm{H.E} = 8$ $\quad y \equiv \mathrm{Y} \pmod{3}$

$x \equiv \mathrm{X} + 3(\mathrm{Y} - y) \pmod{6}$.

Generator $1 + 2i$.

0	1
1	$1+2i$
2	i
3	$4+i$
4	5
5	$2+i$
6	$3+2i$
7	$5+2i$

(8)

i	2
1	0
$1+2i$	1
$2+i$	5
$3+2i$	6
$4+i$	3
5	4
$5+2i$	7

(8)

$$m = 4 + 2i = -i(1+i)^2(2+i) \quad N(m) = 20 \quad \Phi(m) = 8 = 2.\,2^2 \quad H.\,E. = 4$$

$$y \equiv Y \pmod{2} \quad x = X + 3(Y - y) \pmod{10}.$$

Generators 7, 8 + i.

0	0	1	i	3	1
0	1	$8+i$	1	0	0
1	0	7	3	3	0
1	1	$4+i$	$4+i$	1	1
2	0	9	$6+i$	2	1
2	1	$6+i$	7	1	0
3	0	3	$8+i$	0	1
3	1	i	9	2	0
(4)	(2)			(4)	(2)

$$m = 2 + 4i = -i(1+i)^2(1+2i) \quad N(m) = 20 \quad \Phi(m) = 8 = 2.\,2^2 \quad H.\,E. = 4$$

$$y \equiv Y \pmod{2} \quad x \equiv X + 7(Y - y) \pmod{10}.$$

Generators 7, 4 + i.

0	0	1	i	1	1
0	1	$4+i$	1	0	0
1	0	7	$2+i$	2	1
1	1	i	3	3	0
2	0	9	$4+i$	0	1
2	1	$2+i$	$6+i$	3	1
3	0	3	7	1	0
3	1	$6+i$	9	2	0
(4)	(2)			(4)	(2)

$$m = 5 = -i(2+i)(1+2i) \quad N(m) = 25 \quad \Phi(m) = 16 = 2^2.\,2^2 \quad H.\,E. = 4$$

$$x \equiv X \pmod{5} \quad y \equiv Y \pmod{5}.$$

Generators 4 + i, 4 + 4i.

0	0	1	i	3	1
0	1	$4+4i$	$2i$	0	2
0	2	$2i$	$3i$	2	0
0	3	$2+3i$	$4i$	1	3
1	0	$4+i$	1	0	0
1	1	2	$1+i$	2	3
1	2	$3+3i$	$1+4i$	3	2
1	3	$4i$	2	1	1
2	0	$3i$	$2+2i$	3	0
2	1	$3+2i$	$2+3i$	0	3
2	2	4	3	3	3
2	3	$1+i$	$3+2i$	2	1
3	0	$2+2i$	$3+3i$	1	2
3	1	i	4	2	2
3	2	$1+4i$	$4+i$	1	0
3	3	3	$4+4i$	0	1
(4)	(4)			(4)	(4)

$m = 4 + 3i = -i(1 + 2i)^2$ $\quad N(m) = 25$ $\quad \Phi(m) = 20 = 2^2.\,5$ $\quad$ H. E. $= 20$

$$x \equiv X + 7Y \pmod{25}.$$

$m = 3 + 4i = \quad (2 + i)^2$ $\quad N(m) = 25$ $\quad \Phi(m) = 20 = 2^2.\,5$ $\quad$ H. E. $= 20$

$$x \equiv X + 18Y \pmod{25}.$$

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(20)
N	1	2	4	8	16	7	14	3	6	12	24	23	21	17	9	18	11	22	19	13	

N	1	2	3	4	6	7	8	9	11	12	13	14	16	17	18	19	21	22	23	24	
I	0	1	7	2	8	5	3	14	16	9	19	6	4	13	15	18	12	17	11	10	(20)

$m = 5 + i = -i(1 + i)(2 + 3i)$ $\quad N(m) = 26$ $\quad \Phi(m) = 12 = 2^2.\,3$ $\quad$ H. E. $= 12$

$$x \equiv X + 21Y \pmod{26}.$$

$m = 1 + 5i = \quad (1 + i)(3 + 2i)$ $\quad N(m) = 26$ $\quad \Phi(m) = 12 = 2^2.\,3$ $\quad$ H. E. $= 12$

$$x \equiv X + 5Y \pmod{26}.$$

Generator 7.

I	0	1	2	3	4	5	6	7	8	9	10	11	(12)
N	1	7	23	5	9	11	25	19	3	21	17	15	

N	1	3	5	7	9	11	15	17	19	21	23	25	
I	0	8	3	1	4	5	11	10	7	9	2	6	(12)

$m = 5 + 2i \quad N(m) = 29 \quad \Phi(m) = 28 = 2^2 . 7 \quad H.E. = 28 \quad x \equiv X + 12Y \pmod{29}.$

$m = 2 + 5i \quad N(m) = 29 \quad \Phi(m) = 28 = 2^2 . 7 \quad H.E. = 28 \quad x \equiv X + 17Y \pmod{29}.$

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	(28)
N	1	2	4	8	16	3	6	12	24	19	9	18	7	14	
I	14	15	16	17	18	19	20	21	22	23	24	25	26	27	(28)
N	28	27	25	21	13	26	23	17	5	10	20	11	22	15	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
I	0	1	5	2	22	6	12	3	10	23	25	7	18	13	(28)
N	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
I	27	4	21	11	9	24	17	26	20	8	16	19	15	14	(28)

$m = 5 + 3i = -i(1 + i)(1 + 4i) \quad N(m) = 34 \quad \Phi(m) = 16 = 2^4 \quad H.E. = 16$

$x \equiv X + 21Y \pmod{34}.$

$m = 3 + 5i = \quad (1 + i)(4 + i) \quad N(m) = 34 \quad \Phi(m) = 16 = 2^4 \quad H.E. = 16$

$x \equiv X + 13Y \pmod{34}.$

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	(16)
N	1	3	9	27	13	5	15	11	33	31	25	7	21	29	19	23	

N	1	3	5	7	9	11	13	15	19	21	23	25	27	29	31	33	
I	0	1	5	11	2	7	4	6	14	12	15	10	3	13	9	8	(16)

$$m = 6 = -i(1+i)^2 . 3 \quad N(m) = 36 \quad \Phi(m) = 16 = 2 . 2^3 \quad \text{H. E.} = 8$$

$$x \equiv X \pmod 6 \quad y \equiv Y \pmod 6.$$

Generators $1 + 4i$, $4 + 3i$.

0	0	1
0	1	$4+3i$
1	0	$1+4i$
1	1	$4+i$
2	0	$3+2i$
2	1	$5i$
3	0	$1+2i$
3	1	$4+5i$
4	0	5
4	1	$2+3i$
5	0	$5+2i$
5	1	$2+5i$
6	0	$3+4i$
6	1	i
7	0	$5+4i$
7	1	$2+i$
(8)	(2)	

i	6	1
$5i$	2	1
1	0	0
$1+2i$	3	0
$1+4i$	1	0
$2+i$	7	1
$2+3i$	4	1
$2+5i$	5	1
$3+2i$	2	0
$3+4i$	6	0
$4+i$	1	1
$4+3i$	0	1
$4+5i$	3	1
5	4	0
$5+2i$	5	0
$5+4i$	7	0
	(8)	(2)

$$m = 6 + i \quad N(m) = 37 \quad \Phi(m) = 36 = 2^2 . 3^2 \quad \text{H. E.} = 36 \quad x \equiv X + 31Y \pmod{37}.$$

$$m = 1 + 6i \quad N(m) = 37 \quad \Phi(m) = 36 = 2^2 . 3^2 \quad \text{H. E.} = 36 \quad x \equiv X + 6Y \pmod{37}.$$

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	(36)
N	1	2	4	8	16	32	27	17	34	31	25	13	26	15	30	23	9	18	
I	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	(36)
N	36	35	33	29	21	5	10	20	3	6	12	24	11	22	7	14	28	19	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
I	0	1	26	2	23	27	32	3	16	24	30	28	11	33	13	4	7	17	(36)
N	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	
I	35	25	22	31	15	29	10	12	6	34	21	14	9	5	20	8	19	18	(36)

$m = 6 + 2i = -(1 + i)^3 (1 + 2i) \quad N(m) = 40 \quad \Phi(m) = 16 = 2^2 . 2^2 \quad \text{H. E.} = 4.$

$y \equiv Y \pmod 2 \quad x \equiv X + 7(Y - y) \pmod{20}.$

Generators 3, i.

0	0	1
0	1	i
0	2	19
0	3	$6+i$
1	0	3
1	1	$14+i$
1	2	17
1	3	$12+i$
2	0	9
2	1	$16+i$
2	2	11
2	3	$10+i$
3	0	7
3	1	$2+i$
3	2	13
3	3	$4+i$
(4)	(4)	

i	0	1
1	0	0
$2+i$	3	1
3	1	0
$4+i$	3	3
$6+i$	0	3
7	3	0
9	2	0
$10+i$	2	3
11	2	2
$12+i$	1	3
13	3	2
$14+i$	1	1
$16+i$	2	1
17	1	2
19	0	2
	(4)	(4)

$m = 2 + 6i = -i(1 + i)^3 (2 + i) \quad N(m) = 40 \quad \Phi(m) = 16 = 2^2 . 2 \quad \text{H. E.} = 4.$

$y \equiv Y \pmod 2 \quad x \equiv X + 13(Y - y) \pmod{20}.$

Generators 3, $14 + i$.

0	0	1
0	1	$14+i$
0	2	19
0	3	i
1	0	3
1	1	$8+i$
1	2	17
1	3	$6+i$
2	0	9
2	1	$10+i$
2	2	11
2	3	$4+i$
3	0	7
3	1	$16+i$
3	2	13
3	3	$18+i$
(4)	(4)	

i	0	3
1	0	0
3	1	0
$4+i$	2	3
$6+i$	1	3
7	3	0
$8+i$	1	1
9	2	0
$10+i$	2	1
11	2	2
13	3	2
$14+i$	0	1
$16+i$	3	1
17	1	2
$18+i$	3	3
19	0	2
	(4)	(4)

$m = 5 + 4i$ $\mathrm{N}(m) = 41$ $\Phi(m) = 40 = 2^3.5$ H. E. = 40 $x \equiv \mathrm{X} + 9\mathrm{Y}$ (mod 41).

$m = 4 + 5i$ $\mathrm{N}(m) = 41$ $\Phi(m) = 40 = 2^3.5$ H. E. = 40 $x \equiv \mathrm{X} + 32\mathrm{Y}$ (mod 41).

Generator 6.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
N	1	6	36	11	25	27	39	29	10	19	32	28	4	24	21	3	18	26	33	34	(40)
I	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	
N	40	35	5	30	16	14	2	12	31	22	9	13	37	17	20	38	23	15	8	7	(40)

I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
N	0	26	15	12	22	1	39	38	30	8	3	27	31	25	37	24	33	16	9	34	(40)
I	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	
N	14	29	36	13	4	17	5	11	7	23	28	10	18	19	21	2	32	35	6	20	(40)

$$m = 6 + 3i = (2 + i)\,3 \qquad \mathrm{N}(m) = 45 \qquad \Phi(m) = 32 = 2^2 . 2^3$$

$$\mathrm{H.\,E.} = 8 \qquad y \equiv \mathrm{Y} \pmod 3 \qquad x \equiv \mathrm{X} + 3(\mathrm{Y} - y) \pmod{15}.$$

Generators $1 + i$, i.

0	0	1
0	1	i
0	2	14
0	3	$6+2i$
1	0	$1+i$
1	1	$14+i$
1	2	$5+2i$
1	3	$7+2i$
2	0	$2i$
2	1	13
2	2	$6+i$
2	3	2
3	0	$13+2i$
3	1	$4+i$
3	2	$8+i$
3	3	$2+2i$
4	0	11
4	1	$12+2i$
4	2	4
4	3	$9+i$
5	0	$8+2i$
5	1	$1+2i$
5	2	$13+i$
5	3	$5+i$
6	0	$3+i$
6	1	8
6	2	$3+2i$
6	3	7
7	0	$11+i$
7	1	$11+2i$
7	2	$10+2i$
7	3	$10+i$
(8)	(4)	

i	0	1
$2i$	2	0
1	0	0
$1+i$	1	0
$1+2i$	5	1
2	2	3
$2+2i$	3	3
$3+i$	6	0
$3+2i$	6	2
4	4	2
$4+i$	3	1
$5+i$	5	3
$5+2i$	1	2
$6+i$	2	2
$6+2i$	0	3
7	6	3
$7+2i$	1	3
8	6	1
$8+i$	3	2
$8+2i$	5	0
$9+i$	4	3
$10+i$	7	3
$10+2i$	7	2
11	4	0
$11+i$	7	0
$11+2i$	7	1
$12+2i$	4	1
13	2	1
$13+i$	5	2
$13+2i$	3	0
14	0	2
$14+i$	1	1
	(8)	(4)

$$m = 3 + 6i = (1 + 2i)\,3 \qquad \mathrm{N}(m) = 45 \qquad \Phi(m) = 32 = 2^2 . 2^3$$

$$\mathrm{H.\,E.} = 8 \qquad y \equiv \mathrm{Y} \pmod{3} \qquad x \equiv \mathrm{X} + 12(\mathrm{Y} - y) \pmod{15}.$$

Generators $10 + 2i$, $9 + 2i$.

0	0	1
0	1	$9+2i$
0	2	14
0	3	i
1	0	$10+2i$
1	1	$8+2i$
1	2	$14+i$
1	3	$1+i$
2	0	$9+i$
2	1	13
2	2	$2i$
2	3	2
3	0	$7+i$
3	1	$13+2i$
3	2	$2+2i$
3	3	$11+i$
4	0	11
4	1	$6+i$
4	2	4
4	3	$3+2i$
5	0	$2+i$
5	1	$10+i$
5	2	$7+2i$
5	3	$14+2i$
6	0	$12+2i$
6	1	8
6	2	$12+i$
6	3	7
7	0	$5+2i$
7	1	$5+i$
7	2	$4+i$
7	3	$4+2i$
(8)	(4)	

i	0	3
$2i$	2	2
1	0	0
$1+i$	1	3
2	2	3
$2+i$	5	0
$2+2i$	3	2
$3+2i$	4	3
4	4	2
$4+i$	7	2
$4+2i$	7	3
$5+i$	7	1
$5+2i$	7	0
$6+i$	4	1
7	6	3
$7+i$	3	0
$7+2i$	5	2
8	6	1
$8+2i$	1	1
$9+i$	2	0
$9+2i$	0	1
$10+i$	5	1
$10+2i$	1	0
11	4	0
$11+i$	3	3
$12+i$	6	2
$12+2i$	6	0
13	2	1
$13+2i$	3	1
14	0	2
$14+i$	1	2
$14+2i$	5	3
	(8)	(4)

$$m = 7 \quad \mathrm{N}(7) = 49 \quad \Phi(7) = 48 = 2^4.3. \quad \mathrm{H.\,E.} = 48. \quad x \equiv \mathrm{X} \pmod 7$$
$$y \equiv \mathrm{Y} \pmod 7.$$

Generator $2 + i$.

0	1	24	6
1	$2+i$	25	$5+6i$
2	$3+4i$	26	$4+3i$
3	$2+4i$	27	$5+3i$
4	$3i$	28	$4i$
5	$4+6i$	29	$3+i$
6	$2+2i$	30	$5+5i$
7	$2+6i$	31	$5+i$
8	5	32	2
9	$3+5i$	33	$4+2i$
10	$1+6i$	34	$6+i$
11	$3+6i$	35	$4+i$
12	i	36	$6i$
13	$6+2i$	37	$1+5i$
14	$3+3i$	38	$4+4i$
15	$3+2i$	39	$4+5i$
16	4	40	3
17	$1+4i$	41	$6+3i$
18	$5+2i$	42	$2+5i$
19	$1+2i$	43	$6+5i$
20	$5i$	44	$2i$
21	$2+3i$	45	$5+4i$
22	$1+i$	46	$6+6i$
23	$1+3i$	47	$6+4i$
(48)		(48)	

i	12	$3+4i$	2
$2i$	44	$3+5i$	9
$3i$	4	$3+6i$	11
$4i$	28	4	16
$5i$	20	$4+i$	35
$6i$	36	$4+2i$	33
1	0	$4+3i$	26
$1+i$	22	$4+4i$	38
$1+2i$	19	$4+5i$	39
$1+3i$	23	$4+6i$	5
$1+4i$	17	5	8
$1+5i$	37	$5+i$	31
$1+6i$	10	$5+2i$	18
2	32	$5+3i$	27
$2+i$	1	$5+4i$	45
$2+2i$	6	$5+5i$	30
$2+3i$	21	$5+6i$	25
$2+4i$	3	6	24
$2+5i$	42	$6+i$	34
$2+6i$	7	$6+2i$	13
3	40	$6+3i$	41
$3+i$	29	$6+4i$	47
$3+2i$	15	$6+5i$	43
$3+3i$	14	$6+6i$	46
	(48)		(48)

$$m = 7 + i = -i(1+i)(2+i)^2 \quad \mathrm{N}(m) = 50 \quad \Phi(m) = 20 = 2^2.5 \quad \mathrm{H.\,E.} = 20$$
$$x \equiv \mathrm{X} + 43\mathrm{Y} \pmod{50}.$$

$$m = 1 + 7i = -i(1+i)(1+2i)^2 \quad \mathrm{N}(m) = 50 \quad \Phi(m) = 20 = 2^2.5 \quad \mathrm{H.\,E.} = 20$$
$$x \equiv \mathrm{X} + 7\mathrm{Y} \pmod{50}.$$

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(20)
N	1	3	9	27	31	43	29	37	11	33	49	47	41	23	19	7	21	13	39	17	

N	1	3	7	9	11	13	17	19	21	23	27	29	31	33	37	39	41	43	47	49	
I	0	1	15	2	8	17	19	14	16	13	3	6	4	9	7	18	12	5	11	10	(20)

$$m = 5 + 5i = -i(1+i)(2+i)(1+2i) \quad \mathrm{N}(m) = 50 \quad \Phi(m) = 16 = 2^2 . 2^2$$

$$\text{H.E.} = 4 \quad y \equiv \mathrm{Y} \pmod{5} \quad x \equiv \mathrm{X} + \mathrm{Y} - y \pmod{10}.$$

Generators $4 + i$, $9 + 4i$.

0	0	1
0	1	$9+4i$
0	2	$5+2i$
0	3	$2+3i$
1	0	$4+i$
1	1	7
1	2	$8+3i$
1	3	$5+4i$
2	0	$3i$
2	1	$3+2i$
2	2	9
2	3	$6+i$
3	0	$7+2i$
3	1	i
3	2	$1+4i$
3	3	3
(4)	(4)	

i	3	1
$3i$	2	0
1	0	0
$1+4i$	3	2
$2+3i$	0	3
3	3	3
$3+2i$	2	1
$4+i$	1	0
$5+2i$	0	2
$5+4i$	1	3
$6+i$	2	3
7	1	1
$7+2i$	3	0
$8+3i$	1	2
9	2	2
$9+4i$	0	1
	(4)	(4)

$$m = 6 + 4i = -i(1+i)^2(3+2i) \quad \mathrm{N}(m) = 52 \quad \Phi(m) = 24 = 2 . 2^2 . 3 \quad \text{H.E.} = 12$$

$$y \equiv \mathrm{Y} \pmod{2} \quad x \equiv \mathrm{X} + 5(\mathrm{Y} - y) \pmod{26}.$$

Generators 15, $22 + i$.

0	0	1
0	1	$22+i$
1	0	15
1	1	$10+i$
2	0	17
2	1	$12+i$
3	0	21
3	1	$16+i$
4	0	3
4	1	$24+i$
5	0	19
5	1	$14+i$
6	0	25
6	1	$20+i$
7	0	11
7	1	$6+i$
8	0	9
8	1	$4+i$
9	0	5
9	1	i
10	0	23
10	1	$18+i$
11	0	7
11	1	$2+i$
(12)	(2)	

i	9	1
1	0	0
$2+i$	11	1
3	4	0
$4+i$	8	1
5	9	0
$6+i$	7	1
7	11	0
9	8	0
$10+i$	1	1
11	7	0
$12+i$	2	1
$14+i$	5	1
15	1	0
$16+i$	3	1
17	2	0
$18+i$	10	1
19	5	0
$20+i$	6	1
21	3	0
$22+i$	0	1
23	10	0
$24+i$	4	1
25	6	0
	(12)	(2)

$m = 4 + 6i = -i\,(1+i)^2(2+3i)$ $\mathrm{N}(m) = 52$ $\Phi(m) = 24 = 2.\,2^2.\,3$ H. E. = 12

$y \equiv \mathrm{Y} \pmod{2}$ $x \equiv \mathrm{X} + 21\,(\mathrm{Y} - y) \pmod{26}$.

Generators 15, $6 + i$.

0	0	1
0	1	$6+i$
1	0	15
1	1	$20+i$
2	0	17
2	1	$22+i$
3	0	21
3	1	i
4	0	3
4	1	$8+i$
5	0	19
5	1	$24+i$
6	0	25
6	1	$4+i$
7	0	11
7	1	$16+i$
8	0	9
8	1	$14+i$
9	0	5
9	1	$10+i$
10	0	23
10	1	$2+i$
11	0	7
11	1	$12+i$
(12)	(2)	

i	3	1
1	0	0
$2+i$	10	1
3	4	0
$4+i$	6	1
5	9	0
$6+i$	0	1
7	11	0
$8+i$	4	1
9	8	0
$10+i$	9	1
11	7	0
$12+i$	11	1
$14+i$	8	1
15	1	0
$16+i$	7	1
17	2	0
19	5	0
$20+i$	1	1
21	3	0
$22+i$	2	1
23	10	0
$24+i$	5	1
25	6	0
	(12)	(2)

$m = 7 + 2i$ $\mathrm{N}(m) = 53$ $\Phi(m) = 52 = 2^2.13$ H. E. = 52 $x \equiv \mathrm{X} + 23\mathrm{Y} \pmod{53}$.

$m = 2 + 7i$ $\mathrm{N}(m) = 53$ $\Phi(m) = 52 = 2^2.13$ H. E. = 52 $x \equiv \mathrm{X} + 30\mathrm{Y} \pmod{53}$.

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	(52)
N	1	3	9	27	28	31	40	14	42	20	7	21	10	30	37	5	15	45	29	
I	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36		(52)
N	34	49	41	17	51	47	35	52	50	44	26	25	22	13	39	11	33	46		
I	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51					(52)
N	32	43	23	16	48	38	8	24	19	4	12	36	2	6	13					

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
I	0	49	1	46	15	50	10	43	2	12	34	47	32	7	16	40	22	51	45	(52)

N	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	
I	9	11	31	39	44	30	29	3	4	18	13	5	37	35	19	25	48	14	(52)

N	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	
I	42	33	6	21	8	38	28	17	36	24	41	20	27	23	26	(52)

$m = 7 + 3i = -i(1+i)(2+5i)$ $\quad \mathrm{N}(m) = 58$ $\quad \Phi(m) = 28 = 2^2.7$ $\quad$ H. E. $= 28$

$$x \equiv \mathrm{X} + 17\mathrm{Y} \pmod{58}.$$

$m = 3 + 7i = (1+i)(5+2i)$ $\quad \mathrm{N}(m) = 58$ $\quad \Phi(m) = 28 = 2^2.7$ $\quad$ H. E. $= 28$

$$x \equiv \mathrm{X} + 41\mathrm{Y} \pmod{58}.$$

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
N	1	3	9	27	23	11	33	41	7	21	5	15	45	19	(28)
I	14	15	16	17	18	19	20	21	22	23	24	25	26	27	
N	57	55	49	31	35	47	25	17	51	37	53	43	13	39	(28)

N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	
I	0	1	10	8	2	5	26	11	21	13	9	4	20	3	(28)
N	31	33	35	37	39	41	43	45	47	49	51	53	55	57	
I	17	6	18	23	27	7	25	12	19	16	22	24	15	14	(28)

$m = 6 + 5i \quad N(m) = 61 \quad \Phi(m) = 60 = 2^2.3.5 \quad$ H. E. $= 60 \quad x \equiv X + 11Y \pmod{61}$.

$m = 5 + 6i \quad N(m) = 61 \quad \Phi(m) = 60 = 2^2.3.5 \quad$ H. E. $= 60 \quad x \equiv X + 50Y \pmod{61}$.

Generator 2.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(60)
N	1	2	4	8	16	32	3	6	12	24	48	35	9	18	36	11	22	44	27	54	
I	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	(60)
N	47	33	5	10	20	40	19	38	15	30	60	59	57	53	45	29	58	55	49	37	
I	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	(60)
N	13	26	52	43	25	50	39	17	34	7	14	28	56	51	41	21	42	23	46	31	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
I	0	1	6	2	22	7	49	3	12	23	15	8	40	50	28	4	47	13	26	24	(60)
N	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	
I	55	16	57	9	44	41	18	51	35	29	59	5	21	48	11	14	39	27	46	25	(60)
N	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	
I	54	56	43	17	34	58	20	10	38	45	53	42	33	19	37	52	32	36	31	30	(60)

$m = 8 + i = -i(1 + 2i)(3 + 2i) \quad \mathrm{N}(m) = 65 \quad \Phi(m) = 48 = 2^2 . 2^2 . 3$

H. E. = 12 $\quad x \equiv \mathrm{X} + 57\mathrm{Y} \pmod{65}$.

$m = 1 + 8i = (2 + i)(2 + 3i) \quad \mathrm{N}(m) = 65 \quad \Phi(m) = 48 = 2^2 . 2^2 . 3$

H. E. = 12 $\quad x \equiv \mathrm{X} + 8\mathrm{Y} \pmod{65}$.

$m = 7 + 4i = -i(1 + 2i)(2 + 3i) \quad \mathrm{N}(m) = 65 \quad \Phi(m) = 48 = 2^2 . 2^2 . 3$

H. E. = 12 $\quad x \equiv \mathrm{X} + 47\mathrm{Y} \pmod{65}$.

$m = 4 + 7i = (2 + i)(3 + 2i) \quad \mathrm{N}(m) = 65 \quad \Phi(m) = 48 = 2^2 . 2^2 . 3$

H. E. = 12 $\quad x \equiv \mathrm{X} + 18\mathrm{Y} \pmod{65}$.

Generators 27, 2.

0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5	(12)
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	(4)
1	27	14	53	2	54	28	41	4	43	56	17	8	21	47	34	16	42	29	3	32	19	58	6	
6	6	6	6	7	7	7	7	8	8	8	8	9	9	9	9	10	10	10	10	11	11	11	11	(12)
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	(4)
64	38	51	12	63	11	37	24	61	22	9	48	57	44	18	31	49	23	36	62	33	46	7	59	

1	2	3	4	6	7	8	9	11	12	14	16	17	18	19	21	22	23	24	27	28	29	31	32	
0	1	4	2	5	11	3	8	7	6	0	4	2	9	5	3	8	10	7	0	1	4	9	5	(12)
0	0	3	0	3	2	0	2	1	3	2	0	3	2	1	1	1	1	3	1	2	2	3	0	(4)
33	34	36	37	38	41	42	43	44	46	47	48	49	51	53	54	56	57	58	59	61	62	63	64	
11	3	10	7	6	1	4	2	9	11	3	8	10	6	0	1	2	9	5	11	8	10	7	6	(12)
0	3	2	2	1	3	1	1	1	1	2	3	0	2	3	1	2	0	2	3	0	3	0	0	(4)

$$m = 8 + 2i = -i(1+i)^2(4+i) \quad N(m) = 68 \quad \Phi(m) = 32 = 2.2^4 \quad \text{H. E.} = 16$$

$$y \equiv Y \pmod{2} \quad x \equiv X + 13(Y - y) \pmod{34}.$$

Generators 3, i.

0	0	1
0	1	i
1	0	3
1	1	$26+i$
2	0	9
2	1	$2+i$
3	0	27
3	1	$32+i$
4	0	13
4	1	$20+i$
5	0	5
5	1	$18+i$
6	0	15
6	1	$12+i$
7	0	11
7	1	$28+i$
8	0	33
8	1	$8+i$
9	0	31
9	1	$16+i$
10	0	25
10	1	$6+i$
11	0	7
11	1	$10+i$
12	0	21
12	1	$22+i$
13	0	29
13	1	$24+i$
14	0	19
14	1	$30+i$
15	0	23
15	1	$14+i$
(16)	(2)	

i	0	1
1	0	0
$2+i$	2	1
3	1	0
5	5	0
$6+i$	10	1
7	11	0
$8+i$	8	1
9	2	0
$10+i$	11	1
11	7	0
$12+i$	6	1
13	4	0
$14+i$	15	1
15	6	0
$16+i$	9	1
$18+i$	5	1
19	14	0
$20+i$	4	1
21	12	0
$22+i$	12	1
23	15	0
$24+i$	13	1
25	10	0
$26+i$	1	1
27	3	0
$28+i$	7	1
29	13	0
$30+i$	14	1
31	9	0
$32+i$	3	1
33	8	0
	(16)	(2)

$$m = 2 + 8i = -i(1+i)^2(1+4i) \quad \mathrm{N}(m) = 68 \quad \Phi(m) = 32 = 2.\,2^4 \quad \text{H. E.} = 16$$

$$y \equiv \mathrm{Y} \pmod{2} \quad x \equiv \mathrm{X} + 21(\mathrm{Y} - y) \pmod{34}.$$

Generators 3, $26 + i$.

0	0	1
0	1	$26+i$
1	0	3
1	1	$18+i$
2	0	9
2	1	$28+i$
3	0	27
3	1	$24+i$
4	0	13
4	1	$12+i$
5	0	5
5	1	$10+i$
6	0	15
6	1	$4+i$
7	0	11
7	1	$20+i$
8	0	33
8	1	i
9	0	31
9	1	$8+i$
10	0	25
10	1	$32+i$
11	0	7
11	1	$2+i$
12	0	21
12	1	$14+i$
13	0	29
13	1	$16+i$
14	0	19
14	1	$22+i$
15	0	23
15	1	$6+i$
(16)	(2)	

i	8	1
1	0	0
$2+i$	11	1
3	1	0
$4+i$	6	1
5	5	0
$6+i$	15	1
7	11	0
$8+i$	9	1
9	2	0
$10+i$	5	1
11	7	0
$12+i$	4	1
13	4	0
$14+i$	12	1
15	6	0
$16+i$	13	1
$18+i$	1	1
19	14	0
$20+i$	7	1
21	12	0
$22+i$	14	1
23	15	0
$24+i$	3	1
25	10	0
$26+i$	0	1
27	3	0
$28+i$	2	1
29	13	0
31	9	0
$32+i$	10	1
33	8	0
	(16)	(2)

$$m = 6 + 6i = -i(1+i)^3 . 3 \quad N(m) = 72 \quad \Phi(m) = 32 = 2^2 . 2^3 \quad \text{H. E.} = 8$$

$$y \equiv Y \pmod{6} \quad x \equiv X + Y - y \pmod{12}.$$

Generators $2 + i$, i.

0	0	1
0	1	i
0	2	11
0	3	$6+5i$
1	0	$2+i$
1	1	$11+2i$
1	2	$4+5i$
1	3	$7+4i$
2	0	$3+4i$
2	1	$8+3i$
2	2	$3+2i$
2	3	$10+3i$
3	0	$8+5i$
3	1	$1+2i$
3	2	$10+i$
3	3	$5+4i$
4	0	5
4	1	$5i$
4	2	7
4	3	$6+i$
5	0	$10+5i$
5	1	$1+4i$
5	2	$8+i$
5	3	$5+2i$
6	0	$9+2i$
6	1	$4+3i$
6	2	$9+4i$
6	3	$2+3i$
7	0	$4+i$
7	1	$11+4i$
7	2	$2+5i$
7	3	$7+2i$
(8)	(4)	

i	0	1
$5i$	4	1
1	0	0
$1+2i$	3	1
$1+4i$	5	1
$2+i$	1	0
$2+3i$	6	3
$2+5i$	7	2
$3+2i$	2	2
$3+4i$	2	0
$4+i$	7	0
$4+3i$	6	1
$4+5i$	1	2
5	4	0
$5+2i$	5	3
$5+4i$	3	3
$6+i$	4	3
$6+5i$	0	3
7	4	2
$7+2i$	7	3
$7+4i$	1	3
$8+i$	5	2
$8+3i$	2	1
$8+5i$	3	0
$9+2i$	6	0
$9+4i$	6	2
$10+i$	3	2
$10+3i$	2	3
$10+5i$	5	0
11	0	2
$11+2i$	1	1
$11+4i$	7	1
	(8)	(4)

$m = 8 + 3i$ $\mathrm{N}(m) = 73$ $\Phi(m) = 72 = 2^3 . 3^2$ H. E. = 72 $x \equiv \mathrm{X} + 46\mathrm{Y} \pmod{73}$

$m = 3 + 8i$ $\mathrm{N}(m) = 73$ $\Phi(m) = 72 = 2^3 . 3^2$ H. E. = 72 $x \equiv \mathrm{X} + 27\mathrm{Y} \pmod{73}$

Generator 5.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	(72)
N	1	5	25	52	41	59	3	15	2	10	50	31	9	45	6	30	4	20	
I	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	(72)
N	27	62	18	17	12	60	8	40	54	51	36	34	24	47	16	7	35	29	
I	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	(72)
N	72	68	48	21	32	14	70	58	71	63	23	42	64	28	67	43	69	53	
I	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	(72)
N	46	11	55	56	61	13	65	33	19	22	37	39	49	26	57	66	38	44	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
I	0	8	6	16	1	14	33	24	12	9	55	22	59	41	7	32	21	20	(72)
N	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	
I	62	17	39	63	46	30	2	67	18	49	35	15	11	40	61	29	34	28	(72)
N	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	
I	64	70	65	25	4	47	51	71	13	54	31	38	66	10	27	3	53	26	(72)
N	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	
I	56	57	68	43	5	23	58	19	45	48	60	69	50	37	52	42	44	36	(72)

$m = 7 + 5i = -i(1+i)(1+6i) \quad \mathrm{N}(m) = 74 \quad \Phi(m) = 36 = 2^2 . 3^2 \quad \mathrm{H.E.} = 36$

$$x \equiv \mathrm{X} + 43\mathrm{Y} \pmod{74}$$

$m = 5 + 7i = \quad (1+i)(6+i) \quad \mathrm{N}(m) = 74 \quad \Phi(m) = 36 = 2^2 . 3^2 \quad \mathrm{H.E.} = 36$

$$x \equiv \mathrm{X} + 31\mathrm{Y} \pmod{74}.$$

Generator 5.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	(36)
N	1	5	25	51	33	17	11	55	53	43	67	39	47	13	65	29	71	59	
I	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	(36)
N	73	69	49	23	41	57	63	19	21	31	7	35	27	61	9	45	3	15	

N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	
I	0	34	1	28	32	6	13	35	5	25	26	21	2	30	15	27	4	29	(36)
N	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	
I	11	22	9	33	12	20	3	8	7	23	17	31	24	14	10	19	16	18	(36

$m = 8 + 4i = -(1+i)^4(2+i) \quad \mathrm{N}(m) = 80 \quad \Phi(m) = 32 = 2.2^2.2^2 \quad \mathrm{H.E.} = 4$

$y \equiv \mathrm{Y} \pmod 4 \quad x \equiv \mathrm{X} + 3(\mathrm{Y} - y) \pmod{20}.$

Generators 17, $8 + i$, $5 + 2i$.

0	0	0	1
0	0	1	$5+2i$
0	1	0	$8+i$
0	1	1	$18+i$
0	2	0	11
0	2	1	$15+2i$
0	3	0	$12+3i$
0	3	1	$2+3i$
1	0	0	17
1	0	1	$1+2i$
1	1	0	$4+i$
1	1	1	$14+i$
1	2	0	7
1	2	1	$11+2i$
1	3	0	$8+3i$
1	3	1	$18+3i$
2	0	0	9
2	0	1	$13+2i$
2	1	0	$16+i$
2	1	1	$6+i$
2	2	0	19
2	2	1	$3+2i$
2	3	0	$3i$
2	3	1	$10+3i$
3	0	0	13
3	0	1	$17+2i$
3	1	0	i
3	1	1	$10+i$
3	2	0	3
3	2	1	$7+2i$
3	3	0	$4+3i$
3	3	1	$14+3i$
(4)	(4)	(2)	

i	3	1	0
$3i$	2	3	0
1	0	0	0
$1+2i$	1	0	1
$2+3i$	0	3	1
3	3	2	0
$3+2i$	2	2	1
$4+i$	1	1	0
$4+3i$	3	3	0
$5+2i$	0	0	1
$6+i$	2	1	1
7	1	2	0
$7+2i$	3	2	1
$8+i$	0	1	0
$8+3i$	1	3	0
9	2	0	0
$10+i$	3	1	1
$10+3i$	2	3	1
11	0	2	0
$11+2i$	1	2	1
$12+3i$	0	3	0
13	3	0	0
$13+2i$	2	0	1
$14+i$	1	1	1
$14+3i$	3	3	1
$15+2i$	0	2	1
$16+i$	2	1	0
17	1	0	0
$17+2i$	3	0	1
$18+i$	0	1	1
$18+3i$	1	3	1
19	2	2	0
	(4)	(4)	(2)

$$m = 4 + 8i = -(1+i)^4(1+2i) \quad \mathrm{N}(m) = 80 \quad \Phi(m) = 32 = 2.2^2.2^3 \quad \mathrm{H.E.} = 4$$

$$y \equiv \mathrm{Y} \pmod 4 \quad x \equiv \mathrm{X} + 17(\mathrm{Y} - y) \pmod{20}.$$

Generators 17, $3i$, $17 + 2i$.

0	0	0	1
0	0	1	$17+2i$
0	1	0	$3i$
0	1	1	$10+3i$
0	2	0	11
0	2	1	$7+2i$
0	3	0	$4+i$
0	3	1	$14+i$
1	0	0	17
1	0	1	$13+2i$
1	1	0	$16+3i$
1	1	1	$6+3i$
1	2	0	7
1	2	1	$3+2i$
1	3	0	i
1	3	1	$10+i$
2	0	0	9
2	0	1	$5+2i$
2	1	0	$8+3i$
2	1	1	$18+3i$
2	2	0	19
2	2	1	$15+2i$
2	3	0	$12+i$
2	3	1	$2+i$
3	0	0	13
3	0	1	$9+2i$
3	1	0	$12+3i$
3	1	1	$2+3i$
3	2	0	3
3	2	1	$19+2i$
3	3	0	$16+i$
3	3	1	$6+i$
(4)	(4)	(2)	

i	1	3	0
$3i$	0	1	0
1	0	0	0
$2+i$	2	3	1
$2+3i$	3	1	1
3	3	2	0
$3+2i$	1	2	1
$4+i$	0	3	0
$5+2i$	2	0	1
$6+i$	3	3	1
$6+3i$	1	1	1
7	1	2	0
$7+2i$	0	2	1
$8+3i$	2	1	0
9	2	0	0
$9+2i$	3	0	1
$10+i$	1	3	1
$10+3i$	0	1	1
11	0	2	0
$12+i$	2	3	0
$12+3i$	3	1	0
13	3	0	0
$13+2i$	1	0	1
$14+i$	0	3	1
$15+2i$	2	2	1
$16+i$	3	3	0
$16+3i$	1	1	0
17	1	0	0
$17+2i$	0	0	1
$18+3i$	2	1	1
19	2	2	0
$19+2i$	3	2	1
	(4)	(4)	(2)

$m = 9 = 3^2$ $\quad$ N $(m) = 81$ $\quad$ $\Phi(m) = 72 = 2^3 . 3 . 3$ $\quad$ H. E. $= 24$

$x \equiv$ X (mod 9) $\quad$ $y \equiv$ Y (mod 9).

Generators $2 + i$, $4 + 3i$.

0	0	1	8	0	$4+6i$	16	0	$7+3i$
0	1	$4+3i$	8	1	7	16	1	$1+6i$
0	2	$7+6i$	8	2	$1+3i$	16	2	4
1	0	$2+i$	9	0	$2+7i$	17	0	$2+4i$
1	1	$5+i$	9	1	$5+7i$	17	1	$5+4i$
1	2	$8+i$	9	2	$8+7i$	17	2	$8+4i$
2	0	$3+4i$	10	0	$6+7i$	18	0	i
2	1	$7i$	10	1	$3+i$	18	1	$6+4i$
2	2	$6+i$	10	2	$4i$	18	2	$3+7i$
3	0	$2+2i$	11	0	$5+2i$	19	0	$8+2i$
3	1	$2+5i$	11	1	$5+5i$	19	1	$8+5i$
3	2	$2+8i$	11	2	$5+8i$	19	2	$8+8i$
4	0	$2+6i$	12	0	8	20	0	$5+3i$
4	1	$8+3i$	12	1	$5+6i$	20	1	2
4	2	5	12	2	$2+3i$	20	2	$8+6i$
5	0	$7+5i$	13	0	$7+8i$	21	0	$7+2i$
5	1	$4+5i$	13	1	$4+8i$	21	1	$4+2i$
5	2	$1+5i$	13	2	$1+8i$	21	2	$1+2i$
6	0	$8i$	14	0	$6+5i$	22	0	$3+2i$
6	1	$3+5i$	14	1	$2i$	22	1	$6+8i$
6	2	$6+2i$	14	2	$3+8i$	22	2	$5i$
7	0	$1+7i$	15	0	$7+7i$	23	0	$4+7i$
7	1	$1+4i$	15	1	$7+4i$	23	1	$4+4i$
7	2	$1+i$	15	2	$7+i$	23	2	$4+i$
(24)	(3)		(24)	(3)		(24)	(3)	

i	18	0	$3+i$	10	1	$6+i$	2	2
$2i$	14	1	$3+2i$	22	0	$6+2i$	6	2
$4i$	10	2	$3+4i$	2	0	$6+4i$	18	1
$5i$	22	2	$3+5i$	6	1	$6+5i$	14	0
$7i$	2	1	$3+7i$	18	2	$6+7i$	10	0
$8i$	6	0	$3+8i$	14	2	$6+8i$	22	1
1	0	0	4	16	2	7	8	1
$1+i$	7	2	$4+i$	23	2	$7+i$	15	2
$1+2i$	21	2	$4+2i$	21	1	$7+2i$	21	0
$1+3i$	8	2	$4+3i$	0	1	$7+3i$	16	0
$1+4i$	7	1	$4+4i$	23	1	$7+4i$	15	1
$1+5i$	5	2	$4+5i$	5	1	$7+5i$	5	0
$1+6i$	16	1	$4+6i$	8	0	$7+6i$	0	2
$1+7i$	7	0	$4+7i$	23	0	$7+7i$	15	0
$1+8i$	13	2	$4+8i$	13	1	$7+8i$	13	0
2	20	1	5	4	2	8	12	0
$2+i$	1	0	$5+i$	1	1	$8+i$	1	2
$2+2i$	3	0	$5+2i$	11	0	$8+2i$	19	0
$2+3i$	12	2	$5+3i$	20	0	$8+3i$	4	1
$2+4i$	17	0	$5+4i$	17	1	$8+4i$	17	2
$2+5i$	3	1	$5+5i$	11	1	$8+5i$	19	1
$2+6i$	4	0	$5+6i$	12	1	$8+6i$	20	2
$2+7i$	9	0	$5+7i$	9	1	$8+7i$	9	2
$2+8i$	3	2	$5+8i$	11	2	$8+8i$	19	2
	(24)	(3)		(24)	(3)		(24)	(3)

$$m = 9 + i = -i(1+i)(4+5i) \quad \mathrm{N}(m) = 82 \quad \Phi(m) = 40 = 2^3 . 5$$

$$\text{H. E.} = 40 \quad x \equiv \mathrm{X} + 73\mathrm{Y} \pmod{82}.$$

$$m = 1 + 9i = \quad (1+i)(5+4i) \quad \mathrm{N}(m) = 82 \quad \Phi(m) = 40 = 2^3 . 5$$

$$\text{H. E.} \equiv 40 \quad x \equiv \mathrm{X} + 9\mathrm{Y} \pmod{82}.$$

Generator 7.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(40)
N	1	7	49	15	23	79	61	17	37	13	9	63	31	53	43	55	57	71	5	35	
I	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	(40)
N	81	75	33	67	59	3	21	65	45	69	73	19	51	29	39	27	25	11	77	47	

N	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	
I	0	25	18	1	10	37	9	3	7	31	26	4	36	35	33	12	22	19	8	34	(40)
N	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79	81	
I	14	28	39	2	32	13	15	16	24	6	11	27	23	29	17	30	21	38	5	20	(40)

$$m = 9+2i = -i(2+i)(1+4i) \quad 2+9i = (1+2i)(4+i) \quad 7+6i = (2+i)(4+i)$$
$$\text{or } 6+7i = -i(1+2i)(1+4i).$$

For all $N(m) = 85 \quad \Phi(m) = 64 = 2^2 . 2^4 \quad$ H.E. $= 16$.

$$m = 9+2i \quad x \equiv X + 38Y \pmod{85} \quad m = 2+9i \quad x \equiv X + 47Y \pmod{85}$$
$$m = 7+6i \quad x \equiv X + 13Y \pmod{85} \quad m = 6+7i \quad x \equiv X + 72Y \pmod{85}.$$

Generators 3, 52.

0	0	1
0	1	52
0	2	69
0	3	18
1	0	3
1	1	71
1	2	37
1	3	54
2	0	9
2	1	43
2	2	26
2	3	77
3	0	27
3	1	44
3	2	78
3	3	61
4	0	81
4	1	47
4	2	64
4	3	13
5	0	73
5	1	56
5	2	22
5	3	39
6	0	49
6	1	83
6	2	66
6	3	32
7	0	62
7	1	79
7	2	28
7	3	11
(16)	(4)	

8	0	16
8	1	67
8	2	84
8	3	33
9	0	48
9	1	31
9	2	82
9	3	14
10	0	59
10	1	8
10	2	76
10	3	42
11	0	7
11	1	24
11	2	58
11	3	41
12	0	21
12	1	72
12	2	4
12	3	38
13	0	63
13	1	46
13	2	12
13	3	29
14	0	19
14	1	53
14	2	36
14	3	2
15	0	57
15	1	74
15	2	23
15	3	6
(16)	(4)	

1	0	0
2	14	3
3	1	0
4	12	2
6	15	3
7	11	0
8	10	1
9	2	0
11	7	3
12	13	2
13	4	3
14	9	3
16	8	0
18	0	3
19	14	0
21	12	0
22	5	2
23	15	2
24	11	1
26	2	2
27	3	0
28	7	2
29	13	3
31	9	1
32	6	3
33	8	3
36	14	2
37	1	2
38	12	3
39	5	3
41	11	3
42	10	3
	(16)	(4)

43	2	1
44	3	1
46	13	1
47	4	1
48	9	0
49	6	0
52	0	1
53	14	1
54	1	3
56	5	1
57	15	0
58	11	2
59	10	0
61	3	3
62	7	0
63	13	0
64	4	2
66	6	2
67	8	1
69	0	2
71	1	1
72	12	1
73	5	0
74	15	1
76	10	2
77	2	3
78	3	2
79	7	1
81	4	0
82	9	2
83	6	1
84	8	2
	(16)	(4)

$m = 8 + 5i \quad N(m) = 89 \quad \Phi(m) = 88 = 2^3.11 \quad$ H. E. $= 88 \quad x \equiv X + 34\,Y \pmod{89}$.

$m = 5 + 8i \quad N(m) = 89 \quad \Phi(m) = 88 = 2^3.11 \quad$ H. E. $= 88 \quad x \equiv X + 55\,Y \pmod{89}$.

Generator 3.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	(88)
N	1	3	9	27	81	65	17	51	64	14	42	37	22	66	20	60	2	6	18	54	73	41	
I	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	(88)
N	34	13	39	28	84	74	44	43	40	31	4	12	36	19	57	82	68	26	78	56	79	59	
I	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	(88)
N	88	86	80	62	8	24	72	38	25	75	47	52	67	23	69	29	87	83	71	35	16	48	
I	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	(88)
N	55	76	50	61	5	15	45	46	49	58	85	77	53	70	32	7	21	63	11	33	10	30	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	
I	0	16	1	32	70	17	81	48	2	86	84	33	23	9	71	64	6	18	35	14	82	12	(88)
N	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	
I	57	49	52	39	3	25	59	87	31	80	85	22	63	34	11	51	24	30	21	10	29	28	(88)
N	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	
I	72	73	54	65	74	68	7	55	78	19	66	41	36	75	43	15	69	47	83	8	5	13	(88)
N	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	
I	56	38	58	79	62	50	20	27	53	67	77	40	42	46	4	37	61	26	76	45	60	44	(88)

$$m = 9 + 3i = -i(1+i)(1+2i)\,3 \quad \mathrm{N}(m) = 90 \quad \Phi(m) = 32 = 2^2 . 2^3$$

$$\text{H. E.} = 8 \quad y \equiv \mathrm{Y} \pmod{3} \quad x \equiv \mathrm{X} + 7(\mathrm{Y} - y) \pmod{30}.$$

Generators $2 + i$, i.

0	0	1
0	1	i
0	2	29
0	3	$9+2i$
1	0	$2+i$
1	1	$29+2i$
1	2	$7+2i$
1	3	$10+i$
2	0	$24+i$
2	1	17
2	2	$15+2i$
2	3	13
3	0	$5+2i$
3	1	$19+2i$
3	2	$4+i$
3	3	$20+i$
4	0	11
4	1	$3+2i$
4	2	19
4	3	$6+i$
5	0	$25+2i$
5	1	$16+i$
5	2	$14+i$
5	3	$23+2i$
6	0	$27+2i$
6	1	7
6	2	$12+i$
6	3	23
7	0	$22+i$
7	1	$26+i$
7	2	$17+2i$
7	3	$13+2i$
(8)	(4)	

i	0	1
1	0	0
$2+i$	1	0
$3+2i$	4	1
$4+i$	3	2
$5+2i$	3	0
$6+i$	4	3
7	6	1
$7+2i$	1	2
$9+2i$	0	3
$10+i$	1	3
11	4	0
$12+i$	6	2
13	2	3
$13+2i$	7	3
$14+i$	5	2
$15+2i$	2	2
$16+i$	5	1
17	2	1
$17+2i$	7	2
19	4	2
$19+2i$	3	1
$20+i$	3	3
$22+i$	7	0
23	6	3
$23+2i$	5	3
$24+i$	2	0
$25+2i$	5	0
$26+i$	7	1
$27+2i$	6	0
29	0	2
$29+2i$	1	1
	(8)	(4)

$$m = 3 + 9i = (1 + i)(2 + i)\,3 \quad \mathrm{N}(m) = 90 \quad \Phi(m) = 32 = 2^3.\ 2^3$$

$$\text{H.E.} = 8 \quad y \equiv \mathrm{Y} \pmod{3} \quad x \equiv \mathrm{X} + 23(\mathrm{Y} - y) \pmod{30}.$$

Generators $23 + 2i$, $21 + 2i$.

0	0	1
0	1	$21+2i$
0	2	29
0	3	i
1	0	$23+2i$
1	1	$20+i$
1	2	$28+i$
1	3	$1+2i$
2	0	$15+2i$
2	1	17
2	2	$6+i$
2	3	13
3	0	$26+i$
3	1	$10+i$
3	2	$25+2i$
3	3	$11+2i$
4	0	11
4	1	$24+i$
4	2	19
4	3	$27+2i$
5	0	$16+i$
5	1	$7+2i$
5	2	$5+2i$
5	3	$14+i$
6	0	$18+i$
6	1	7
6	2	$3+2i$
6	3	23
7	0	$13+2i$
7	1	$17+2i$
7	2	$8+i$
7	3	$4+i$
(8)	(4)	

i	0	3
1	0	0
$1+2i$	1	3
$3+2i$	6	2
$4+i$	7	3
$5+2i$	5	2
$6+i$	2	2
7	6	1
$7+2i$	5	1
$8+i$	7	2
$10+i$	3	1
11	4	0
$11+2i$	3	3
13	2	3
$13+2i$	7	0
$14+i$	5	3
$15+2i$	2	0
$16+i$	5	0
17	2	1
$17+2i$	7	1
$18+i$	6	0
19	4	2
$20+i$	1	1
$21+2i$	0	1
23	6	3
$23+2i$	1	0
$24+i$	4	1
$25+2i$	3	2
$26+i$	3	0
$27+2i$	4	3
$28+i$	1	2
29	0	2
	(8)	(4)

$m = 9 + 4i \quad \mathrm{N}(m) = 97 \quad \Phi(m) = 96 = 2^5.\,3 \quad \mathrm{H.E.} = 96 \quad x \equiv \mathrm{X} + 22\mathrm{Y} \pmod{97}.$

$m = 4 + 9i \quad \mathrm{N}(m) = 97 \quad \Phi(m) = 96 = 2^5.\,3 \quad \mathrm{H.E.} = 96 \quad x \equiv \mathrm{X} + 75\mathrm{Y} \pmod{97}.$

Generator 5.

I	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	(96)
N	1	5	25	28	43	21	8	40	6	30	53	71	64	29	48	46	36	83	27	38	
I	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	(96)
N	93	77	94	82	22	13	65	34	73	74	79	7	35	78	2	10	50	56	86	
I	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	(96)
N	42	16	80	12	60	9	45	31	58	96	92	72	69	54	76	89	57	91	67	
I	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	(96)
N	44	26	33	68	49	51	61	14	70	59	4	20	3	15	75	84	32	63	24	
I	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	(96)
N	23	18	90	62	19	95	87	47	41	11	55	81	17	85	37	88	52	66	39	

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
I	0	34	70	68	1	8	31	6	44	35	86	42	25	65	71	40	89	78	81	69	(96)
N	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39		
I	5	24	77	76	2	59	18	3	13	9	46	74	69	27	32	16	91	19	95		(96)
N	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58		
I	7	85	39	4	58	45	15	84	14	62	36	63	93	10	52	87	37	55	47		(96)
N	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77		
I	67	43	64	80	75	12	26	94	57	61	51	66	11	50	28	29	72	53	21		(96)
N	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96		
I	33	30	41	88	23	17	73	90	38	83	92	54	79	56	49	20	22	82	48		(96)

$$m = 7 + 7i = (1 + i)\,7 \quad \mathrm{N}\,(m) = 98 \quad \Phi\,(m) = 48 = 2^4.\,3 \quad \mathrm{H.\,E.} = 48$$

$$y \equiv \mathrm{Y} \pmod{7} \quad x \equiv \mathrm{X} + \mathrm{Y} - y \pmod{14}.$$

Generator $2 + i$.

0	1	24	13	i	12	$7+2i$	44
1	$2+i$	25	$5+6i$	$3i$	4	$7+4i$	28
2	$3+4i$	26	$4+3i$	$5i$	20	$7+6i$	36
3	$9+4i$	27	$12+3i$	1	0	$8+i$	22
4	$3i$	28	$7+4i$	$1+2i$	19	$8+3i$	23
5	$11+6i$	29	$10+i$	$1+4i$	17	$8+5i$	37
6	$9+2i$	30	$12+5i$	$1+6i$	10	9	32
7	$9+6i$	31	$12+i$	$2+i$	1	$9+2i$	6
8	5	32	9	$2+3i$	21	$9+4i$	3
9	$10+5i$	33	$11+2i$	$2+5i$	42	$9+6i$	7
10	$1+6i$	34	$6+i$	3	40	$10+i$	29
11	$3+6i$	35	$4+i$	$3+2i$	15	$10+3i$	14
12	i	36	$7+6i$	$3+4i$	2	$10+5i$	9
13	$13+2i$	37	$8+5i$	$3+6i$	11	11	16
14	$10+3i$	38	$11+4i$	$4+i$	35	$11+2i$	33
15	$3+2i$	39	$4+5i$	$4+3i$	26	$11+4i$	38
16	11	40	3	$4+5i$	39	$11+6i$	5
17	$1+4i$	41	$6+3i$	5	8	$12+i$	31
18	$5+2i$	42	$2+5i$	$5+2i$	18	$12+3i$	27
19	$1+2i$	43	$6+5i$	$5+4i$	45	$12+5i$	30
20	$5i$	44	$7+2i$	$5+6i$	25	13	24
21	$2+3i$	45	$5+4i$	$6+i$	34	$13+2i$	13
22	$8+i$	46	$13+6i$	$6+3i$	41	$13+4i$	47
23	$8+3i$	47	$13+4i$	$6+5i$	43	$13+6i$	46
(48)		(48)			(48)		(48)

$$m = 8 + 6i = -(1+i)^2(1+2i)^2 \quad \mathrm{N}(m) = 100 \quad \Phi(m) = 40 = 2.2^2.5.$$

$$\mathrm{H.\,E.} = 20 \quad y \equiv \mathrm{Y} \pmod 2 \quad x \equiv \mathrm{X} + 7(\mathrm{Y} - y) \pmod{50}.$$

Generators 3, $44+i$.

0	0	1
0	1	$44+i$
1	0	3
1	1	$46+i$
2	0	9
2	1	$2+i$
3	0	27
3	1	$20+i$
4	0	31
4	1	$24+i$
5	0	43
5	1	$36+i$
6	0	29
6	1	$22+i$
7	0	37
7	1	$30+i$
8	0	11
8	1	$4+i$
9	0	33
9	1	$26+i$

(20) (2)

10	0	49
10	1	$42+i$
11	0	47
11	1	$40+i$
12	0	41
12	1	$34+i$
13	0	23
13	1	$16+i$
14	0	19
14	1	$12+i$
15	0	7
15	1	i
16	0	21
16	1	$14+i$
17	0	13
17	1	$6+i$
18	0	39
18	1	$32+i$
19	0	17
19	1	$10+i$

(20) (2)

i	15	1
1	0	0
$2+i$	2	1
3	1	0
$4+i$	8	1
$6+i$	17	1
7	15	0
9	2	0
$10+i$	19	1
11	8	0
$12+i$	14	1
13	17	0
$14+i$	16	1
$16+i$	13	1
17	19	0
19	14	0
$20+i$	3	1
21	16	0
$22+i$	6	1
23	13	0

(20) (2)

$24+i$	4	1
$26+i$	9	1
27	3	0
29	6	0
$30+i$	7	1
31	4	0
$32+i$	18	1
33	9	0
$34+i$	12	1
$36+i$	5	1
37	7	0
39	18	0
$40+i$	11	1
41	12	0
$42+i$	10	1
43	5	0
$44+i$	0	1
$46+i$	1	1
47	11	0
49	10	0

(20) (2)

$$m = 6 + 8i = -i(1+i)^2(2+i)^2 \quad \mathrm{N}(m) = 100 \quad \Phi(m) = 40 = 2.2^2.5$$

$$\mathrm{H.\,E.} = 20 \quad y \equiv \mathrm{Y} \pmod 2 \quad x \equiv \mathrm{X} + 43(\mathrm{Y} - y) \pmod{50}.$$

Generators 3, $8+i$.

0	0	1
0	1	$8+i$
1	0	3
1	1	$10+i$
2	0	9
2	1	$16+i$
3	0	27
3	1	$34+i$
4	0	31
4	1	$38+i$
5	0	43
5	1	i
6	0	29
6	1	$36+i$
7	0	37
7	1	$44+i$
8	0	11
8	1	$18+i$
9	0	33
9	1	$40+i$

(20) (2)

10	0	49
10	1	$6+i$
11	0	47
11	1	$4+i$
12	0	41
12	1	$48+i$
13	0	23
13	1	$30+i$
14	0	19
14	1	$26+i$
15	0	7
15	1	$14+i$
16	0	21
16	1	$28+i$
17	0	13
17	1	$20+i$
18	0	39
18	1	$46+i$
19	0	17
19	1	$24+i$

(20) (2)

i	5	1
1	0	0
3	1	0
$4+i$	11	1
$6+i$	10	1
7	15	0
$8+i$	0	1
9	2	0
$10+i$	1	1
11	8	0
13	17	0
$14+i$	15	1
$16+i$	2	1
17	19	0
$18+i$	8	1
19	14	0
$20+i$	17	1
21	16	0
23	13	0
$24+i$	19	1

(20) (2)

$26+i$	14	1
27	3	0
$28+i$	16	1
29	6	0
$30+i$	13	1
31	4	0
33	9	0
$34+i$	3	1
$36+i$	6	1
37	7	0
$38+i$	4	1
39	18	0
$40+i$	9	1
41	12	0
43	5	0
$44+i$	7	1
$46+i$	18	1
47	11	0
$48+i$	12	1
49	10	0

(20) (2)

$m = 10 = -(1+i)^2(1+2i)(2+i)$ $\mathrm{N}(m)=100$ $\Phi(m) = 32 = 2.2^2.2^2.$ H. E. $=4$

$x \equiv \mathrm{X} \pmod{10}$ $y \equiv \mathrm{Y} \pmod{10}$.

Generators $9+4i$, $9+6i$, $6+5i$.

0	0	0	1
0	0	1	$6+5i$
0	1	0	$9+6i$
0	1	1	$4+i$
0	2	0	$5+8i$
0	2	1	$3i$
0	3	0	$7+2i$
0	3	1	$2+7i$
1	0	0	$9+4i$
1	0	1	$4+9i$
1	1	0	7
1	1	1	$2+5i$
1	2	0	$3+2i$
1	2	1	$8+7i$
1	3	0	$5+6i$
1	3	1	i
2	0	0	$5+2i$
2	0	1	$7i$
2	1	0	$3+8i$
2	1	1	$8+3i$
2	2	0	9
2	2	1	$4+5i$
2	3	0	$1+4i$
2	3	1	$6+9i$
3	0	0	$7+8i$
3	0	1	$2+3i$
3	1	0	$5+4i$
3	1	1	$9i$
3	2	0	$1+6i$
3	2	1	$6+i$
3	3	0	3
3	3	1	$8+5i$
(4)	(4)	(2)	

i	1	3	1
$3i$	0	2	1
$7i$	2	0	1
$9i$	3	1	1
1	0	0	0
$1+4i$	2	3	0
$1+6i$	3	2	0
$2+3i$	3	0	1
$2+5i$	1	1	1
$2+7i$	0	3	1
3	3	3	0
$3+2i$	1	2	0
$3+8i$	2	1	0
$4+i$	0	1	1
$4+5i$	2	2	1
$4+9i$	1	0	1
$5+2i$	2	0	0
$5+4i$	3	1	0
$5+6i$	1	3	0
$5+8i$	0	2	0
$6+i$	3	2	1
$6+5i$	0	0	1
$6+9i$	2	3	1
7	1	1	0
$7+2i$	0	3	0
$7+8i$	3	0	0
$8+3i$	2	1	1
$8+5i$	3	3	1
$8+7i$	1	2	1
9	2	2	0
$9+4i$	1	0	0
$9+6i$	0	1	0
	(4)	(4)	(2)

Mod $(1 + i)$.

1	1

Mod $(1 + i)^2$.

Generator i.

i	1
1	0
	(2)

Mod $(1 + i)^3$.

Generator i.

i	1
3	2
$2+i$	3
1	0
	(4)

Mod $(1 + i)^4$

Generators i, $1 + 2i$.

0	0	1
0	1	$1+2i$
1	0	i
1	1	$2+i$
2	0	3
2	1	$3+2i$
3	0	$3i$
3	1	$2+3i$
(4)	(2)	

i	1	0
$3i$	3	0
1	0	0
$1+2i$	0	1
$2+i$	1	1
$2+3i$	3	1
3	2	0
$3+2i$	2	1
	(4)	(2)

Mod $(1 + i)^5$.

Generators i, $1 + 2i$, 3.

0	0	0	1
0	0	1	3
0	1	0	$1+2i$
0	1	1	$7+2i$
1	0	0	i
1	0	1	$3i$
1	1	0	$6+i$
1	1	1	$2+3i$
2	0	0	7
2	0	1	5
2	1	0	$3+2i$
2	1	1	$5+2i$
3	0	0	$4+3i$
3	0	1	$4+i$
3	1	0	$6+3i$
3	1	1	$2+i$
(4)	(2)	(2)	

i	1	0	0
$3i$	1	0	1
1	0	0	0
$1+2i$	0	1	0
$2+i$	3	1	1
$2+3i$	1	1	1
3	0	0	1
$3+2i$	2	1	0
$4+i$	3	0	1
$4+3i$	3	0	0
5	2	0	1
$5+2i$	2	1	1
$6+i$	1	1	0
$6+3i$	3	1	0
7	2	0	0
$7+2i$	0	1	1
	(4)	(2)	(2)

Mod $(1+i)^6$.

Generators i, $1+2i$, 3.

0	0	0	1
0	0	1	3
0	1	0	$1+2i$
0	1	1	$3+6i$
0	2	0	$5+4i$
0	2	1	$7+4i$
0	3	0	$5+6i$
0	3	1	$7+2i$
1	0	0	i
1	0	1	$3i$
1	1	0	$6+i$
1	1	1	$2+3i$
1	2	0	$4+5i$
1	2	1	$4+7i$
1	3	0	$2+5i$
1	3	1	$6+7i$
2	0	0	7
2	0	1	5
2	1	0	$7+6i$
2	1	1	$5+2i$
2	2	0	$3+4i$
2	2	1	$1+4i$
2	3	0	$3+2i$
2	3	1	$1+6i$
3	0	0	$7i$
3	0	1	$5i$
3	1	0	$2+7i$
3	1	1	$6+5i$
3	2	0	$4+3i$
3	2	1	$4+i$
3	3	0	$6+3i$
3	3	1	$2+i$
(4)	(4)	(2)	

i	1	0	0
$3i$	1	0	1
$5i$	3	0	1
$7i$	3	0	0
1	0	0	0
$1+2i$	0	1	0
$1+4i$	2	1	1
$1+6i$	2	3	1
$2+i$	3	3	1
$2+3i$	1	1	1
$2+5i$	1	3	0
$2+7i$	3	1	0
3	0	0	1
$3+2i$	2	3	0
$3+4i$	2	2	0
$3+6i$	0	1	1
$4+i$	3	2	1
$4+3i$	3	2	0
$4+5i$	1	2	0
$4+7i$	1	2	1
5	2	0	1
$5+2i$	2	1	1
$5+4i$	0	2	0
$5+6i$	0	3	0
$6+i$	1	1	0
$6+3i$	3	3	0
$6+5i$	3	1	1
$6+7i$	1	3	1
7	2	0	0
$7+2i$	0	3	1
$7+4i$	0	2	1
$7+6i$	2	1	0
	(4)	(4)	(2)

Mod $(1+i)^7$.

Generators i, 3, $1+2i$.

0	0	0	1	2	0	0	15
0	0	1	$1+2i$	2	0	1	$7+6i$
0	0	2	$13+4i$	2	0	2	$11+4i$
0	0	3	$13+6i$	2	0	3	$11+2i$
0	1	0	3	2	1	0	13
0	1	1	$3+6i$	2	1	1	$5+2i$
0	1	2	$15+4i$	2	1	2	$9+4i$
0	1	3	$7+2i$	2	1	3	$1+6i$
0	2	0	9	2	2	0	7
0	2	1	$9+2i$	2	2	1	$15+6i$
0	2	2	$5+4i$	2	2	2	$3+4i$
0	2	3	$5+6i$	2	2	3	$3+2i$
0	3	0	11	2	3	0	5
0	3	1	$11+6i$	2	3	1	$13+2i$
0	3	2	$7+4i$	2	3	2	$1+4i$
0	3	3	$15+2i$	2	3	3	$9+6i$
1	0	0	i	3	0	0	$8+7i$
1	0	1	$14+i$	3	0	1	$10+7i$
1	0	2	$4+5i$	3	0	2	$4+3i$
1	0	3	$2+5i$	3	0	3	$6+3i$
1	1	0	$3i$	3	1	0	$8+5i$
1	1	1	$10+3i$	3	1	1	$14+5i$
1	1	2	$4+7i$	3	1	2	$4+i$
1	1	3	$14+7i$	3	1	3	$10+i$
1	2	0	$8+i$	3	2	0	$7i$
1	2	1	$6+i$	3	2	1	$2+7i$
1	2	2	$12+5i$	3	2	2	$12+3i$
1	2	3	$10+5i$	3	2	3	$14+3i$
1	3	0	$8+3i$	3	3	0	$5i$
1	3	1	$2+3i$	3	3	1	$6+5i$
1	3	2	$12+7i$	3	3	2	$12+i$
1	3	3	$6+7i$	3	3	3	$2+i$
(4)	(4)	(4)		(4)	(4)	(4)	

i	1	0	0	$8+i$	1	2	0
$3i$	1	1	0	$8+3i$	1	3	0
$5i$	3	3	0	$8+5i$	3	1	0
$7i$	3	2	0	$8+7i$	3	0	0
1	0	0	0	9	0	2	0
$1+2i$	0	0	1	$9+2i$	0	2	1
$1+4i$	2	3	2	$9+4i$	2	1	2
$1+6i$	2	1	3	$9+6i$	2	3	3
$2+i$	3	3	3	$10+i$	3	1	3
$2+3i$	1	3	1	$10+3i$	1	1	1
$2+5i$	1	0	3	$10+5i$	1	2	3
$2+7i$	3	2	1	$10+7i$	3	0	1
3	0	1	0	11	0	3	0
$3+2i$	2	2	3	$11+2i$	2	0	3
$3+4i$	2	2	2	$11+4i$	2	0	2
$3+6i$	0	1	1	$11+6i$	0	3	1
$4+i$	3	1	2	$12+i$	3	3	2
$4+3i$	3	0	2	$12+3i$	3	2	2
$4+5i$	1	0	2	$12+5i$	1	2	2
$4+7i$	1	1	2	$12+7i$	1	3	2
5	2	3	0	13	2	1	0
$5+2i$	2	1	1	$13+2i$	2	3	1
$5+4i$	0	2	2	$13+4i$	0	0	2
$5+6i$	0	2	3	$13+6i$	0	0	3
$6+i$	1	2	1	$14+i$	1	0	1
$6+3i$	3	0	3	$14+3i$	3	2	3
$6+5i$	3	3	1	$14+5i$	3	1	1
$6+7i$	1	3	3	$14+7i$	1	1	3
7	2	2	0	15	2	0	0
$7+2i$	0	1	3	$15+2i$	0	3	3
$7+4i$	0	3	2	$15+4i$	0	1	2
$7+6i$	2	0	1	$15+6i$	2	2	1
	(4)	(4)	(4)		(4)	(4)	(4)

Mod $(1 + i)^8$.

Generators $1 + 2i$, 3, i.

0	0	0	1	2	0	0	$13+4i$	4	0	0	$9+8i$	6	0	0	$5+12i$
0	0	1	i	2	0	1	$12+13i$	4	0	1	$8+9i$	6	0	1	$4+5i$
0	0	2	15	2	0	2	$3+12i$	4	0	2	$7+8i$	6	0	2	$11+4i$
0	0	3	$15i$	2	0	3	$4+3i$	4	0	3	$8+7i$	6	0	3	$12+11i$
0	1	0	3	2	1	0	$7+12i$	4	1	0	$11+8i$	6	1	0	$15+4i$
0	1	1	$3i$	2	1	1	$4+7i$	4	1	1	$8+11i$	6	1	1	$12+15i$
0	1	2	13	2	1	2	$9+4i$	4	1	2	$5+8i$	6	1	2	$1+12i$
0	1	3	$13i$	2	1	3	$12+9i$	4	1	3	$8+5i$	6	1	3	$4+i$
0	2	0	9	2	2	0	$5+4i$	4	2	0	$1+8i$	6	2	0	$13+12i$
0	2	1	$9i$	2	2	1	$12+5i$	4	2	1	$8+i$	6	2	1	$4+13i$
0	2	2	7	2	2	2	$11+12i$	4	2	2	$15+8i$	6	2	2	$3+4i$
0	2	3	$7i$	2	2	3	$4+11i$	4	2	3	$8+15i$	6	2	3	$12+3i$
0	3	0	11	2	3	0	$15+12i$	4	3	0	$3+8i$	6	3	0	$7+4i$
0	3	1	$11i$	2	3	1	$4+15i$	4	3	1	$8+3i$	6	3	1	$12+7i$
0	3	2	5	2	3	2	$1+4i$	4	3	2	$13+8i$	6	3	2	$9+12i$
0	3	3	$5i$	2	3	3	$12+i$	4	3	3	$8+13i$	6	3	3	$4+9i$
1	0	0	$1+2i$	3	0	0	$5+14i$	5	0	0	$9+10i$	7	0	0	$13+6i$
1	0	1	$14+i$	3	0	1	$2+5i$	5	0	1	$6+9i$	7	0	1	$10+13i$
1	0	2	$15+14i$	3	0	2	$11+2i$	5	0	2	$7+6i$	7	0	2	$3+10i$
1	0	3	$2+15i$	3	0	3	$14+11i$	5	0	3	$10+7i$	7	0	3	$6+3i$
1	1	0	$3+6i$	3	1	0	$15+10i$	5	1	0	$11+14i$	7	1	0	$7+2i$
1	1	1	$10+3i$	3	1	1	$6+15i$	5	1	1	$2+11i$	7	1	1	$14+7i$
1	1	2	$13+10i$	3	1	2	$1+6i$	5	1	2	$5+2i$	7	1	2	$9+14i$
1	1	3	$6+13i$	3	1	3	$10+i$	5	1	3	$14+5i$	7	1	3	$2+9i$
1	2	0	$9+2i$	3	2	0	$13+14i$	5	2	0	$1+10i$	7	2	0	$5+6i$
1	2	1	$14+9i$	3	2	1	$2+13i$	5	2	1	$6+i$	7	2	1	$10+5i$
1	2	2	$7+14i$	3	2	2	$3+2i$	5	2	2	$15+6i$	7	2	2	$11+10i$
1	2	3	$2+7i$	3	2	3	$14+3i$	5	2	3	$10+15i$	7	2	3	$6+11i$
1	3	0	$11+6i$	3	3	0	$7+10i$	5	3	0	$3+14i$	7	3	0	$15+2i$
1	3	1	$10+11i$	3	3	1	$6+7i$	5	3	1	$2+3i$	7	3	1	$14+15i$
1	3	2	$5+10i$	3	3	2	$9+6i$	5	3	2	$13+2i$	7	3	2	$1+14i$
1	3	3	$6+5i$	3	3	3	$10+9i$	5	3	3	$14+13i$	7	3	3	$2+i$
(8)	(4)	(4)		(8)	(4)	(4)		(8)	(4)	(4)		(8)	(4)	(4)	

Mod $(1 + i)^8$.

i	0	0	1	$4+i$	6	1	3	$8+i$	4	2	1	$12+i$	2	3	3
$3i$	0	1	1	$4+3i$	2	0	3	$8+3i$	4	3	1	$12+3i$	6	2	3
$5i$	0	3	3	$4+5i$	6	0	1	$8+5i$	4	1	3	$12+5i$	2	2	1
$7i$	0	2	3	$4+7i$	2	1	1	$8+7i$	4	0	3	$12+7i$	6	3	1
$9i$	0	2	1	$4+9i$	6	3	3	$8+9i$	4	0	1	$12+9i$	2	1	3
$11i$	0	3	1	$4+11i$	2	2	3	$8+11i$	4	1	1	$12+11i$	6	0	3
$13i$	0	1	3	$4+13i$	6	2	1	$8+13i$	4	3	3	$12+13i$	2	0	1
$15i$	0	0	3	$4+15i$	2	3	1	$8+15i$	4	2	3	$12+15i$	6	1	1
1	0	0	0	5	0	3	2	9	0	2	0	13	0	1	2
$1+2i$	1	0	0	$5+2i$	5	1	2	$9+2i$	1	2	0	$13+2i$	5	3	2
$1+4i$	2	3	2	$5+4i$	2	2	0	$9+4i$	2	1	2	$13+4i$	2	0	0
$1+6i$	3	1	2	$5+6i$	7	2	0	$9+6i$	3	3	2	$13+6i$	7	0	0
$1+8i$	4	2	0	$5+8i$	4	1	2	$9+8i$	4	0	0	$13+8i$	4	3	2
$1+10i$	5	2	0	$5+10i$	1	3	2	$9+10i$	5	0	0	$13+10i$	1	1	2
$1+12i$	6	1	2	$5+12i$	6	0	0	$9+12i$	6	3	2	$13+12i$	6	2	0
$1+14i$	7	3	2	$5+14i$	3	0	0	$9+14i$	7	1	2	$13+14i$	3	2	0
$2+i$	7	3	3	$6+i$	5	2	1	$10+i$	3	1	3	$14+i$	1	0	1
$2+3i$	5	3	1	$6+3i$	7	0	3	$10+3i$	1	1	1	$14+3i$	3	2	3
$2+5i$	3	0	1	$6+5i$	1	3	3	$10+5i$	7	2	1	$14+5i$	5	1	3
$2+7i$	1	2	3	$6+7i$	3	3	1	$10+7i$	5	0	3	$14+7i$	7	1	1
$2+9i$	7	1	3	$6+9i$	5	0	1	$10+9i$	3	3	3	$14+9i$	1	2	1
$2+11i$	5	1	1	$6+11i$	7	2	3	$10+11i$	1	3	1	$14+11i$	3	0	3
$2+13i$	3	2	1	$6+13i$	1	1	3	$10+13i$	7	0	1	$14+13i$	5	3	3
$2+15i$	1	0	3	$6+15i$	3	1	1	$10+15i$	5	2	3	$14+15i$	7	3	1
3	0	1	0	7	0	2	2	11	0	3	0	15	0	0	2
$3+2i$	3	2	2	$7+2i$	7	1	0	$11+2i$	3	0	2	$15+2i$	7	3	0
$3+4i$	6	2	2	$7+4i$	6	3	0	$11+4i$	6	0	2	$15+4i$	6	1	0
$3+6i$	1	1	0	$7+6i$	5	0	2	$11+6i$	1	3	0	$15+6i$	5	2	2
$3+8i$	4	3	0	$7+8i$	4	0	2	$11+8i$	4	1	0	$15+8i$	4	2	2
$3+10i$	7	0	2	$7+10i$	3	3	0	$11+10i$	7	2	2	$15+10i$	3	1	0
$3+12i$	2	0	2	$7+12i$	2	1	0	$11+12i$	2	2	2	$15+12i$	2	3	0
$3+14i$	5	3	0	$7+14i$	1	2	2	$11+14i$	5	1	0	$15+14i$	1	0	2
	(8)	(4)	(4)		(8)	(4)	(4)		(8)	(4)	(4)		(8)	(4)	(4)

V. *Ionic Velocities.*

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THE most obvious and regular result of passing a current of electricity through the solution of a salt is the simultaneous appearance at the two electrodes of products of chemical decomposition. Since the intervening solution is unaltered, and the products appear only at the electrodes, the motion of the parts of the salt in opposite directions through the solution is a necessary hypothesis, and the idea of ionic velocity is at once suggested.

The facts of ordinary chemical action, as well as those of electrolysis, require us to suppose that a continual molecular interchange of partners is going on, but how this takes place, whether by means of "free ions" or by means of the fractional number of collisions which the mechanical theory teaches us will occur with sufficient violence to bring about separation and possible rearrangement, remains an open question, and one which it is not necessary to consider for our present purpose. The point is that such a motion in opposite directions must occur, and the ions move with a definite average velocity through any given solution. This is independent of any particular hypothesis as to the nature of the active electrolytic molecule, and need not commit us either to the "dissociation" or to the "chemical" theory.

In order to explain the facts of "migration" HITTORF ('Pogg. Ann.,' vols. 89, 98, 103, 106, 1853 to 1859) supposed that the velocities of opposite ions were different, which would produce the observed alterations of concentration round the electrodes according to the obvious and well-known law.

Later, KOHLRAUSCH introduced the idea of a definite specific ionic velocity for each ion, independent of the nature of its combination ('Wied. Ann.,' vols. 6 and 26), except in so far as its motion is affected by the different resistance offered by the different media through which it travels. These differences will disappear at infinite dilution, and the values obtained by extrapolation then show that the specific ionic velocities of the same ion, as deduced from observations on various salts containing it, are, as nearly as can be observed, identical. As it is in examination of this idea that most of the experiments, presently to be described, were made, it will be convenient to reproduce KOHLRAUSCH's theory in some detail. The experimental portion of his work consisted of a splendid series of determinations of the conductivities of salt solutions. Assuming that all the salt molecules dissolved are actively concerned in conveying the current, from simple notions of convection combined with **Ohm**'s law he calculated the relative velocity of the two ions which would be necessary to give the observed conductivity.

Thus (following Dr. **Lodge**'s abstract in the 'B.A. Report,' 1886), let n^3 be the number of active molecules in 1 cub. centim., q the total charge of electricity of whichever kind possessed by the ions of each active molecule, and U the relative velocity with which the opposite ions are sheared past each other by a potential gradient, dv/dx; we shall then get from considerations of convection $n^3q\text{U}$ for the intensity of current (or quantity of electricity conveyed per second through unit area normal to the flow), and from **Ohm**'s law $k(d\text{V}/dx)$ for the same quantity, where k is the conductivity of the unit cube, therefore

$$k\frac{d\text{V}}{dx} = n^3q\text{U}.$$

If N be the number of monad gramme equivalents of the active substance present in the unit cube, and η the electro-chemical equivalent of hydrogen

$$\text{N}\frac{1}{\eta} = n^3q;$$

therefore

$$\gamma = k\frac{d\text{V}}{dx} = n^3q\text{U} = \frac{\text{NU}}{\eta};$$

therefore

$$\text{U} = \frac{k}{\text{N}}\eta\frac{d\text{V}}{dx} = \frac{\eta}{\text{N}}\gamma.$$

Whence the arithmetical sum of the opposite velocities of anion and kation (u and v) when urged by a slope of potential of one volt per centimetre through a solution containing N monad gramme equivalents of active substance per cubic centimetre, and of specific conductivity k, is

$$\text{U} = u + v = k/\text{N} \times \cdot 00010352 \times 10^8 = 10352k/\text{N} \text{ centims. per second.}$$

In order to apply this to particular cases **Kohlrausch** assumes that all the dissolved salt is active, so that N is known from the strength of the solutions. The ratio u/v is taken from **Hittorf**'s migration experiments, and thus the absolute values of u and v can be obtained.

Kohlrausch expresses his experimental results in terms of the specific molecular conductivity (*i.e.*, k/N), and finds that for increasing dilution this tends to a limiting value. Thus, by drawing curves with $\text{N}^{\frac{1}{3}}$ (the reciprocal of the average distance between the molecules) as abscissæ, and k/N as ordinates, he is able to obtain the specific molecular conductivity for infinite dilution. When the values for u and v are worked out for these ideal solutions, it is found that the velocity of any given ion is independent of the nature of the salt of which it forms part, and hence KOHLRAUSCH assigns numbers to various ions which represent their specific ionic velocity.

From a table of these numbers we can at once deduce the conductivity of any given salt solution, and the agreement of the values so obtained with those experimentally observed, furnishes the first confirmation of KOHLRAUSCH's theory. He calculates the conductivity of various salt solutions of strength 0·1 gramme equivalent per litre, and then experimentally determines it for solutions of that strength. Some of his results are as follows :—

Name of salt.	Molecular conductivity calculated from the ionic velocities.	Molecular conductivity observed.
KCl	1060	1047
NH_4Cl	1040	1035
$NaCl$	860	865
$\frac{1}{2}BaCl_2$	840	861
$\frac{1}{2}ZnCl_2$	780	768
KI	1070	1069
KNO_3	1000	983
$AgNO_3$	900	886
$KC_2H_3O_2$	780	784
HCl	3260	3244
KOH	1950	1986
$NaOH$	1750	1700

The tabulated numbers are $k/m \times 10^8$ where k is the conductivity in terms of that of mercury, and m is the number of gramme equivalents of salt per litre. The general agreement is very remarkable, though several exceptions show that the law is not of universal application ; for instance, acetic acid should give 2980, while the observed number is 43.

The next confirmation of the theory was given by Dr. OLIVER LODGE ('B. A. Report,' 1886), who actually observed the velocity of the hydrogen ion as it travelled along a glass tube and discolourized a solution of phenol-phthallein as it went. The tube contained a jelly in which was dissolved a little phenol-phthallein and common salt, with just enough soda to bring out the colour. The tube joined two vessels filled with sulphuric acid, and when a current was passed between them the hydrogen travelled with it, and liberated HCl as it went along the tube. This decolourized the phenol-phthallein, and could thus be traced. The velocity per unit potential gradient came out as ·0029, ·0026, and ·0024 centim. per second, while KOHLRAUSCH's theoretical number was ·0030.

This agreement is striking evidence in favour of KOHLRAUSCH's theory and leads us carefully to consider the data on which it is based. The only objection which can be raised is to the assumption that all the molecules are active, and on this the accuracy of the results deduced apparently depends, for if we suppose only a part of the salt to be active, it seems necessary to allow a greater ionic velocity *in order that the same current may be carried.* Now the work of ARRHENIUS, VAN'T HOFF, OSTWALD, and others, gives very strong evidence that only a certain fraction of the number of dissolved molecules are active (whatever we may assign as the cause of this activity), and it seems to have been generally supposed that this was quite inconsistent with KOHLRAUSCH's hypothesis (see LODGE, 'B. A. Report,' 1886, p. 391, *et seq.*). I have shown, however, that by examining the matter a little more closely the two ideas can be reconciled. The proof has already been published ('Phil. Mag.,' July, 1891), but it will be convenient to reproduce it here.

Suppose that the ratio of the number of the active to the whole number of molecules, which measures the "ionization" (to use Professor FITZGERALD's convenient name), represents in reality the fraction of each second during which each molecule is on the average active. Each molecule is in turn active, but *at any instant* only a certain fraction of the whole number of molecules are active. (In terms of the dissociation hypothesis this ratio measures the "mean free time" of each ion.) As far as statical effects, such as osmotic pressure are concerned, this is, of course, equivalent to supposing a certain fixed fraction of the whole number of molecules to be active, but when we consider the velocities of the ions the case is different.

KOHLRAUSCH gets the relative veloctity of the ions from the relation

$$U_1 = u + v = k/m\,;$$

but if we suppose that only $1/n^{\text{th}}$ of the molecules are active, we should apparently have to put $U_2 = kn/m$ to get the same current through the solution, which would give $U_2 = nU_1$.

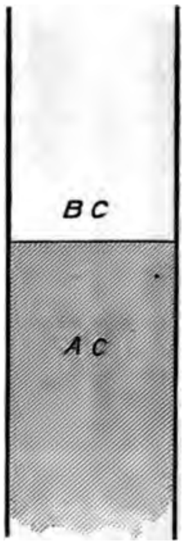
But this U_2 represents the actual velocity of the ions while they are active (or "free"), and if we take a dynamical view of the ionization equilibrium, they are active only for $1/n^{\text{th}}$ part of their time. While inactive (or "combined") they have no relative velocity, and so their average velocity for any *long* time is $1/n^{\text{th}}\ U_2 = U_1$—the same as in KOHLRAUSCH's hypothesis. The view that all the salt is active supposes a uniform ionic velocity—the view above detailed supposes a series of rests interposed with a series of intervals during which the ion is moving forward with a velocity which is, while it lasts, on the average $= nU_1$; but the final result is the same, the *effective* velocity is U_1.

We can, therefore, combine KOHLRAUSCH's theory with the supposition that some of the molecules present in the solution are inactive, the result being that the presence of the inactive or non-electrolytic molecules, which decreases the molecular conductivity of the solution, shows itself by diminishing the effective velocities of the ions.

The agreement between Dr. LODGE's observed value for the velocity of hydrogen and that deduced by KOHLRAUSCH is too close to be accidental, but some further experiments in which the velocities of various ions were determined by watching the formation of precipitates as they travelled along did not give such good results. Thus, Dr. LODGE found Ba = ·00012, Sr = ·00015, and Cl = Br = I = ·00024 centims. per second, while KOHLRAUSCH gives Ba = ·00033, Cl = ·00053, I = ·00060 ('Wied. Ann.,' vol. 6, p. 206).

There are several disadvantages in this method. The greatest appears to be the impossibility of keeping the solutions of a definite strength—the mere fact of producing a precipitate must withdraw a certain amount of salt from the sphere of action, and so reduce the strength of the solution. This will increase the resistance of that portion of the tube in which the precipitate has been formed, and the potential gradient will cease to be uniform and calculable. When jelly is used, the experiments are easy to make, but it has a considerable conductivity of its own, and its great viscosity must offer a high resistance to the passage of the ions through it. Moreover, it gradually exudes from the tube in a direction opposite to that of the current —perhaps owing to electric endosmose. If ordinary liquid solutions are employed, the precipitates cause serious mechanical disturbances which prevent good results, while the use of indicators restricts the experiments to the cases of acids and alkalis and introduces a foreign substance which may itself serve to convey some of the current.

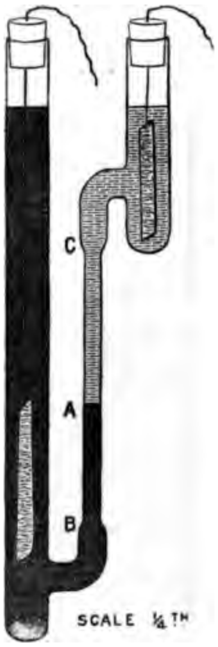
Fig. 1.



In order to get over these difficulties, and to submit KOHLRAUSCH's important and beautiful theory to adequate experimental tests, I undertook a series of experiments in which the substances used acted as their own indicators. The solutions can then be kept at a definite strength, no third substance need be introduced, no precipitate need disturb the uniform course of the ions, and no jelly bring in unknown and irregular conditions.

Consider the boundary of two salt solutions of slightly different density, which have one ion in common, but are of different colours (fig. 1). Let us denote the salts by AC and BC. When a current passes across the boundary there will be a transference of C ions in one direction (*against* the current, if we suppose the C's to be acid radicles, and, therefore, anions), and of A and B ions in the other (in the case supposed, *with* the current). Since the colour in this case depends on the kation A or B, the colour boundary will move with the current, and its velocity will indicate that of the ion causing the change in colour.

Fig. 2.



Postponing for the present the consideration of certain complications which may arise, I will describe the method of working and some preliminary observations. As small quantities of gas may be evolved at the electrodes they could not be sealed up, and finally a very simple apparatus was found to be the best (fig. 2). Two vertical glass tubes about 2 centims. in diameter were joined by a third, considerably narrower, which was bent parallel to the others for the greater part of its length. The longer tube was filled with the denser solution to about the level A, and then the lighter solution was run into the other tube from a burette, till it just began to trickle over the shoulder and run down to A. The proper solutions were then run into the two limbs at rates just sufficient to keep the junction near A; after some practice this could be done quite well, and the tubes filled without much disturbing the junction.

The current was passed from platinum electrodes, which could be connected with a battery of twenty-six accumulators by means of platinum wires passing through the corks. The corks fitted loosely, and a vertical groove, filed in each, allowed any gas which might be evolved to escape. The junction tube had a length, from B to C, of 12·6 centims., and a diameter of about 1·2 centims. If we add one-half the diameter as a correction for each end we get 13·8 centims. as the effective length of the tube, which must be divided into the total difference of potential between the electrodes to give the potential gradient.

The advantages of such an apparatus are shortly as follows :—

1. Any gas evolved at once escapes without interfering with the experiment.

2. Since the liberation of ions and decomposition of salt occur only at the electrodes, it is (in general) only near them that alterations of concentration occur. Since they are surrounded by a large mass of liquid, these alterations will not appreciably change the resistance of the tube as a whole, and will be entirely without effect on the resistance of the column of liquid in the narrow tube, which is the important point.

3. Changes in density at the electrodes will not appreciably accelerate or retard the velocity of the junction. Take the case of copper chloride when the current is passing down as a fairly typical example. The average current was $\frac{1}{2550}$ in C.G.S. units. The quantity of electricity which passed in one hour was, therefore, $\frac{60 \times 60}{2550}$, and the weight of copper deposited,

$$31{\cdot}5 \times \frac{60 \times 60}{2550} \times 10^{-4} = {\cdot}0045 \text{ grm.}$$

Supposing that all this copper comes from the kathode vessel, and that when deposited it occupies no volume (both of which suppositions would increase the disturbing effect we are considering), half this weight of solution would pass from anode to kathode to bring the liquids in the two tubes to their original levels. The area of cross-section of the tube is 0·430 sq. centim. (see p. 347). Therefore the passage of ·0023 cub. centim. of solution from anode to kathode in one hour would mean a velocity of ·0054 centim. per hour. The actual velocity observed in this case, due to the ionic velocities, was 1·6 centims. per hour. The correction for differences in density is thus utterly negligible.

4. A more important objection was pointed out by Professor J. J. THOMSON. With salt solutions of equivalent strength there is, in general, a difference of specific resistance, although this is small in most cases. At the junction there will thus be a discontinuity of potential gradient, and, therefore, a distribution of electricity, which will be acted on by the electric force and introduce disturbances at the boundary. By a little consideration we shall see that this is a non-reversible effect, and should be eliminated by reversing the current and taking the mean of observations in each direction. Suppose that the current is passing in the direction which gives a distribution of positive electricity over the boundary. This will tend to make the

boundary move under the influence of the electric forces in the same direction as the current. Suppose the current reversed. It will now cause a distribution of negative electricity over the boundary, which will tend to move it in the direction opposite to that of the current. But the current has now also been reversed, and, therefore, the boundary tends to move in the same direction relatively to the tube as at first. This effect is therefore eliminated by reversing the current, and when the velocity is observed to be the same in each direction, within the limits of experimental error, it shows that the effect we are considering is negligible.

The first solutions used were those of copper and ammonium chlorides, with just enough ammonia added to each to bring out the deep blue colour of the copper. Their strength was about 0·18 grm. equivalent per litre.

The current was first sent upwards from the copper to the ammonium solution—the junction travelled upwards, with the current. The following readings of its position were made by a kathetometer:—

Voltmeter at grade 23·8.	Time.	Position.	
8·8	12.45	27·20	= 3·62 in 90′ = ·0402 in 1′
	2.15	23·58	
8·8	3.55	19·49	= 4·09 in 100′ = ·0409 in 1′

The current was then reversed—the junction travelled downwards.

Voltmeter at grade 23·8.	Time.	Position.	
7·5	12.30	20·74	= 6·09 in 160′ = ·0381 in 1′
7·7	3.10	26·83	

If we calculate this for the same E.M.F. as before we get ·0441 centim. per minute for the velocity coming down. The mean is ·0423 centim. per minute.

The E.M.F. as measured by a voltmeter comes out $\frac{88}{23{\cdot}8} \times 10{\cdot}2 = 37{\cdot}7$ volts, which makes the potential gradient 2·73.

The velocity of the copper ion, under a potential gradient of 1 volt per centimetre, and through a solution of strength 0·18 grm. equivalent per litre, at a temperature of 15° C., comes out

0·00026 centim. a second.

The value deduced by KOHLRAUSCH from theory for a solution of infinite dilution at 18° C. is ('Wied. Ann.,' vol. 6, p. 206)

0·00031 centim. a second.

The next solutions were chosen to show the motion of the acid radicles, and potassium chloride and potassium permanganate were taken. The junction in this case, depending on the acid radicles, should, if its motion really shows the course of the ions, travel against the current. Such was found to be the case. The solutions had a strength of about ·046 grm. equivalent per litre.

Voltmeter at grade 23·8.	Time.	Position.	
10·8	4.11	21·42	·93 in 10′ = ·093 in 1′
	4.21	20·35	·62 in 5′ = ·124 in 1′
	4.26	19·73	
10·8	4.40	20·89	·55 in 5′ = ·110 in 1′
	4.45	20·34	·57 in 5′ = ·114 in 1′
	4.50	19·77	

A downward current from chloride to permanganate thus gives an upward velocity to the junction of 0·110 centim. per minute.

CURRENT reversed.

Voltmeter at grade 23·8.	Time.	Position.	
10·8	4.28	19·72	·61 in 5′ = ·122 in 1′
	4.33	20·33	·58 in 5′ = ·116 in 1′
	4.38	20·91	

The downward velocity of the junction is thus 0·119 centim. per minute.

The mean of the two is therefore 0·115 centim. per minute.

The potential gradient is 3·36, which makes the velocity for unit gradient

0·00057 centim. per second.

As the red permanganate disappears its place must be taken by chlorine, and hence this ought to give an indication of the velocity of the chlorine ion. For this KOHLRAUSCH gives

0·00053 centim. per second.

Solutions of one-tenth this strength were then set up (·0046 grm. equivalent per litre).

	Centim. per minute.
Average upward velocity	= ·1195
,, downward velocity	= ·1205
Mean	= ·120

Voltmeter, 10·9.

Therefore specific ionic velocity

$$= 0·00059 \text{ centim. per second,}$$

which is appreciably greater than for the stronger solutions.

The success of these preliminary determinations and the closeness of their agreement with theory led me to consider whether any improvement in the method was possible, by which further and more accurate results could be obtained.

In order to eliminate entirely the effect of the discontinuity of potential gradient it is necessary to use pairs of salts which have the same specific resistance for the same strength of solution. This very much limits one's choice, but an example will be found below. In such cases as that of copper and ammonium chlorides where the velocities in opposite directions are sensibly the same, it is fair to conclude that this effect is negligible, but it was worth while to find one case at any rate where it could not in the least affect the result.

The next improvement was in devising a means of getting over the necessity of directly estimating the potential gradient.

Consider the kation of one salt AC at the junction. Let its specific ionic velocity be v_1, and its actual velocity v, under a slope of potential $= d\mathrm{V}/dx$. Let r be the specific resistance of the solution. Let γ be the total current passing, and R the total resistance of the column of liquid from the anode to the junction.

Then we have $v = v_1\, d\mathrm{V}/dx$.

Since γ is constant along the tube we get by OHM's law

$$v = v_1 \gamma \frac{d\mathrm{R}}{dx}.$$

Now since the tube is of uniform bore near the junction $d\mathrm{R}/dx =$ the increase of resistance which would occur if we supposed the column of liquid from the electrode to the junction, whose resistance we have called R, to be increased in length by unity.

This increase of resistance $= r/\mathrm{A}$, where A represents the area of cross-section of the tube at the junction.

Therefore

$$\frac{d\mathrm{R}}{dx} = \frac{r}{\mathrm{A}},$$

and we get

$$v = v_1 \gamma\, r/\mathrm{A},$$

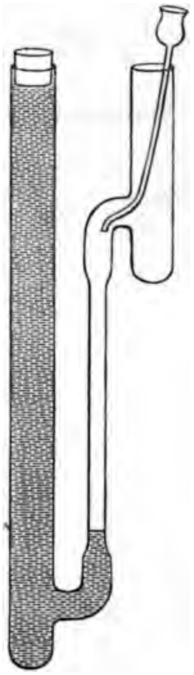
or

$$v_1 = v\mathrm{A}/\gamma r.$$

Now in this equation A can be determined once for all, γ can be read off in a graduated galvanometer of proper sensitiveness, r can be measured by separate experiments for the solution in question, and v, the velocity of the junction, is directly observed. Hence v_1, the specific ionic velocity, can be deduced without estimating the potential gradient directly, and introducing all the consequent uncertainties.

The method was first employed for the case of copper and ammonium chlorides, where the differences in specific resistance are not great, so that the velocities in opposite directions are approximately equal, and the correction for discontinuity is not important. The area of cross-section of the tube was determined as follows.

Fig. 3.



The longer limb was corked up so as to be air-tight, and filled with water till the surface stood just above the beginning of the narrow part of the junction tube. Its level was read off on the kathetometer. A flask containing water was then weighed, and liquid from it poured into the junction-tube through a weighed thistle-tube, till the surface nearly reached the top of the narrow part. The level was then again read off, giving the length of the column of liquid added, and the volume of water used was estimated by weighing the flask again, a correction for the small quantity retained on the walls of the thistle-tube being obtained by noticing the increase in its weight. Two determinations of the area of cross-sections by this section gave (1) 0·4300 sq. centim. and (2) 0·4297 sq. centim. Mean 0·4299, or correct to three places, **0·430** sq. centim.

The solutions were prepared by weighing out 0·670 grm. of pure cupric chloride and 0·534 grm. of ammonium chloride into two 100 cub. centims. flasks, and dissolving them in about 70 cub. centims. of water. An aqueous solution of ammonia was then added to the copper solution till the deep blue colour was formed. The same volume of ammonia was next run into the ammonium chloride solution, and both were made up to 100 cub. centims. This gave solutions whose strengths were 0 1 monad grm. equivalent per litre. Their specific resistances were found by the method used by KOHLRAUSCH and FITZPATRICK, and came out 157×10^9 (for the $CuCl_2$) and 117×10^9 (for the NH_4Cl) in C.G.S. units for the cubic centimetre.

The solutions were carefully run into the velocity tubes, and the current sent through them and a low-resistance galvanometer in series. The galvanometer had previously been graduated by sending the current from a single freshly-prepared Daniell cell through it and through a known resistance, and noticing the deflections as the resistance was varied from 11,000 ohms to 150 ohms.

GRADUATION of Galvanometer. Zero 1°·9.

Ohms.	Deflection.	Ohms.	Deflection.
11,000	= 5°·8	1,600	= 39°·5
10,000	6·1	1,400	42·5
9,000	6·7	1,200	45·9
8,000	7·4	1,000	49·8
7,000	8·2	900	52·1
6,000	9·8	800	54·5
5,000	11·5	700	57·0
4,000	15·4	600	59·9
3,500	18·7	500	63·2
3,200	21·0	400	67·1
3,000	22·9	350	69·1
2,700	25·6	300	71·7
2,500	28·0	250	74·1
2,200	31·3	200	76·8
2,000	33·7	150	79·4
1,800	36·6		

The resistances were then again increased to 400 ohms = 67°·1, 1200 = 45°·9, and 3500 = 18°·7—numbers identical with those first obtained.

COPPER and Ammonium Chlorides. Velocities. Current upwards.

Galvanometer.	Time.	Position.	
74°·7	10.50	$\left.\begin{array}{l}32\cdot4\\32\cdot2\end{array}\right\}32\ 3$	1·70 centims. per hour.
	4.0	$\left.\begin{array}{l}23\cdot4\\23\cdot5\end{array}\right\}23\cdot5$	

CURRENT reversed at 4.0 downwards.

Galvanometer.	Time.	Position.
74°·9	6.25	26·9 27·2 } 27·0 = 1·45 centims. per hour.
72·8	11.30 12.45	27·2 29·2 } 1·65 centims. per hour.
	1.30 1.45	30·5 30·9 } 1·70 „ „

CURRENT upwards.

Galvanometer.	Time.	Position.
73°·9	2.0 4.15	30·9 27·3 } 1·53 centims. per hour.
73·0	5.0 7.15	26·3 23·0 } 1·43 „ „

Means going up G. = 73°·9 v = 1·57 centims. per hour.
„ „ down 73°·9 1·60 „ „

Means of all observations G. = 73°·9
v = 1·59 centims. per hour.

We get γ from the table of graduations to be $\frac{1}{255}$, and find from our equation (p. 346)

$$v_1 = \frac{vA}{\gamma r} = \frac{1·59 \times ·430 \times 255}{157 \times 10^9 \times 60 \times 60} = 3·09 \times 10^{-12}.$$

This is for unit potential gradient on the C.G.S. system, and gives for a gradient of 1 volt per centim. a velocity of

$3·09 \times 10^{-4}$ = **0·000309** centim. per second.

This is the velocity of the copper ion in a solution of 0·1 grm. equivalent per litre, while KOHLRAUSCH deduced for infinite dilution

0·00031 centim. per second.

The next case I shall describe is that of a pair of salts whose 0·1 grm. equivalent solutions have specific resistances which are nearly the same. Mr. FITZPATRICK kindly supplied me with a long list of conductivities reduced to comparable units, and from those I was able to select two solutions which fulfilled the necessary conditions (viz., (i.) different colours; (ii.) different specific gravities; (iii.) nearly equal specific resistances). Potassium bichromate has a specific molecular conductivity (*i.e.*, k/m) of $9{\cdot}10 \times 10^{-12}$ (LENZ), and potassium carbonate $9{\cdot}34 \times 10^{-12}$ (KOHLRAUSCH), a difference of less than 3 per cent.

I thought it better to use solutions of slightly different strengths and exactly equal conductivities to get rid of even this small variation, and therefore adjusted the strength of the carbonate till its resistance just equalled that of the bichromate. Thus, in the resistance cell which I used, the bichromate gave 1485 ohms and 1490 ohms at $17^\circ{\cdot}4$, and the carbonate 1475 ohms and 1480 ohms at $17^\circ{\cdot}5$.

The first point I investigated with these solutions was the influence of change of potential gradient on the velocity. These should, on KOHLRAUSCH'S theory, obviously be proportional to each other.

When the current was passed downwards, so that the junction travelled up the tube, two distinct surfaces of separation appeared, one between the orange of the strong bichromate and the yellow of that portion of the carbonate solution into which a little bichromate had passed by diffusion, and the other between this and the colourless carbonate. These kept fairly distinct and travelled upwards at equal rates. I append full particulars of the first series of observations.

VELOCITIES.—Potassium Bichromate and Carbonate. Current Downwards, Movement Upwards.

	Galvanometer.	Time.	Position.	Velocity in centims. per hour.
Orange	35·0°	11.45	24·70	
		.46	·65	
		.47	·60	
Yellow	35·1	11.48	23·03	
		.49	22·92	
		.50	·85	
Orange	35·1	12.0	23·80	3·63
		.1	·74	
		.2	·69	
Yellow	35·1	12.3	22·18	3·39
		.4	·08	
		.5	·00	
Orange	35·4	12.15	22.89	3·65
		.16	·83	
		.17	·77	
Yellow	35·6	12.18	21·32	3·24
		.19	·28	
		.20	·23	

Mean upward velocity . . . . = 3·48 centims. per hour.
„ galvanometer reading . . = 35°·3.

In the same way the observations of downward velocity came out 3·28 and 3·55 for galvanometer readings of 35·2 and 35·1, and when new solutions were set up 3·45 for 36·4.

Mean downward velocity . . . = 3·43 centims. per hour.
„ galvanometer reading . . = 35°·6.

The means in each direction are

$$v = 3{\cdot}45 \qquad \mathrm{G} = 35^\circ{\cdot}4,$$

which gives

$$v_1 = \frac{v\mathrm{A}}{\gamma r} = \frac{3{\cdot}45 \times {\cdot}430 \times 129 \times {\cdot}00910}{60 \times 60}$$

$$= \mathbf{0{\cdot}00048} \text{ centim. per second.}$$

The galvanometer was too sensitive, and had to be shunted and regraduated with a Daniell cell.

Resistance.	Resistance.
70 = 49°·0	140 = 32°·7
75 = 47·7	150 = 31·9
80 = 46·1	160 = 30·0
85 = 44·9	180 = 26·9
90 = 43·7	200 = 23·8
95 = 42·3	230 = 20·2
100 = 41·1	260 = 17·3
110 = 38·9	300 = 14·1
120 = 36·9	320 = 12·7
130 = 35·1	350 = 11·4

Resistance of battery + galvanometer + wires = 1 ohm.

An E.M.F. of about one-third that used above was then applied with the following results :—

CURRENT upwards. Movement downwards.

Galvanometer.	Time.	Position.	Galvanometer.	Time.	Position.	Result.	Centims. per hour.
12°·5	12.30	23·65	12°·5	12.45	24·04	·39	1·48
	.31	·67		.46	·04	·37	
	.32	·70		.47	·05	·35	
12·7	1.4	24·92	12·7	1.19	25·27	·35	1·39
	.5	·95		.20	·30	·35	
	.6	·98		.21	·32	·34	

CURRENT Downwards. Movement Upwards.

Galvanometer.	Time.	Position.	Galvanometer.	Time.	Position.	Result.	Centims. per hour.
°			°				
14·0	11.40	24·14	14·0	11.55	23·81	·33	1·28
	.41	·10		.56	·78	·32	
	.42	·07		.57	·76	·31	
13·8	3.25	24·14	13·7	3.55	23·46	·68	1·37
	.26	·11		.56	·43	·68	
	.27	·10		.57	·41	·69	
13·6	4.20	22·04	13·1	4.50	21·44	·60	1·23
	.21	·02		.51	·39	·63	
	.22	·00		.52	·38	·62	

Means—Downward velocity . . . . = 1·44 centims. per hour.
G. . . . . = 12°·6
Upward velocity . . . . . = 1·29 centims. per hour.
G. . . . . = 13°·7

This gives

$$v = 1{\cdot}37 \qquad G = 13^\circ{\cdot}2,$$

and

$$v_1 = vA/\gamma r = \mathbf{0{\cdot}00047} \text{ centim. per second.}$$

Thus the value obtained for the specific ionic velocity is independent of the E.M.F. applied, *i.e.*, *the velocity of the ions is proportional to the potential gradient.*

New solutions were afterwards set up and the experiments with the large E.M.F. repeated. The results were—

Movement upwards:

$$v = 4{\cdot}98,\ 4{\cdot}84,\ 4{\cdot}62,\ 4{\cdot}66,\ 4{\cdot}62.$$

$$G = 43{\cdot}8,\ 43{\cdot}8,\ 43{\cdot}0,\ 43{\cdot}2,\ 43{\cdot}2.$$

Means:

$$v = 4{\cdot}74 \qquad G = 43{\cdot}4 \qquad v_1 = {\cdot}00044.$$

Movement downwards:

$$v = 5{\cdot}02,\ 5{\cdot}16,\ 4{\cdot}78,\ 4{\cdot}70,\ 4{\cdot}66.$$

$$G = 42{\cdot}5,\ 43{\cdot}4,\ 43{\cdot}0,\ 43{\cdot}0,\ 42{\cdot}8.$$

Means:

$$v = 4{\cdot}86, \qquad G = 42^\circ{\cdot}0, \quad \text{therefore, } v_1 = {\cdot}00049.$$

Mean of v_1 in both directions = **0·00046.**

Taking the mean value deduced from all three series of separate experiments, we get

$$v_1 = 0{\cdot}00047 \text{ centim. per second,}$$

as the specific ionic velocity of the bichromate group.

This is not given by **Kohlrausch**, but we can calculate it by his method from known observations. Thus, **Lenz** has found the specific molecular conductivity of the solution to be $9{\cdot}10 \times 10^{-12}$, and the migration constant was determined by **Hittorf**, and came out ·502.

By **Kohlrausch**'s theory (see p. 338),

$$\begin{aligned} \text{U} = u + v &= 10352 \frac{k}{\text{N}} \\ &= 10352 \times 9{\cdot}10 \times 10^{-8} = {\cdot}000942, \end{aligned}$$

therefore,

$$\begin{aligned} v &= {\cdot}000942 \times {\cdot}502 \\ &= 0{\cdot}000473. \end{aligned}$$

This close agreement between theory and observation led me to examine a case in which the specific resistances of the solutions were not the same, in order to see whether the method could be used when no solutions could be found with the exact resistance needed. I used potassium bichromate and potassium chloride, whose conductivities are $9{\cdot}10 \times 10^{-12}$ and $11{\cdot}13 \times 10^{-12}$ respectively. If the extension of method is practicable, these solutions should give the same velocity for the bichromate group as the former pair—bichromate and carbonate. The galvanometer was moved, and, therefore, had to be regraduated.

Resistance.	Resistance.
60 = 54°·0	90 = 45°·4
65 = 52·4	100 = 43·0
70 = 51·0	110 = 40·7
75 = 49·4	120 = 38·6
80 = 48·0	130 = 36·5
85 = 46·7	

The following results were obtained—

Current Downwards. Motion Upwards.

$v =$ 4·04 4·18 4·18 4·22. Mean, 4·15.

$\text{G} =$ 40·0 40·0 40·2 40·2. „ 40°·1.

Therefore

$$v_1 = 0{\cdot}000516.$$

Current Upwards. Motion Downwards.

$v =$ 5·19 5·18. Mean, 5·19.

$\text{G} =$ 50·5 49·2. „ 49°·9.

$$v_1 = 0{\cdot}000394.$$

Thus, the effect of the discontinuity of potential gradient is to increase the velocity in one direction and to decrease it in the other, as it should do (see p. 343). The mean of the two numbers comes out

$$v_1 = 0{\cdot}000455 \text{ centim. a second,}$$

a value which, though it is of the same order of magnitude as the one obtained from the other pair of solutions, is appreciably less. A second series of observations gave

0·000483 and 0·000402. Mean, 0·000443.

These results show that the mean value of the velocities in opposite directions gives a number which is nearly, but not quite, the same as that obtained from a pair of solutions whose specific resistances are equal to each other. We may, therefore, use solutions whose resistances are not identical, to give, at any rate, some indication of the value of the specific ionic velocity, provided the differences are not great. But this extension of the method must be used with caution.

While working with solutions of different resistances it was often observed that when travelling in one direction the boundary got vague and uncertain, and when travelling in the other hard and sharp. Similar phenomena at the junction of liquids through which a current is passing have been previously described (see GORE, 'Roy. Soc. Proc.,' 1880 and 1881). Many of them can be explained as follows:—

Suppose that the coloured solution has greater resistance than the other, and that the junction is travelling from the coloured to the colourless solution. Any wandering ion which happens to be in advance of the main body, finds itself in a region where the potential gradient is less. It is therefore gradually overtaken, and the boundary becomes sharp. When the current is reversed, so that the junction travels in the opposite direction, any straggling ion, which lags behind the retreating column and so gets into the region of smaller potential gradient, finds itself left further and further behind, while others are continually falling out of the ranks. In this way the boundary becomes vague. If the coloured solution has less resistance than the other, a solitary ion is acted on by a *greater* potential gradient, and the order of these phenomena is reversed.

As far as I am aware, no attempt has hitherto been made to apply KOHLRAUSCH's theory to the case of solutions of salts in solvents other than water. The conductivity of alcoholic solutions is much less than that of the corresponding aqueous ones, and the question whether KOHLRAUSCH's theory still held good seemed of great interest.

The method described above can easily be applied, but the comparison of the results with theory offered some difficulty, as no data for the migration constants are available. In order to prevent the necessity of experimentally determining these constants, I thought it better to observe the velocities of both ions of a salt and compare their sum with the value deduced from the conductivity. The salts used were (i.) the chloride and nitrate of cobalt (the colours of which in alcoholic solutions are blue and red respectively) giving the velocity of the acid radicles, and (ii.) cobalt and calcium chlorides (blue and colourless) and cobalt and calcium nitrates (red and colourless), giving the velocity of the cobalt ion.

It was found that the observed velocities were very small, and in order to increase them a shorter tube was used of rather greater cross-section. This enabled a greater potential gradient to be applied. The area of cross-section was determined as before and came out 0·746 sq. centim.

A series of solutions of cobalt chloride and cobalt nitrate of different strengths was made up and the specific resistances determined. They were then placed in pairs of equivalent strength in the velocity apparatus, and a long investigation made on their behaviour. It was found that with weak solutions, whose strength was below 0·08 grm. equivalent per litre, the phenomena were quite regular. The junction travelled against the current, as it ought, since the change of colour depends on the acid radicles. As the strength of solution increases new phenomena appear. When the concentration reaches 0·1 grm. equivalent per litre the junction sometimes divides into two parts, which often travel in opposite directions, producing a broad purple band between the red and the blue solutions.

The junction sometimes seems to travel entirely the wrong way, but there is always either a fainter band which goes the right way, or, if this cannot be seen, the colour of the solution in that direction gradually changes. These phenomena more and more disturb the normal course of the experiment as the concentration of the solutions becomes greater, and with the strongest solutions which can conveniently be used (0·15 grm. equivalent) are very marked indeed.

In order to examine the applicability of the theory to these cases, it is necessary to choose solutions so dilute that the motion of the ions is normal, but still strong enough to show their colours plainly, and give an appreciable difference of density. The best solutions to use were found to be those whose strength was about 0·05 grm. equivalent per litre.

Solutions of cobalt nitrate and cobalt chloride in alcohol of ·05 strength were, therefore, set up in the velocity tube, and the following results were obtained :—

CURRENT from red to blue. Temperature, 17°·8.

$v =$	·39	·35	·43	·43.	Mean,	·40 centim. per hour.
$G =$	49·5	49·4	48·8	48·5.	„	49°·1

CURRENT from blue to red.

$v =$	·42	·41.	Mean,	·42 centim. per hour.
$G =$	49·1	48·6.	„	48°·9

CURRENT from red to blue.

$v =$ ·50 ·47 ·41 ·40. Mean, ·44 centim. per hour.
$G =$ 49·8 49·8. „ 49°·8

CURRENT from red to blue.

$v =$ ·41 ·45 ·48 ·47 ·34 ·43. Mean, ·43 centim. per hour.
$G =$ 49·6 49·1 48·8 49·0 48·7. „ 49°·0

Thus the velocity is practically the same in both directions, and the mean values are $v = 0{\cdot}42$ centim. per hour and $G = 49^{\circ}{\cdot}2$.

The conductivity of the chloride was found by FITZPATRICK's method to be $2{\cdot}86 \times 10^{-13}$ C.G.S. units, which gives

$$v_1 = vA/\gamma r_1 = 0{\cdot}000026 \text{ centim. per second}$$

for the velocity of chlorine in this solution. We see at once from this result, that conductivity and specific ionic velocity for different solvents are, at any rate approximately, proportional to each other. The conductivity of these alcohol solutions is about one-tenth, or rather less, that of an aqueous solution of a neutral salt of equivalent strength, and we see that the ionic velocity of chlorine is reduced to rather less than one-tenth of its usual value for aqueous solution. This confirms the result at which we have already arrived, viz., that all the molecules are concerned, at any rate in turn, in the process of electrolysis, and that the cause of the increased resistance is a decrease in the *average* ionic velocity.

In order to obtain the velocity of the cobalt ion, solutions of cobalt and calcium chlorides were set up of strength 0·05 grm. equivalent.

CURRENT upwards.

$v =$ ·30 ·26 ·20 ·24. Mean, 0·25.
$G =$ 45·8 45·4 45·6 45·4. „ 45°·6.

CURRENT downwards.

$v =$ ·26 ·24. Mean, 0.25.
$G =$ 46·9 48·0. „ 47°·5.

CURRENT upwards.

$v =$ ·30 ·21 ·26 ·28. Mean, 0·26.
$G =$ 47·9 47·0 „ 47°·4.

(After this, the velocity began to fall off, and the kathode solution ($CaCl_2$) got thick and faintly blue, perhaps indicating that some current is conveyed by the water of crystallisation of the $CoCl_2 6H_2O$, which deposits a hydrate when it meets the calcium, and that some cobalt goes faster than the rest and colours the solution. Finally, the junction went the wrong way, and other secondary actions occurred. All this shows that the effect of too great concentration has not yet been eliminated.)

Taking the mean velocity while it remained constant we get $G = 46^\circ\cdot 8$, $v = 0\cdot 25$ centim. per hour.

Therefore $v_1 = vA/\gamma r = 0\cdot 000022$ centim. per second.

Adding this to the velocity of the chlorine (viz., 0·000026) we get

$$V = u + v = 0\cdot 000048 \text{ centim. per second.}$$

The conductivity is $2\cdot 86 \times 10^{-13}$, and on KOHLRAUSCH's theory this gives

$$V = 10352 \frac{k}{N} = 0\cdot 000060 \text{ centim. per second.}$$

Considering the difficulties of the method and the effect of too great concentration, these numbers must be considered to agree within the limits of experimental error.

The conductivity of the nitrate of cobalt is rather greater than that of the chloride. This means that the number of active molecules is larger, and that the salt more nearly approaches the normal state. We should therefore expect the agreement to be closer than in the case of the chloride.

The velocity of the cobalt was found by using cobalt and calcium nitrate solutions, the first being red, the second colourless.

Upward velocity—Means: $G = 50^\circ\cdot 3$ $v = 0\cdot 44.$
Downward " " $G = 43^\circ\cdot 6$ $v = 0\cdot 41.$
Means of both directions: $G = 46^\circ\cdot 9$ $v = 0\cdot 43.$

No irregularities (such as reversals, &c.) were detected with these salts. This is what would be expected from the better conductivity.

The result is

$$v_1 = vA/\gamma r = 0\cdot 000044 \text{ centim. per second.}$$

The velocity of the NO_3 group can be calculated from the numbers in p. 356. The conductivity is $3\cdot 60 \times 10^{-13}$, and this gives

$$v_1 = 0\cdot 000035 \text{ centim. per second.}$$

Therefore

$$V = u + v = 0\cdot 000079 \text{ centim. per second.}$$

From KOHLRAUSCH's theory we can calculate

$$V = 10352\,\frac{k}{N} = 10352\,\frac{3{\cdot}80 \times 10^{-13}}{{\cdot}05} = 0{\cdot}000079 \text{ centim. per second.}$$

From the close agreement of these numbers, it is, I think, fair to conclude that KOHLRAUSCH's theory holds good in the case of alcoholic solutions, at all events when they are moderately dilute.

The case of the chloride, where the agreement is not so good, seems even more interesting than that of the well-behaved nitrate, and to be worthy of closer examination. The irregularities only appear when the concentration of the solutions is considerable, and we may assume that the agreement with theory would be closer for more dilute solutions if the difficulties of observing them could be surmounted. But how are we to explain the phenomena shown by strong solutions? The velocity of the junction becomes uncertain, several distinct lines often appear, bounding different shades between red and blue, and some of these may travel in the opposite direction to that indicated by theory and actually observed in the case of dilute solutions. On any theory of ionic charges, it seems impossible that an ion, as such, should travel in a direction against the electric forces, and we are forced to the supposition that the part of the salt which we see travelling thus is composed of non-electrolytic molecules, and is in reality attached to the opposite ion to form a complex structure which moves with the electric forces. The only other possible explanation seems to be the supposition that there is a proper motion of the solvent on which the ionic velocities are superposed, and that in some cases the motion of the solvent in one direction is greater than that of the ion in the other. Even neglecting the mechanical disturbances which would then appear, this view is untenable, firstly, because a motion of the solvent is disproved by another series of experiments, which I hope soon to publish, and, secondly, because the reversed motion is observed in strong solutions of both cobalt nitrate-cobalt chloride, and cobalt chloride-calcium chloride. The motion of the boundary in these two cases is in opposite directions with reference to the current, hence we cannot explain matters by a motion of the solvent in one direction.

The hypothesis of complex ions, advanced to explain the abnormal migration constant of cadmium iodide in alcohol must, therefore, be extended to these solutions. The cases are not quite parallel, for the phenomena shown by cadmium iodide can be explained if we imagine molecules of CdI_2 joined to the anion only, while in our case, since both pairs of solutions used show reversed motion when sufficiently strong, we must suppose that molecules of undecomposed salt are connected both to the anion and to the kation.

I hope soon to be able to publish an account of some further experiments bearing on the question of complex ions, and of others connected with several points in the subject of electrolysis.

The following table shows at a glance the results of this paper, and the figures obtained by calculation from KOHLRAUSCH's theory :—

SPECIFIC Ionic Velocities.

I. Aqueous Solutions.

Ion.	Velocity observed.	Velocity calculated from KOHLRAUSCH's theory.
Copper	·00026* ·000309	·00031
Chlorine	·00057* ·00059*	·00063
Bichromate group (Cr_2O_7) . .	·00048 ·00047 ·00046	·000473

II. Alcoholic Solutions.

Salt.	Velocity of anion (observed).	Velocity of kation (observed).	Sum of velocities (observed).	Sum of velocities (calculated).
Cobalt chloride .	·000026	·000022	·000048	·000060
„ **nitrate** .	·000035	·000044	·000079	·000079

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* Preliminary determinations.

VI. *The Value of the Mechanical Equivalent of Heat, deduced from some Experiments performed with the view of establishing the relation between the Electrical and Mechanical Units; together with an Investigation into the Capacity for Heat of Water at different Temperatures.*

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Communicated by R. T. GLAZEBROOK, *F.R.S.*

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[PLATES 2–4.]

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INTRODUCTION.

THE necessity for a re-determination of the value of the mechanical equivalent may not be obvious at first sight. The classic determinations by JOULE have undergone but little alteration at the hands of succeeding observers, and the researches of ROWLAND (1879) into this matter were of such an exhaustive nature that there would appear to be little room left for further investigation. It should, however, be remembered that even JOULE's later determinations differ by as much as 1 part in 100*; and that marvellous as is the agreement, amongst themselves, of the results obtained by ROWLAND, they, since his method of investigation was unaltered throughout, stand in need of confirmation by different methods of observation. Again, ROWLAND, as far as I have been able to ascertain, stands practically alone in his conclusion that the specific heat of water diminishes as the temperature rises from 0° to 30° C. It is difficult to conceive of a more important investigation (for the purposes of accurate physical measurements) than the determination of the capacity for heat of water at different temperatures, and it is to me a matter of extreme surprise that greater efforts have not been made to trace the variation (if any) in its value. The science of calorimetry must be regarded as in its infancy so long as its fundamental unit is a matter of doubt.

Other observers who have attempted to obtain the value of the mechanical equivalent, by means of the work done by an electric current, have been hampered by constant perplexities as to the absolute values of the electrical units adopted. The science of electrical measurements has now arrived at such a stage that its units may be regarded as sufficiently established,† and, therefore, the time seems particularly appropriate for an enquiry into the relation between those units and the mechanical ones.

The difficulties of such an investigation are, of course, great, as is shown by the divergence of the results obtained by those observers who, in recent years, have adopted the electrical method. One cause of inaccuracy has been present in all determinations I have examined, viz., the increase in temperature of the conductor above the temperature of the medium in which it was placed, and the consequent alteration in its resistance. ROWLAND‡ writes as follows :—"There can be no doubt that experiments depending on the heating of a wire give too small a value of the equivalent, seeing that the temperature of the wire during heating must always be higher than that of the water surrounding it, and hence more heat will be generated than there should be."

An account is given in the following pages of the manner in which this difficulty has been overcome, and I think it will be seen that this objection to the electrical method of investigation is now removed.

The difficulties with regard to measurement of temperature are not peculiar to the electrical method of investigation, and, therefore, I need not dwell upon them. I would, however, venture to add my expressions of astonishment to those of ROWLAND that so many enquirers have attached so little importance to this point; many investigators, whose methods have otherwise been of a high order of accuracy, having contented themselves with the mercurial thermometer as a standard. The importance of close attention to the thermometry is shown by the fact that the difference between the air and mercury readings, even at low temperatures, will more than account for a change of decrease into increase in the specific heat of water as the temperature rises. An indirect result of my present investigation is to add further proof (if that is necessary) to the accuracy of the measurements of temperature by platinum thermometers, and it is remarkable that so exact and easy a method is not more generally applied to that most difficult subject, thermometry.

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* See Table XLV., *infra*.

† 'Report Brit. Assoc.,' 1892.

‡ 'Proceedings American Academy,' June, 1879, p. 153.

Another difficulty which meets any enquirer into calorimetric measurements, is the uncertain nature of the thermal unit ordinarily adopted. I take it that the thermal unit is usually defined as the "amount of heat required to raise unit mass of water from 0° to 1° C." The obvious objections to this unit are that no one has more than a vague idea as to its magnitude, and that in choosing such a range of temperature we have selected, without doubt, the most difficult of all temperatures at which to ascertain it. Again, it is not improbable that over such a range water may exhibit greater changes in its properties than it does at other temperatures. The consequence is that investigators, such as ROWLAND, are compelled to express their results in terms of a variable unit, the theoretical one being an unascertained and indefinite quantity. To escape this difficulty I have taken as my standard the quantity of heat required to raise unit mass of water through 1° C. of the air-thermometer at 15° C. (*i.e.*, from 14°·5 to 15°·5 C.); in other words, I assume the specific heat of water at 15° C. to be 1, and I venture to suggest that (for want of a better) the above definition be accepted as that of the thermal unit.*

I have not included in this paper any prolonged analysis of the work of previous observers. ROWLAND has given a most complete summary, with references, of all results anterior to 1880, and the agreement between later observers is not sufficient to warrant any modification of his conclusions.†

The main portion of this communication is devoted to an account of the experiments performed in my laboratory during 1892, and, in order to avoid the introduction of redundant matter into that account, I here give a brief summary of my previous investigations.

I commenced this work in 1887, and the general principle on which I proposed to proceed was that of eliminating the effects of conduction, radiation, &c., rather than of ascertaining the actual loss or gain due to such causes. If a calorimeter is suspended in a chamber whose walls are kept at a constant temperature, it is obvious that if the initial temperature of the calorimeter is below that of the external envelope, and if the calorimeter has its temperature steadily raised by means of a current, a time will come when the loss by radiation, &c., must equal the gain experienced by the calorimeter during the time it was below the temperature of the surrounding space. The temperature at which the effects of radiation, &c., are eliminated, I term the "null point," and an investigation into its position indicates that if t_1 be the time that the calorimeter was beneath the external temperature, and t_2 the time required to raise it from its initial temperature to the same number of degrees above, that it was previously beneath, the temperature of the surrounding envelope, then the calorimeter would arrive at its null point at a time somewhere between $2t_1$ and t_2. In the Appendix will be found an investigation into the exact position of this point.

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* Personally I should have preferred to select a higher temperature, since a temperature above that of our ordinary surroundings is, at all times, more easily obtainable than a lower one; but as 15° C. has become so commonly accepted as a standard of temperature in electrical measurements, I have adopted it as my standard.

† MICULESCU gives all results published before January, 1892. See 'Annales de Chimie et de Physique,' vol. 27, p. 206.

1887–1889.—During these years my whole attention was directed to an effort to ascertain the heat developed by the current while the calorimeter was passing from its initial temperature to this null point. Attractive as the method appeared, I ultimately relinquished it, for I wished to direct my attention as much to the investigation of the changes in the specific heat of water as to the determination of the mechanical equivalent, and, although the method is an admirable one for the latter, it is not so suitable for the purposes of the former enquiry.

My general method of conducting the experiments was the same as that observed in 1892 and described in the succeeding pages. From the outset, I had determined to vary all the conditions as much as possible, believing that it was only by such means that constant sources of error could be detected. The agreement amongst individual experiments taken under the same conditions was, if anything, at times more marked than in the experiments of 1892; nevertheless, when the final reduction of the results took place, fatal discrepancies invariably showed themselves. For example, the water equivalent, as deduced from the experiments, increased when the mass of the water was increased, and experiments conducted with a high electromotive force invariably gave too great a value for the time as compared with that obtained when a lower electromotive force was used. These discrepancies, as it now appears, may be attributed to two causes:—

(1.) To insufficient stirring;
(2.) To the rise in temperature of the wire, previously referred to.

Throughout these experiments I was conscious of the error introduced by ignorance of the actual temperature of the wire, but it was not until the summer of 1892 that a satisfactory method of estimating this difference of temperature was adopted.

Unfortunately, many of my experiments in past years were conducted with wires coated with a thick insulating covering, and it is, therefore, impossible to apply the correction obtained this year to the reduction of the earlier results. The observations were excellent in themselves, and some hundreds of experiments were performed. They all give too high a value of J (almost invariably above $4 \cdot 2 \times 10^7$), and are of no value except for the experience gained by their means and the improvements they suggested in the apparatus.

Again, the thermometry was not of a sufficiently satisfactory nature. The thermometers were standardized by direct comparison with B.A. thermometers calibrated at Kew.* As these B.A. thermometers were only divided into $\frac{1}{10}$ of a degree, the $\frac{1}{100}$ of a degree had to be estimated,† and the order of accuracy was not sufficient.

1890.—Throughout this year I devoted my attention entirely to the measurement of temperatures by platinum thermometers, and I, at that time, proposed to use such thermometers in place of mercurial ones during my J experiments. Excellent, however, as platinum thermometers are, for the accurate determination of temperature, they are not suitable when the observations have to be taken at exact intervals of time. A thermometer, whose readings are at all times visible, must be used in such a case.

1891.—By the kind permission of the Master and Fellows of Sidney College, the Chemical Laboratory was placed at my disposal during this summer. This building was unsuitable for physical work, for the traffic in the adjoining street was so great that galvanometers became at times unmanageable, and many of the observations had to be taken during the night, under conditions somewhat conducive to personal errors.

The steel regulating chamber, described in Section III., was first used in this year (1891), and the results are therefore of more value than the preceding ones; but the same persistent error in the measurement of R still remained, and the form of stirrer adopted was unsatisfactory. The summer's work was, however, of some value, as it gave sufficient data for calibrating the mercury thermometer *when rising*.

A platinum wire coil, having a single coating of amber varnish, was used, and it is reasonable to suppose that its increase of temperature above the surrounding water was of a similar order to that ascertained to be the case in the coil of 1892. Assuming the increase of its resistance to bear the same ratio to its total resistance as in the case of the 1892 coil, and, applying a similar correction to our results, we obtain

$$J = 4{\cdot}188 \times 10^7$$

expressed in terms of water at 18°·88 C, and assuming the coefficient of the change in specific heat of water obtained in 1892, we get

$$4{\cdot}192 \times 10^7‡$$

as the value expressed in terms of the thermal unit as previously defined.

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* The accuracy of the Kew corrections for the inequality of the bore has been previously discussed, see 'B.A. Report,' 1890.

† I find that many observers, by means of such thermometers, determine temperatures (without using a micrometer scale) to $\frac{1}{1000}$ of a degree. I envy, but cannot lay claim to, such powers of observation.

‡ The wires connecting the lid of the calorimeter with the steel chamber (see fig. 2, *infra*) had (in 1891) a slightly higher resistance than those used in 1892. I am unable to make the necessary correction, as I have not sufficient data. The effect of the correction (which would certainly be less than in 5000) would be to slightly increase the value of J here given.

This agrees within 1 in 2000 with our later determinations.*

1892.—The difficulties experienced in obtaining an altogether suitable room for the work compelled me to have a special building erected in which to carry on the investigation.

The apparatus is so involved and complicated that it requires weeks to put it together and to get it into working order, and such spare time as I had in the early part of this year was devoted to these preparations. On previous occasions the constant day and night changes in temperature of the rooms in which experiments were conducted had been a cause of uncertainty; for the numbers, obtained by using resistance boxes or Clark cells whose temperature is constantly changing, are always of doubtful accuracy. I therefore designed a regulator which, except in cases of severe frost or extreme heat, automatically maintained the room at a constant temperature. I found this arrangement so effective that I have, in the hope that it may be useful to others, given a short description of it in Appendix II.

If I endeavoured to adequately describe how much I owe to the assistance of others during this investigation I should unduly increase the length of this introduction. I must, however, express my thanks to the Council of the Royal Society, to Lord KELVIN, Lord RAYLEIGH, Professor G. DARWIN, Professor HICKS, Professor J. J. THOMSON, Dr. FEUSZNER, Dr. GUILLAUME, Mr. CALLENDAR, Mr. VERNON HARCOURT, Mr. HEYCOCK, Mr. LARMOR, and Mr. NEVILLE for the encouragement and assistance they have afforded me. More especially am I indebted to Mr. GLAZEBROOK, not only for his advice and help, but for the careful and laborious comparison of my resistance box with the standards of the B.A.; and also to Mr. SKINNER for his repeated comparison of my Clark cells with the Cavendish standards. My thanks are also due to Mr. A. IVATT, B.A., Christ's College, Cambridge, for his assistance in the experimental work during 1887 and part of 1888.

Nearly the whole of the apparatus (some of it of a novel and intricate character) was constructed by, or under the direction of, Mr. F. THOMAS, Jesus Lane, Cambridge. Had it not been for his unremitting attention and skill, the results of my investigation would have been far from satisfactory.

From 1888 onwards I was assisted by Mr. G. M. CLARK, B.A., Sidney College, Cambridge, who has from the commencement of our joint work been indefatigable in his exertions. Our respective contributions are so intimately associated that I find it impossible to make the remainder of this communication in the first person, and although, by his own wish, his name does not appear on the title-page, it should, in justice, be regarded as a joint contribution.

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* The Clark cells used as our standards of E.M.F. are those given in Table IX. of Messrs. GLAZEBROOK and SKINNER's paper, 'Phil. Trans.,' A, 1892, p. 605.

SECTION I.—GENERAL DESCRIPTION.

Let R be the resistance of a conductor at a certain standard temperature θ, then if the ends of the conductor be kept at a constant difference of potential E

$$\frac{dQ_\epsilon}{dt} = \frac{E^2}{J.R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where Q_ϵ is the quantity of heat generated by the current, t is time, and J the mechanical equivalent of heat.

If the wire is immersed in water, and if the capacity for heat of the water and the calorimeter at the standard temperature θ is M, then

$$\frac{dQ}{dt} = M.\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

It is necessary to distribute the heat generated in the wire throughout the whole calorimeter as quickly as possible, otherwise we cannot accurately observe $d\theta/dt$, hence, since work must be done when stirring, we have to deal with a mechanical as well as with an electrical supply. Let the rise in temperature per second due to the mechanical supply be σ, and let $Q_{\epsilon,\sigma}$ denote the quantity of heat generated by both electrical and mechanical sources. Then

$$\frac{dQ_{\epsilon,\sigma}}{dt} = \frac{E^2}{J.R} + \sigma M = M.\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

It is certain that some heat will be gained or lost by radiation, conduction, and convection. Let ρ be the rise or fall per second in temperature due to radiation, &c., when the difference between the external and internal temperatures is 1° C. Denote the temperature of the surrounding envelope by θ_0, and the temperature of the calorimeter at any time by θ_1; also suppose that M becomes M′ and R becomes R′ when θ becomes θ_1.

We now have

$$\frac{dQ_{\epsilon,\sigma,\rho}}{dt} = \frac{E^2}{J.R'} + \sigma M' - \rho M'(\theta_1 - \theta_0)^* = M'\frac{d\theta_1}{dt} \quad . \quad . \quad . \quad . \quad (4),$$

where $Q_{\epsilon,\sigma,\rho}$ denotes the gain of heat due to all causes, hence

$$\frac{E^2}{J.R'.M'} + \{\sigma - \rho(\theta_1 - \theta_0)\} = \frac{d\theta_1}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

---

* It is shown in Section XII. that we are justified in assuming that NEWTON's law of cooling holds true over our range of temperature.

If $E = 0$, this becomes

$$\sigma - \rho(\theta_1 - \theta_0) = \left(\frac{d\theta_1}{dt}\right)^*_{\sigma,\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Provided that the mechanical supply is regular, and θ_0 is kept constant, we can, by a sufficient number of observations over small ranges, determine the values of $(d\theta_1/dt)_{\sigma,\rho}$ for different values of θ_1, and by substituting in equation 5, we obtain the value of $E^2/J.R'.M' = A$, where A is known.

Assuming that over small ranges the values of R′ and M′ are linear functions of θ_1,† we have

$$R' = R\{1 + k(\theta_1 - \theta)\}‡, \text{ and } M' = M\{1 + l(\theta_1 - \theta)\},$$

hence

$$\frac{E^2}{J.R.\{1 + k(\theta_1 - \theta)\}\,M.\{1 + l(\theta_1 - \theta)\}} = A \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Now the values of E and R at the standard temperature can be ascertained by comparisons with the standards, and the value of k can be ascertained by direct measurements of R′ at different temperatures, hence

$$J.M.\{1 + l(\theta_1 - \theta)\} = \frac{E^2}{A.R.\{1 + k(\theta_1 - \theta)\}} = B, \text{ where B is known} . \quad (8),$$

we have thus one equation connecting the three quantities, J, M, and l.

Let w and w_x be the capacities for heat of the water and the calorimeter respectively at the standard temperature θ, and let the temperature coefficients of their specific heats be f and g respectively, then

$$M\{1 + l(\theta_1 - \theta)\} = w\{1 + f(\theta_1 - \theta)\} + w_x\{1 + g(\theta_1 - \theta)\},$$

hence equation (8) becomes

$$J[w\{1 + f(\theta_1 - \theta)\} + w_x\{1 + g(\theta_1 - \theta)\}] = B \quad . \quad . \quad . \quad . \quad (9).$$

If observations are taken with different weights (w_1 and w_2) of water, we obtain B_1, B_2 the corresponding values of B, thus

$$J[w_1\{1 + f(\theta_1 - \theta)\} + w_x\{1 + g(\theta_1 - \theta)\}] = B_1$$

and

$$J[w_2\{1 + f(\theta_1 - \theta)\} + w_x\{1 + g(\theta_1 - \theta)\}] = B_2;$$

---

* In this, and similar cases, we use the suffix to denote the sources of heat.

† The value of $d\theta$, during our experiments, was about 1° C.

‡ The true value of $R' = R\{1 + k(\theta_1 + \beta - \theta)\}$ where β is the excess of the temperature of the wire above θ_1 the temperature of the calorimeter. See Section XIV., p. 478.

by subtraction we obtain

$$J (w_1 - w_2)(1 + f\overline{\theta_1 - \theta}) = B_1 - B_2 \quad . \ . \ . \ . \ . \ . \ . \quad (10).$$

Hence, when $\theta_1 = \theta$, the value of J can be found without ascertaining the value of the water equivalent or the temperature coefficient of the specific heat of water. As J is a constant quantity we can, by repeating the observations at different temperatures, obtain from equation (10) the value of f.

Or we can obtain f without finding J, for repeating the observations at temperatures θ_1 and θ_2

$$J (w_1 - w_2)(1 + f\,\overline{\theta_1 - \theta}) = B_{1,1} - B_{2,1}$$

and

$$J (w_1 - w_2)(1 + f\,\overline{\theta_2 - \theta}) = B_{1,2} - B_{2,2}$$

the second suffix of B denoting the temperatures.

Hence, by division, we obtain

$$\frac{1 + f\overline{\theta_1 - \theta}}{1 + f\overline{\theta_2 - \theta}} = \frac{B_{1,1} - B_{2,1}}{B_{1,2} - B_{2,2}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (11).$$

An alternative method of finding J is to first find f by equation (11), and then to find the value of w_x and g.* The value of J can then be deduced from a single experiment by means of equations (8) or (9), and as this method enables us more easily to compare the results of individual experiments we have, as a rule, adopted it in our reductions.

In the remainder of this section we briefly indicate the manner in which we have determined the various quantities grouped together and denoted in the above equations by B.

The calorimeter was suspended by means of glass tubes in an air-tight chamber whose walls were kept at a constant temperature. The pressure of the dry air in this chamber was reduced to under ·5 millim. The water in the calorimeter was stirred at a very rapid rate throughout the experiments, and the value of $(d\theta/dt)$ ascertained at all parts of the range 14° to 26° C.; firstly, when the work was done by the stirrer only, secondly, when the rise was due to both the electrical and the mechanical supply of heat.

The masses of water used during the experiments varied in the ratio of about 1 to 3, and the difference of potential at the ends of the coil was so altered that the heat developed by the current was changed in the proportion of 1 to 9.

As shown by equation (10) *supra*, there was no necessity to ascertain the water equivalent of the calorimeter, although it was found convenient, as a check upon the calculations, to deduce it from the differences in time caused by differences in the mass of the contained water.

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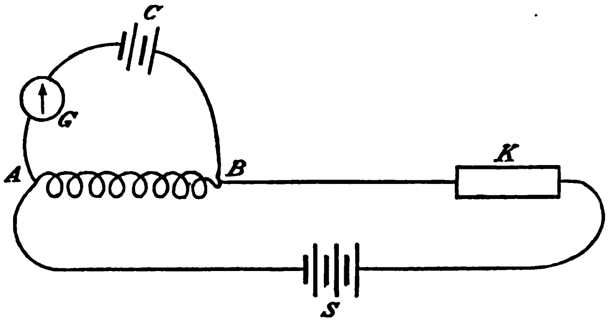
* These values can be obtained from the preceding equations without first determining J. The process is fully explained in Section XIV.

The walls of the chamber containing the calorimeter were maintained at a constant temperature throughout each experiment, this being, in general, the mean temperature of the above-mentioned range. Direct observations of the changes in $(d\theta_1/dt)_{\sigma,\rho}$, caused by changes in θ_1, enabled us to ascertain the values of both σ and ρ.

(1.) *Potential* (E).—A detailed account is given in Section VI. of the method by which the ends of the calorimeter coil were maintained at a constant potential difference. Briefly, the principle on which our method depended was as follows.

The extremities of the wires *CA*, *CB*, will (when the galvanometer *G* shows no deflection) be maintained at a difference of potential equal to that due to the cells at *C*.

Fig. 1.



Let *K* be an adjustable resistance placed in the circuit which communicates with *S* (the storage cells), and let R be the resistance of the wire *AB*. Whatever variations may take place in R and in the E.M.F. at *S* it is always possible (provided the adjustment of *K* is sufficiently under control) to maintain, by close attention to the indications of the galvanometer, the points *A* and *B* at a constant difference of potential.

A special rheostat was designed by means of which it was found possible, in spite of variations in the resistance of *AB*, to maintain the potential-difference unchanged throughout an experiment, and we believe that in no case did the variations exceed $\frac{1}{10,000}$ of the mean difference of potential during each experiment.

The Clark cells (which were placed at *C*) were constructed by us according to the directions of Messrs. GLAZEBROOK and SKINNER, and have, on several occasions, been directly compared with the Cavendish and indirectly with the Berlin standard.

Their differences from the standard are small, and their mean E.M.F. at 15° C. = 1·4344 volts (see Section VI.).

(2.) *Resistance* (R).—Had it been possible to maintain a constant value for R it would have greatly simplified both the experimental work and the calculations. In the year 1890 we devoted much time to the examination of the various copper-manganese-nickel alloys, and we performed a series of determinations with a coil whose tempera ture-coefficient was practically zero. The reasons which led us to reject these alloys and adopt a platinum wire will, we think, be found sufficient (see section VII.).

The value of R was first determined by a dial-box (legal ohms) constructed by Messrs. Elliott.* Mr. Glazebrook has been so kind as to perform a complete standardization of this box by means of the B.A. standards. The resulting corrections have been applied, and the values of R expressed in true ohms, as defined by the B.A. Report, 1892.

The method by which we ascertained the true temperature, and therefore, the resistance of the wire when traversed by a current, is fully described in Section VII. We found that the difference in temperature between the wire and that indicated by the thermometer could be accurately determined, and, other conditions being equal, varied as C^2.

The arrangement of our connections enabled us to dispense with exterior resistance, as the ends of the coil itself were maintained at a constant potential-difference.

(3.) *Current* (C).—The current was maintained by storage cells of the (L) type, supplied to us by Messrs. Bailey and Grundy. The E.M.F. of these cells was wonderfully constant. This constancy, although not a necessity, was a convenience. As previously pointed out, C was not directly measured.

(4.) *Time* (T).—An electrical clock with a seconds pendulum was used as our standard for time. It was carefully compared at intervals with a chronometer by Dent. A chronograph was controlled by this clock. The rate of the clock was a losing one until August 21st, but after that date its error was less than 1/25,000, and no correction was necessary.

(5.) *Mass* (M).—A balance sensitive to a change of 1/100,000 of the least mass measured by us, and a set of Oertling's weights,† were used in our determination of mass.

(6.) *Temperature* (θ).—At the time of writing (December, 1892) our thermometry is based on measurements made by platinum-thermometers. We propose to make, at an early date, a direct comparison of our standard thermometer with the air thermometer by means of the apparatus described by Mr. Callendar,‡ who has been so kind as to promise his assistance.

In a previous paper§ the details of a careful comparison of the platinum and the air-thermometer have been given. It was then shown that the platinum-air difference curve $\delta\,[\overline{\theta/100}|^{2} - \theta/100]$ gave values of θ at all temperatures from 0° to 100° C., accurate within ·01° of the real value of θ, and that discrepancies appeared to have an experimental origin. The experience of two years' work with platinum-thermometers has but increased our confidence in them.‖ Should a direct comparison with the air-thermometer modify our determinations of differences of temperature (and it is only differences which are important to us), our results will be modified accordingly. This will not, however, necessitate any repetition of the experimental work, as the corrections involved will be of a numerical order only.

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* Particulars of this box have been given in a previous paper. 'Phil. Trans.,' A., 1891, p. 44.

† These weights were re-standardized by Messrs. Oertling in August, 1892.

‡ 'Roy. Soc. Proc.,' Jan., 1891.

§ See 'Phil. Trans.,' 1891, A, p. 155.

‖ 'Philosophical Magazine,' December, 1892.

Measurement of the Heat generated by Stirring.

The difficulties that we have met with in this portion of our work have been very great. As will be shown in Section XI., we have, against our will, been compelled to increase the work done by the stirrer until the heat thus generated became about 1/10 of the whole supply. We found continuous and very rapid stirring to be a necessity. Variations in R and in the thermometer readings and in the water equivalent of the calorimeter invariably followed if the mixing was insufficient. Commencing in 1888 with a stirrer revolving at the rate of about 100 revolutions per minute, the rate was gradually increased until, in our last series, we aimed (with, it is true, a different form of stirrer) at a rate of nearly 2000 revolutions per minute.

The accurate measurement of the heat thus generated is only second in importance to the determination of the heat due to the current, and, accordingly, a great portion of our time and attention was directed to it. Fortunately, we were able to completely establish the law that, with our form of stirrer, the work done varied directly as r^3, r being the rate of rotation. From this time our difficulties rapidly diminished.

Radiation, Conduction, Convection, &c.

We endeavoured, as far as possible, to diminish loss or gain due to the above causes.

BOTTOMLEY* has shown that there is a rapid fall in the rate of loss or gain by radiation and convection when the pressure of the surrounding gas falls below 1 millim. Most of our observations were taken with the surrounding pressure under ·5 millim.

The calorimeter itself had a bright polished surface, and the walls of the enclosing chamber were clothed with bright speculum metal. In Section XI., we show that throughout our range of temperature, NEWTON'S law of radiation holds—*i.e.*, the loss or gain varies directly as the differences of temperature. It must, however, be remembered that we have not endeavoured to separate the partial effects of the radiation, conduction, and convection, and hence we can only say that the combined effect obeys this law. The total loss or gain, due to the above causes, was determined separately for each change in the contained mass.

Any error in this part of our work would have but a small effect on the mean value of J, since, in each experiment (if taken as a whole), the total loss or gain by conduction, convection, and radiation, must have been small.

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* 'Phil. Trans.,' 1887, A, p. 445.

SECTION II.—THE CALORIMETER AND COIL. (See Plate 2, fig. 2.)

The calorimeter, which was of a cylindrical form, was made of "gilding metal"—73 per cent. copper and 27 per cent. zinc.

The weight of the lower part was 501·84 grms.; of the lid 245·67 grms.

The internal diameter and the depth were each 8 centims.

Its capacity was therefore about 400 cub. centims.

The weight of the calorimeter may appear excessive for this capacity. However, as any "panting" of the lid or base would have thrown the stirrer out of adjustment and as the pressure on the flat base of the calorimeter would amount to about 120 lbs., when the space surrounding it was vacuous, it was evident that considerable thickness was necessary to secure rigidity. The two parts of the calorimeter were held together by eight screw bolts; a lead ring previously placed between the parts was crushed by the tightening of the screws and thus the join was rendered air-tight. It was therefore necessary to make both the lid and the projecting lip of the calorimeter sufficiently strong to bear the strain without bending.

As we expected to determine the water equivalent with as much accuracy as other quantities, the large mass of the calorimeter did not appear to be a serious objection.

The whole of the calorimeter was heavily gilded both inside and out, and the surfaces polished. The lead ring* was covered with gold leaf because considerable quantities of $PbCO_3$ were formed if the water had access to it; thus, no metal except gold was exposed to the action of the water. An advantage of the gold exterior was that its surface did not tarnish, and thus its coefficient of radiation remained constant.

Fixed on the interior of the lid were eight metal tubes, 1 centim. in length. Three of these were 18 millims. in diameter and served to hold the glass tubes by which the calorimeter was suspended. The others were 5 millims. in diameter and supported the glass rods on which the coil was wound. These rods reached to within a couple of millimeters of the base when the lid was placed in position. Two copper rods, 7·5 centims. in length (14 B.W.G.) passed from the insulated junctions at *M* and *N* (fig. 2, p. 382) to nearly the bottom of the calorimeter. These were also heavily gilded, and the ends of the platinum wires were fastened to their lower extremities. The last coil used by us was about 13 inches in length, and was wound round the lower end of the glass rods in such a manner, that the difference of potential of any two adjacent points could never be great. Thus the first 6 inches were wound on one side, the remainder on the further side of the calorimeter. A depth of 1·5 centim. of water completely covered the whole coil.

The glass tubes passing through the lid were held in position by perforated india-rubber corks. Great pressure had to be exerted to force the tubes "home," and several were broken in the process. The external diameter of the glass tubes was 14 millims., the interior diameter of the gold tubes 18 millims.; thus the annulus of india-rubber was very thin. The lower ends of the glass tubes were turned outwards so as to nearly cover the lower end of the india-rubber, of which the small exposed surface was originally coated with gutta-percha; but (as explained elsewhere) this was removed after J 19, and replaced by amber varnish.* The india-rubber corks, where they appeared above the lid, were, from the commencement, coated with amber varnish. No diffusion appeared to take place through these corks, and, as other observers seem to have been troubled by such diffusion when using india-rubber, it is probable that the coating of amber varnish (which adheres very firmly to the rubber) is an effective remedy. In any case it prevented any contact between the rubber and the contained water. We may thus assert that the water came into contact with the following substances only—gold, amber, glass, agate, gutta-percha (before experiment J 20) and (after experiment J 20) some hard shellac,† very small quantities of which were used to prevent slipping of the coil on the glass rods, the substance previously used for this purpose being gutta-percha. The platinum coil itself was coated with amber varnish, but it is possible, that owing to cracks in its coating caused by movements of the wire, the water had access to it in places. The small slabs of ebonite used at the junctions *M* and *N* (fig. 4, p. 396) were also covered with several layers of the same invaluable varnish.

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* A pure gold ring was tried, but it was found impossible to crush it sufficiently to render the join air-tight.

SECTION III.—THE METHOD OF MAINTAINING THE WALLS SURROUNDING THE CALORIMETER AT A CONSTANT TEMPERATURE.

The necessity of maintaining at a constant temperature the walls of the chamber containing the calorimeter is so obvious that it seems unnecessary to dwell upon it. In our earlier experiments, the calorimeter was suspended in a copper vessel which was immersed in a large tank, and the uniformity of the temperature depended on the skill of the observer in adjusting the temperature of the tank water by altering the supply of hot and cold water admitted. The method adopted by us in 1891 and 1892 worked automatically, and enabled us to maintain a chamber at any temperature within the range of our experiments, whether that temperature was above or below that of the room. Briefly, the arrangement may be described as an enormous thermometer-bulb within which was the calorimeter.

A section and plan, on a scale of $\frac{1}{8}$, are shown on Plate 3. The steel chamber, *A*, *B*, *C* (a very fine piece of work) was constructed by Messrs. WHITWORTH and Co. A detailed description of its various parts is unnecessary, as all particulars are given in the plate referred to. The annular space (coloured black) was filled with mercury.

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* *See Note*, p. 392, *infra*.

† The shellac was melted on, and not deposited from its solution. A hard surface is rarely obtained by the latter method.

As it was advisable to free that space from air bubbles, the screw plug *P* was removed before the insertion of the mercury, and replaced by a tube leading to a GEISSLER'S mercury pump. A tube from *D* dipped into a basin of hot mercury, and in this tube was a tap which was at first closed. The air in the annular space was withdrawn until the pressure was a small fraction of a millimeter, and the hot mercury was then slowly admitted—the exhaust pump being worked at intervals. When the mercury became visible within the tube at *P*, the tube was cautiously removed, and the air-tight screw plug replaced. The mercury necessary to fill the space weighed 70 lbs.

So true were the surfaces of the two parts of the metal chamber, which were screwed together by the bolts at *A* and *C*, that, although no packing except the finest possible coating of grease had been placed upon them, they proved absolutely air-tight. These two portions of the chamber were each formed from a single block of steel and, when placed together, somewhat resembled a double hat-box. The inner and outer surfaces were turned smooth and true, and then were nickel-plated. Two holes on opposite sides were bored down between the mercury chamber and the inner surface, as shown at *E* and *F*. These holes were filled with mercury and, their inner walls being very thin, the thermometer placed in them would accurately indicate the temperature of the inner surface of the chamber. This inner surface, as also the base of the chamber, was, in our more recent experiments, covered with highly-polished speculum metal. A ring of lead wire was placed round the top edge of the chamber, on which was laid that portion of the lid shown in section at *G* and *H*, and in plan at *G′* and *H′*, and the twelve bolts securing it were screwed home. Many as have been our difficulties with regard to air-tight joints, those made in this manner, viz., by the crushing of a lead ring between two surfaces, have never given us any anxiety. The second lid, from which the calorimeter was suspended, and whose section is shown at *K L*, and plan at *K′ L′*, was secured in a similar manner.

The necessity for this double lid is not, at first sight, obvious, but it must be remembered that it was impossible to separate the calorimeter from the portion of the lid to which it was attached, and, therefore, a heavy steel plate would have been inconvenient, and have rendered any direct weighings impossible. On the other hand, the total pressure on this lid, when the chamber was exhausted, was very great, and, had the whole of it consisted of one thin piece, it is probable that it would not have withstood the strain without "panting," and, as it was necessary on account of the stirring rod, &c., that the whole apparatus should be absolutely rigid, it was essential to avoid any phenomenon of this kind. The first lid, therefore, is of considerable thickness, and the orifice in it only just sufficient to allow for the introduction of the calorimeter. Again, if the lid had been of one piece, it would have been impossible to remove it without disturbing the tubes *M* and *N*.

The tube *M* communicated with a five-way Sprengel pump and a McLeod gauge; the tube *N* with a Geissler pump and a water pump. The arrangements for drying and exhausting by means of these tubes is described in Section V.

The steel chamber, together with the mercury, weighed nearly 2 cwts. It stood on a small tripod in a galvanized iron tank, whose capacity was about 20 gallons. The tank itself was placed on a bed of Portland cement before the cement had hardened, and thus the whole erection was firm and stable. When the outer tank was filled, the surface of the water was about 3 inches above the lid of the steel chamber and was maintained at that level by a constant inflow of water, which entered almost above the screw *Q* and left by an overflow pipe shown at *W*. The screw at *Q* rotated in such a direction that the water was forced downwards, passed under the steel chamber, and returned across the lid. The screw revolved about 800 times a minute, and the stirring was very thorough, so that it was impossible to detect any difference of temperature in different portions of the tank, even when hot water was being admitted, unless the thermometer was held immediately against the entrance pipe. A glass tube with narrow bore (Sprengel tubing) was fixed into the steel tube at *D* and communicated with the regulating apparatus shown in Plate 4, fig. 1. *R* is the regulator—the gas entering at *A* and leaving at *B*. The tube *R* was about 1 inch in diameter and was drawn to a fine opening at its lower end. The thin glass partition *C* passed down the centre of this tube and projected at its lowest extremity into the narrow opening. To render the partition air-tight, its edges were coated with hard shellac, and after it had been thrust into position, the tube was warmed until the shellac melted. The top of the tube was closed with shellac, thus the gas, entering at *A*, could only arrive at *B* by passing round the lower end of this partition, and a very small movement of the mercury in the narrow neck sufficed to cut off the supply. We adopted this form of regulator after considerable experience of the ordinary form, in which the supply passes down an inner tube and up the annular space around it, which labours under the defect that the adhesion between the mercury and the inner tube causes the gas to remain cut off for some time after the mercury has commenced to contract. The form here shown is free from this defect, as it is found that the mercury does not adhere to the thin glass partition. Beneath the narrow tube a small reservoir (capacity about 3 cub. centims.) at *E* allowed considerable contraction to take place, and yet permitted the mercury to be kept in view. A side tube passed from *E* to the three-way taps *F* and *H*, and thus, by means of the tap at *F*, mercury could be withdrawn when necessary. The tubes *HK* and *HL* were about 4 feet in length—*K* terminating in a thistle-funnel, in which a supply of mercury was placed.

The tube *L* was connected with the three-way tap *M*, by which communication could be made with the exhaust water-pump, or with the open air. Behind the tube *L* was fixed a vertical scale of millimeters, and the whole of the tubing was rather wide-bore Sprengel. Through the tap at *H* mercury from the arm *KH* could be passed into the arm *KL* to any convenient height. By means of the taps *H* and *F* any portion of this mercury could be passed into the regulator, and, if necessary, the *same amount* could at any time be withdrawn by connecting *L* with the exhaust (by means of tap *M*) and opening *F* and *H*—independently of the level of the mercury in the regulator. This contrivance, therefore, rendered it possible to "set" the instrument so as to produce any required change of temperature in the tank; for it was found that the transference of about 312 millims. of the column *HL* into the regulator caused a difference of 1° C. in the temperature of the tank. It was thus possible to adjust the temperature to a very small fraction of a degree.

The connecting tube from *E* to *D* was bent into a spring of sufficient length to allow of some play to the lower end, and thus it was possible to withdraw the end from *D* without moving the whole edifice. The glass work was supported on a vertical board, about 5 feet in height, and the various taps were placed within easy reach of the observer. All gas admitted at *A* was dried by passing through tubes containing quicklime, as it was found that the regulator worked in a much more satisfactory manner when the surface of the mercury was clean. After leaving *B*, the gas entered a tube about 2 feet long, containing a very large number of small pin-holes. The tube was connected by two cross pieces containing similar small holes, with two pilot lights fed by an independent supply. A change of $\frac{1}{500}$ of a degree in the tank visibly affected the brilliancy of the tiny gas jets, and a change of less than $\frac{1}{100}$ made all the difference between total eclipse and a complete series of bright blue jets. The supply of gas through the regulator was, however, sufficient when the mercury had receded far enough to maintain the jets at a length of nearly $\frac{3}{4}$-inch.

Placed above this row of jets, but so far from the pilot lights as to be unaffected by them, was a flat silver tube, the section of whose bore was a rectangle of about 1 inch wide and $\frac{1}{16}$ inch deep, and, through this tube, the water passed on its way to the tank. The supply was regulated by a small "constant level" tank of the usual pattern, suspended from the ceiling by a string. Alterations in the level of this tank afforded a fine adjustment—the temperature depending on the rate of flow. The water was supplied direct from the main, and hence, even in the hottest summer weather, it was possible to maintain the tank at a temperature as low as 14°; in winter, as low as 3° or 4° C. Any temperature above this minimum was, of course, obtainable, but we considered it unadvisable to raise it above 40° C. on account of the ebonite, &c. This regulator proved itself to be a most satisfactory instrument. Although, when set to give a new temperature, it was slow in its action (the capacity for heat of the tank and its contents being great) it was none the less sure, and, after about half-an-hour from the readjustment, the temperature would be found to have become steady. If set to give a higher temperature, the thermometer at *E* (Plate 3) would pass that temperature by about $\frac{1}{15}$ of a degree, and, after this preliminary oscillation, would become steady, although it is probable that this first was followed by other oscillations, which were too small to be apparent. It is possible that the temperature of the inner surface of the steel chamber was, to some extent, affected by the temperature of the calorimeter. The change must, however, have been very small, since it was insufficient to affect the thermometer at *E*. It must be remembered that any such change would at once produce an effect on the mercury, and, by its action on the regulator, reproduce the original condition.

SECTION IV.—THE MOTOR AND ITS CONNECTIONS.

A belt passed from the motor to a vertical shaft fixed near the back of the tank. This shaft carried "speed-wheels," from which ran two cords, one to the stirrer in the outer tank, the other to the stirrer in the calorimeter. In order to prevent any "slipping," with which we were at one time much troubled, all these cords ran through loose pulleys hung by strings, which, passing over small fixed pulleys, were connected with pans containing shot. The tension of the cords could thus be regulated by altering the weight of shot, and any change in their length, due to stretching, &c., was of no consequence. The shaft of the calorimeter stirrer was connected with the revolving piece above it by a double HOOKE's joint. Until experiment J 19 a short piece of brass tubing with a cross wire at each end served as the HOOKE's joint, with the exception of J's 17 and 18, when only a single HOOKE's joint was used. A difference in the "stirring supply" was shown in these two experiments (see p. 450), probably due to some difference in pressure between the agate cylinder at the bottom of the stirring-rod and the ring surrounding it. After experiment J 19 the connection was made by a small circle of thin wire, which passed through a hole in the revolving shaft and dropped into a slit in the top of the stirring-rod. Thus all rigidity in the connection was avoided, and the stirring effect was more regular after the change. Above the stirring-shaft was a HARDING's counter, whose axis carried at its upper extremity a V-pulley, and at its lower extremity the HOOKE's joint (see Plate 2, fig. 2). These counters move in steps, and not with a continuous sliding motion, and, as it was essential that we should be able to read with considerable accuracy the time of each 1000 revolutions, this form of counter was the best adapted to the purpose. The wear that these instruments will stand is surprising, for, on many occasions, nearly a quarter of a million revolutions per day were recorded, and the same counter was used by us throughout our experiments of 1891 and 1892, with the exception of experiments J 20 to 34, when the original counter, having shown some slight signs of wear, was replaced by another.

The motor (one of the fan type) was by BAILEY and Co., and of the usual pattern. When working at the Sidney Laboratory in 1891, the changes in pressure, and therefore in the speed of the motor, were so frequent that we were compelled to relinquish our attempts to perform our J experiments during the day, and the observations had, therefore, to be taken between 10 p.m. and 2 a.m. Even during these hours, the regularity was far from satisfactory.

As has been explained in the introduction a special room was built for the experiments of 1892. Here the supply pipe to the motor led directly from the main, but the irregularities of pressure were still so great as to render some improvement imperative. The form of regulator devised, although it did not insure that absolute uniformity which we could have wished, was yet so effective that we venture to describe it at length, in the hope that it may prove useful to others who have met with the same difficulty.

AB (see Plate 4, fig. 2) was a flexible diaphragm made of india-rubber of one ply. This was bolted between two brass saucer-shaped discs with tinned surfaces. The lower of these discs was connected with a tube leading to the motor; the upper, by means of *E*, *F*, *G*, with an iron reservoir (a mercury bottle) placed on the floor. A thin metal rod was attached to a plate in the centre of the diaphragm, and at its lower extremity were fixed two cylinders placed so as to form a balance valve. The ends of these cylinders were grooved in such a manner that any upward movement diminished, and any downward movement increased the size of the channels through which the water had to pass. The water from the main entered at *C*, and after passing through the grooved channels followed the path indicated by the arrows, until it arrived at the injector of the motor. An iron tube about 6 feet high was supported by a vertical plank, which reached from the floor to the roof of the laboratory. The lower end of this tube entered the iron reservoir *G* at the bottom, and to the upper extremity was fastened about 4 feet of high pressure tubing of which the free end was lashed to a mercury jar *H*, similar to the hand one used in a GEISSLER'S pump. This jar rested on a movable shelf which, by means of a rope and pulley, could be raised from about 4 feet to 10 feet above the vessel *G*. Sufficient mercury was placed in this jar to fill the flexible tube and the lower 3 or 4 inches of the iron vessel *G*. The taps at *K*, *F*, and *L*, were then opened and water passed into the tube *E*, *K*, and the vessel *G*. As the pressure increased, the mercury was forced up the tube *M*, *N* until its surface became visible in *H*. The tap at *F* was then closed and the screw plug at *E* opened to allow any air in the pipes, or above the diaphragm, to escape. *E* was then closed and first the tap *L* was opened, then the tap *F*, until the mercury again appeared in *H*. *F* was then finally closed and would not again be used unless it was necessary to refill the apparatus. The pressure above *AB* could now be adjusted at will by altering the elevation of *H*, and the effective pressure could thus be changed from 20 lbs. to 60 lbs. per sq. inch. When the tap *L* was opened, if the pressure due to the tap-water on the lower side of *AB* exceeded the pressure on the top, the valve was lifted and the flow to the motor diminished until the pressure of the water which had passed the valves became the same as that above the diaphragm. Conversely, if the water pressure diminished, the flow to the motor increased. The tap at *K* was only used to remove the pressure of the mercury column from the top of the diaphragm when the instrument was not in use. Within certain limits, this regulator acted excellently; but if the water pressure fell so low that the diaphragm assumed its lowest position, it ceased to be effective. By altering the elevation of the movable shelf at *H*, the rate of the motor could be adjusted at will, and a scale being fixed to the vertical plank, the extent of the adjustment necessary to produce a given alteration was approximately known after a few observations.

[Note by E. H. G. Added April, 1893. Lord RAYLEIGH has discussed the principles which should be observed when using a motor of the fan type ('Proc. Soc. Roy.,' 1881, p. 109), and has shown that when the motor is run at high speed the curves of work and of resistance cut each other at a greater angle than is possible when the rate of revolution is small, and thus a more uniform motion is obtained in the former, than in the latter, case. Our speed wheels were so arranged that the motor under ordinary circumstances ran at about 16 revolutions per second, which appeared to be as high a speed as was necessary.]

SECTION V.—THE PRESSURE IN THE SURROUNDING SPACE.

As already mentioned, the tubes *M* and *N* (see Plate 3) led to the pumps and McLeod gauge. The circuit embraced several yards of glass tubing, a large number of junctions and four taps, of which three had to be absolutely air-tight. In addition there were the four insulated junctions in the steel and the two in the calorimeter lid, and also the six openings (three in the steel and three in the calorimeter), through which the glass tubes which supported the calorimeter passed. All who have worked with low pressures will appreciate the difficulty of making such a series of joints sufficiently air-tight.

The manner in which the difficulty was overcome in the openings through the steel and calorimeter is described in Section VII. In what we may term the external circuit the greatest difficulty was experienced in the joints between the glass and the steel tubes *M* and *N*. Throughout 1891 we used telescopic joints. The glass tubes, which closely fitted into the steel ones, were coated with hard shellac, and, when both glass and steel had been warmed, were pressed into position. There is no doubt that such joints can be made absolutely air-tight; but they cannot be trusted to last for any length of time. It is possible that shellac is sufficiently viscous to yield to long continued pressure, or, on the other hand, it may crack when exposed to sudden changes of temperature. Whatever the cause may be, we have found such joints most treacherous. Their very perfection when first made renders them the more dangerous, for they inspire the observer with a false confidence.

Our experiences during 1891 led us to reject the use of shellac in any form.* In 1892 the connection between the glass and steel was made by platinum tubes of small bore, soldered into the steel at one end and fused into the glass at the other. However, the latter form of junction could not be relied on unless the diameter of the platinum was small, and this constriction of both the passages leading to the pumps greatly increased the labour of exhausting, for the process of diffusion through such narrow tubes is very slow when the difference of pressure at the ends falls below 1 millim. As we could find no way out of this difficulty we had to content ourselves with a higher pressure than we had proposed to work with.

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* From the commencement we avoided the use of india-rubber tubing.

The taps used were of a kind that has a diagonal hole through the stopper and a mercury cup at each end; they appear to be very perfect, and any leakage from either end can be detected by its effect on the mercury. One of these taps was in each glass tube leading from *M* and *N*, and, by their means, all the rest of the circuit could be cut off from the calorimeter.

The tube connected with *M* led to a five-way Sprengel and a McLeod gauge (both by HICKS). The tube from *N*, after passing the first tap, branched into two arms, one of which led to a Geissler pump, while in the other was a three-way tap establishing connection with the water pump, or with a series of drying bottles, containing H_2SO_4. A tube containing $Ba(HO)_2$ was fixed between the drying bottles and the calorimeter, as we thought it just possible that, otherwise, sufficient H_2SO_4 vapour might pass into the chamber to produce some action on the steel surfaces.

When the apparatus was fixed together the temperature of the tank was raised to about 40° C.; the air was then exhausted down to about 16 millims. by the water pump, dry air was passed in slowly, and this process was repeated several times. Finally the pressure was reduced as far as possible by the water pump (to about 12 to 14 millims.), and the Sprengel and Geissler brought into play.

It took about 100 double strokes with these pumps to bring the pressure down to 1 millim., so the labour involved was great, and want of time alone prevented our reducing the pressure further than we did. By increasing the number of the platinum tubes we hope to be able in future to work with much better vacua.

The absence of leakage is indicated by the fact that the lapse of a week made no perceptible difference in the reading of the McLeod gauge, although a change of ·01 millim. could have been easily detected.

It was long before this degree of perfection was attained, for the difficulty of ascertaining the position of a slight leakage, in such a circuit, was very great, and many weary hours were expended in leak hunting and in consequent taking to pieces and putting together again the various portions of the apparatus.

SECTION VI.—THE METHOD OF MAINTAINING A CONSTANT E.M.F.

General plan of the electrical connections* (fig. 2, p. 382):—

Leads passing through points numbered 1 and 3 belong to the Clark cell circuit.

Leads passing through points numbered 2 and 4 belong to the storage cell circuit.

The leads 1 and 2, as also 3 and 4, unite at the lid of the calorimeter itself, as shown in the small sketch.

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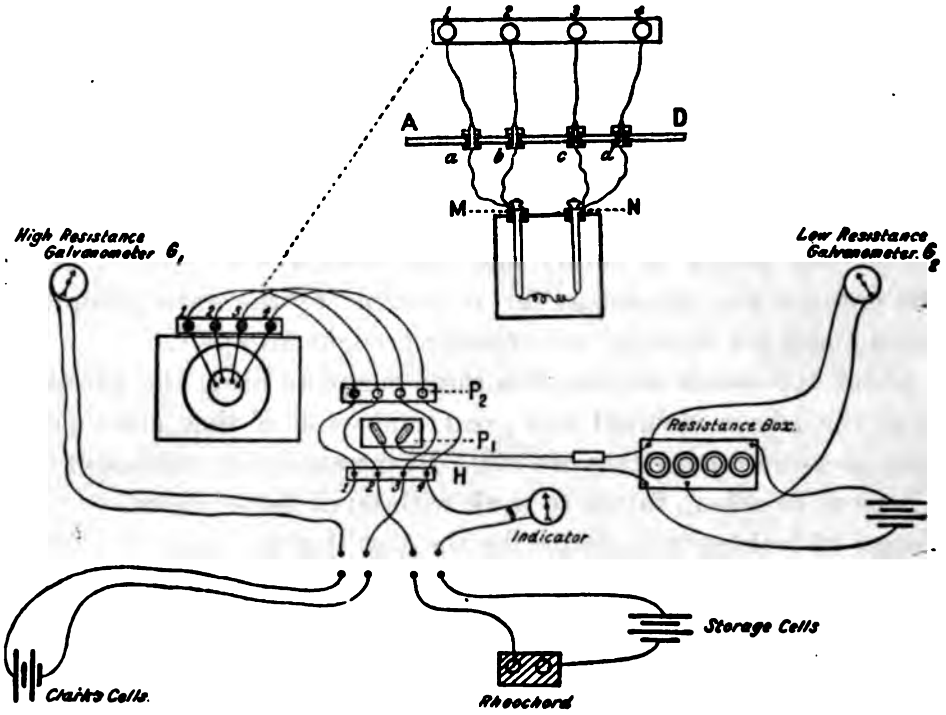
* The actual position of the galvanometers, &c., was different from the sketch, which is only intended to indicate the nature of the arrangement.

All connecting wires of any length were carried overhead, and hung by silk threads from a net suspended about 7 feet above the table. Thus all were visible, and could be traced without difficulty. All keys and instruments through which communication with earth could be made were placed on slabs of paraffin or ebonite, and the accumulators were also insulated. Each lead was placed as close as possible to its return to avoid any electro-magnetic action. The resistance of the galvanometer G_1 was about 9000 ohms. Instruments of such high resistance speedily indicate any leakage in the circuit, and it was not until we had given considerable time and attention to the insulation that we were free from trouble on this account.

Fig. 2.



The galvanometer (an astatic one manufactured by the Scientific Instrument Company, and the field so adjusted that the time of a single oscillation was about 4 seconds) showed no deflection unless connected with the Clark cells, although, at times, the difference of potential between the screws 2 and 4 amounted to 10 volts. If the fingers were placed on the ends of the wires communicating with it, the spot of light could be thrown nearly off the scale. The galvanometer was placed at the farther end of a tunnel (4 feet in length) of which the nearer end was closed by a sheet of ground glass. The beam of light was thrown in through the side of the tunnel, reflected on to the galvanometer mirror and thence on to the ground glass, which was protected by a hood from other illumination. Thus the "spot" was easily visible, even in bright day-light, by either of the observers when in the situations they occupied during the experiments. The extreme sensitiveness of this galvanometer will be illustrated when dealing with the comparison of our Clark cells.

The method of maintaining the ends of the coil at a constant potential difference has been indicated in Section 1. The experimental accuracy of the method depends entirely upon the action of the rheostat at K (fig. 1, p. 370). This instrument should be capable of a fine, as well as a rough, adjustment, and, above all, there should be no possibility of uncertain contact.

The majority of rheostats establish connection by the pressure of a sliding piece upon a wire, which is, at the best, an unreliable form of contact. The instrument devised by us has, however, proved to be so free from this defect that we describe it at some length.

It is well known that the electrical contact between a moving platinum wire and mercury is, under ordinary circumstances, an unsatisfactory one; its uncertain nature being probably due to the air-film attached to the wire.* We found that in vacuo the contact was in every way satisfactory.

A and B (see Plate 2, fig. 3) are two glass tubes about 7 feet long, with a platinum wire running through the interior of each—the ends of the wires projecting at A and B. The resistance of the wire in A is about 6 ohms per foot, and of that in B 1·7 ohm per foot. C and D are two wide tubes about 3½ feet long, closed at the lower ends.

A and B were carefully filled through a capillary tube with hot mercury, and inverted so as to stand in C and D. Practically, A and B are two barometers having tubes of about twice the usual length. C and D are firmly fixed in a stand, and A and B are raised or lowered by handles at the side, connected with gearing wheels which move two racks, fixed upon the rods carrying A and B. In their lowest position the barometer column reaches almost to the top of the tubes; in their highest, about 4 feet of the wire is left above the mercury. Spiral wires are attached to the projecting ends of the platinum wire at the top, and the ends of a copper fork, carrying a screw connector (E) at the bend, communicate with the mercury in C and D.

The current enters at F, passes down the wires in A and B (which are thus in parallel arc), and leaves the instrument at E. The wire from F to B is so arranged that additional resistance can be placed in it if required.

The mean resistance of the calorimeter coil used by us this year was about $8{\cdot}5\omega$; the minimum external resistance about $1{\cdot}3\omega$. The E.M.F. of each storage cell was about 1·4 times the E.M.F. of one Clark cell. Denoting our external resistance by r, and the coil resistance by R, it is evident that (when the number of storage and Clark cells were equal) an approximate balance would be obtained if $(\mathrm{R} + r)/\mathrm{R} = 1{\cdot}4$, which gives $r = 3{\cdot}4$, *i.e.*, when the rheostat resistance was about $2{\cdot}1\omega$.

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* If the wire is amalgamated the mercury is apt to cling to it when the wire is withdrawn.

It was found that by raising *A* about 6 inches, *B* about 24 inches, and introducing a resistance of 4ω into *FB*, an approximate balance was obtained. The resistance of *A* was then about ·3ω, and *B* 7·30ω. *B* thus acted as a shunt, and a considerable alteration in it caused only a small change in *r*. By altering their relative positions the ratio A/B could be changed, and the sensitiveness of the instrument could thus be altered when required. When arranged as above, a movement of 1 inch in *B* would throw the galvanometer spot off the scale, while the effect of a change of 1/25 inch was distinctly visible.

Thus a rise of 1 inch in B would change *r* from $(·3 + 7·3)/(·3 \times 7·3)$ to $(·3 + 7·44)/(·3 \times 7·44)$, *i.e.*, a change of ·0003ω.

And thus the ratio *r*/R would change from 1·3 / 8·5 to 1·3003 / 8·5, a change of less than 1 / 50,000.

The scale on the galvanometer screen had divisions of about 1 inch in length, and a deviation of 1 scale division, during a whole experiment, was exceptional. It thus appears that the variations in E were certainly within 1 / 100,000 of the mean value during the experiments, and, therefore, changes in E, consequent on changes in R or S, might be disregarded.* It was necessary that throughout each experiment one observer should give his entire attention to the adjustment of the rheostat. As the temperature of the coil rose steadily the value of *r* had to be increased, its total increase (over our range) being about ·005ω, *i.e.*, a rise of about 36 inches in B.† No doubt this adjustment took place in steps. The spot was observed to cross the zero line and was then thrown back to the further side—a rise of 1 millim. in B being, as a rule, sufficient for this purpose. As the handle had a large radius, this small change was effected without difficulty. The effect of such small oscillations about the zero position would probably "mean out," and if not, would be negligible.

Other possible causes of alterations in the value of E are—

(1) Polarization, as a consequence of the current required to affect the galvanometer. The high resistance of the galvanometer rendered any perceptible polarization extremely improbable. In our earlier experiments an additional resistance of 10,000ω was placed in the circuit; contact was intermittent and established by a key which was worked by a water motor.

Throughout our 1891 experiments we made it a rule to compare before and immediately after each experiment the E.M.F. of the cells used‡ with others which were kept as standards, and, as we were unable to detect any change, these excessive precautions were abandoned. On twenty-six occasions when no extra resistance was introduced these cells were compared before, and at the close of, an experiment.

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* When the external circuit became heated by the current at the commencement of an experiment considerable adjustment was required, but it was made without difficulty.

† This increase is in *addition* to the increase in *r* due to the heating of the external circuit.

‡ The Clark cells used in 1891 belonged to the Cavendish Laboratory, and are the ones referred to in Table IX., 'Phil. Trans.,' A, 1892, p. 605.

Poggendorff's method was used ; a change of one plug indicated a change of E/6000, each cell was balanced against the standard ones in the manner adopted by Messrs. **Glazebrook** and **Skinner**, and in no case could any alteration whatever be detected. This point having been satisfactorily established, we considered it unnecessary to repeat the observations in a similar manner this year, but contented ourselves with occasional comparison of the cells used with others which we kept as standards.

(2) The second difficulty was not surmounted so satisfactorily. As before mentioned (Introduction) the laboratory was maintained in cold weather at a constant temperature of about 16° C., by an automatic regulator. However, during the months of August and September, 1892, while the nights were cool, the temperature during the day was extremely high, and occasionally rose in the laboratory to over 20° C. The Clark cells were contained in a tank of water, but their temperature at times changed by 3° or 4° C.; and under such circumstances the difficulty was to know what temperature to take.

Messrs. **Glazebrook** and **Skinner** have shown that the E.M.F. does not change with the thermometer, but lags considerably. A difference between the "virtual" and "real" temperature of 1° C. would cause a difference of 1 in 1000 in our corrected times, and we believe that many of the irregularities in our results, especially in experiments before August 15, are attributable to this cause. On that date we got into working order a regulator which much diminished this source of error. A large bulb of mercury was immersed in the tank, and was connected by a narrow tube with a trap of the ordinary gas regulator pattern, through which cold tap-water, instead of gas, was kept flowing. When the mercury expanded, the water ceased to flow through the regulator, and diverged by a bye-way into the tank; when the mercury contracted, the cold water passed into a waste pipe, instead of into the tank. The pressure of the water was kept constant by a small supply cistern, with an overflow pipe, which was suspended by a string, and the head of water could thus be regulated. This apparatus worked most satisfactorily—a change of $\frac{1}{4}$° C. turning the inflowing water from the tank to the waste. Unfortunately the weather became so warm that even the tap-water at that end of the laboratory rose to about 16° C. The changes, after the introduction of the regulator, were, however, much diminished, and rarely amounted to $0^{\circ}\cdot 5$ C. As regards the errors thus introduced, although they seriously affect individual experiments, they must tend to eliminate when a large number of observations are taken.

The Clark cells used for the 1892 experiments were constructed during January and February of that year. Particulars of their construction are given in an Appendix to Messrs. **Glazebrook** and **Skinner**'s paper; therefore it is unnecessary to give here more than a brief summary.

Although the method of preparing the solutions for different batches of six was varied, the E.M.F. of all the cells is practically the same. In all cases, the platinum wires were amalgamated by being dipped in mercury when red hot. The mercury, which was freshly distilled, was about 1 centim. in depth, and was nearly touched by the ends of the zinc rods. About 1 centim. of the "paste" was placed on the mercury, then, about 1·5 centim. of $ZnSO_4$ solution and some crystals of $ZnSO_4$. Above this solution was fixed a thin cork, on which was a thin layer of paraffin wax. On this was placed about 6 centims. of glass-wool and then about 2 centims. of marine glue. The diameter of the cell was about 2·5 centims. The platinum wire was sealed into a thin glass tube, the end of which projected above the marine glue. In most cases the upper end of the zincs was completely buried.

Nos.[*] 31 to 42.—Hg_2SO_4[†] unwashed but shaken up with Hg and the $ZnSO_4$ solution filtered at 30° C., zincs not amalgamated.

Nos. 43 to 48.—Prepared by following out the Board of Trade directions exactly.

Nos. 49 to 54.—Boiled the $ZnSO_4$ solution with ZnO, allowed to cool to 30° C. Filtered at 30° C. Added to this a small quantity of Hg_2SO_4 paste. Filtered off resulting black precipitate, warmed filtrate, and again filtered at 30° C. The cells made with this solution appeared to settle down to their final E.M.F. immediately after their manufacture, unlike the others, which were some time before they became steady.

About thirty comparisons of these cells were made between January and September, 1892. Nos. 31 to 42 remained at the Cavendish Laboratory for some time, and were repeatedly compared with the Rayleigh Standard by Mr. SKINNER, to whom we owe our best thanks for the trouble he took in this matter.

We select as examples comparisons made on three different dates. The results of the comparisons on other dates are as close, but it is unnecessary to give them at length.

No discrepancy showed itself except after rapid changes of temperature, and in the most extreme case the difference from the mean did not amount to 1 in 2000. The comparison of cells 31 to 36 with the Cavendish Standard, on February 13th and 20th, gave precisely similar results. These comparisons were only the last of a series. On both dates the temperature of the standard was slightly higher than that of the cells, and therefore the equality was very close. Mr. SKINNER, writing on February 20th, said, "You may take it that Nos. 31 to 36 are now equal to our standard, and no further comparison is necessary." In our comparisons the cells were, in each case, placed in opposition to No. 31.

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[*] These cells are numbered 131 to 160 in Messrs. GLAZEBROOK and SKINNER's paper, 'Phil. Trans.,' A, 1892, pp. 622 to 624. Numbers 154 to 160 were not used by us during this investigation.

[†] Chemicals supplied by Messrs. HARRINGTON.

TABLE I.

	Feb. 13 and 20, 1892. Comparison with Cavendish Standard. (Diff. in 6000 parts.)	May 1, 1892. Comparison with No. 31. (Diff. in 6000 parts.)	July 3, 1892. Comparison with No. 31. (Diff. in 6000 parts.)	Sept. 6, 1892. Comparison with No. 31.* (By potentiometer.)
31	+ 1 +	0	0	1·00000
32	+ 1	+ 2 +	+ 2 −	+ ·00002
33	+ 1	+ 1	0 +	− ·00003
34	+ 1	− 1	− 1 −	− ·00013
35	+ 2 +	− 1	0	− ·00011
36	+ 2 +	0	+ 1	·0
37		0	+ 1	+ ·00001
38		0	+ 1	+ ·00002
39		0 −	+ 1	+ ·00001
40		0	+ 1	− ·00006
41		0 −	0 +	− ·00006
42		0 −	0 +	− ·00007
43		+ 2	+ 3	+ ·00022
44		+ 1	+ 2	+ ·00013
45		0 +	+ 2 +	+ ·00014
46		+ 2 +	+ 3	+ ·00020
47		+ 4	+ 3	+ ·00014
48		+ 3 +	+ 5	+ ·00016
49		0 +	0 +	− ·00001
50		+ 1	+ 1 −	·0
51		− 1	0 +	− ·00002
52		− 1	0 +	− ·00009
53		0 −	+ 1 −	− ·00007
54		− 1	0 +	·0
	Temperature . . .	12°·2	17°·2	16°·04

On May 1 and July 3 the total resistance of the two boxes used was 12,000ω, so the numbers given represent the differences in 6000 parts. In the comparison on September 6 a CLARK's potentiometer (constructed by Messrs. ELLIOTT) was used. Readings can be taken with an instrument of this kind, with much greater ease and certainty than when using two resistance boxes.

It may be worth mentioning that on September 6, the instrument was adjusted by one observer and the readings taken by the other, who then moved the handle, thereby compelling a fresh adjustment by the first observer. The galvanometer used was the high resistance one, G_1, previously referred to, and gave a distinct deflection for a change of $1/10^{-6}$ in R.

Several comparisons were also made at different dates with some cells belonging to the Cavendish Laboratory.[*] These cells[†] were (early in August) in close agreement with No. 31, and also with the Cavendish. They have since been compared with the Berlin Standard, and appear to exceed that standard by about ·0002 volt.

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* We have not reduced the potentiometer readings to the same form as that adopted in the previous columns, as a difference of ·00017 corresponds to 1 in 6000, and the differences shown on September 6 would only amount to + 1 + in the most extreme case.

It will be noticed that the results of the comparison on September 6th are in much closer agreement than those of previous dates. This is partially due to the potentiometer, since the adjustment in this case has not to proceed by steps. It must be remembered also that the cells had been kept (by the regulator above described) at a steady temperature for a considerable time; and the great importance of a constant temperature is rendered evident by the figures given in Table I., and illustrates the truth of Mr. GLAZEBROOK's remark that many discrepancies are due to the different "lag" of the cells when their temperature is changing.

A consideration of the above facts will, we believe, bear out the conclusion that the mean value of our Clark cells may be regarded as exceeding that of the Cavendish standard by not more than ·0002 volt (*i.e.*, by about 1 in 7000). During our J experiments we always used a considerable number of these cells, placing several in parallel arc, and in no case did we use less than nine of them at any one time. Thus no correction for individual cells is necessary.

A reference to GLAZEBROOK and SKINNER's paper will show that but a short time elapsed between their determination of the absolute value of the Cavendish standard, and the comparison of our cells with that standard. The absolute value of the standard is given by them as 1·4342 volt;[‡] we are therefore justified in assuming the mean value of cells No. 31 to 54, as

$$1{\cdot}4344\ \{1 + {\cdot}00077\,(15 - t)\}.$$

[Note by E. H. G., added April, 1893. All the cells mentioned above practically maintain their relative positions at this date. I have recently taken a series of observations in the hope of throwing some light on the gradual approximation of the cells, as shown by Table I. I find that if the temperature is suddenly raised, and then kept quite constant, the initial differences steadily decrease with lapse of time, and that the cells continue to converge after the lapse of many days, and even weeks. The cells prepared by my method of adding some Hg_2SO_4 to the solution before filtering, appear to settle into their final state more rapidly than the others.]

SECTION VII.—THE MEASUREMENT OF RESISTANCE.

The coil used by us in 1888 was of platinum-iridium wire, and had a mean R of about 19 ω.

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[*] We take this opportunity of thanking Professor THOMSON for his kindness in permitting us to borrow many pieces of apparatus belonging to the Cavendish Laboratory.

[†] These are the cells Nos. 65, 69, 70, 77, 78, and 79, mentioned in Messrs. GLAZEBROOK and SKINNER's paper. No. 65 may be regarded as equal to our No. 31. See 'Phil. Trans.,' A, 1892, pp. 605, 623, and 625.

[‡] 'Phil. Trans.,' A, 1892, p. 582.

In 1889 we used two coils of 2 and 4 ω, one of German silver and one of platinum, both covered with a layer of gutta-percha.

In 1891 we used a coil of about 10 ω covered with amber varnish and made of copper-manganese-nickel alloy, and also a platinum wire of about the same value.

This year we have contented ourselves with a platinum wire of about 8·5 ω.

Since $H \propto E^2/R$ and E is, throughout each experiment, a constant, therefore $H \propto 1/R$· Hence if a coil of large resistance is used, the quantity of heat developed will be small, unless E is great.

It appeared to us desirable to keep the value of E as low as possible, for the greater the difference of potential at the ends of the coil, the greater is the loss arising from imperfect insulation, &c. On the other hand, if the coil is too short the development of heat is too localized, and its even distribution throughout the calorimeter rendered more difficult. Our experience in the earlier experiments, as also a consideration of the masses of water, &c., led us to the conclusion that a difference of potential equal to that of from 3 to 5 Clark cells* and a resistance of 8 to 10 ω would balance the opposing influences in the manner best adapted for our purpose.

Platinum would at first sight appear to be an unsuitable metal of which to form the coil, for its temperature coefficient is very great, and the element of change thus introduced involves much arithmetic in the final calculations. For the same reason the error due to its rise in temperature above the surrounding water is likely to be more serious than if the coil was formed of an alloy. These considerations carried so much weight that we undertook (in the summer of 1890) a series of investigations into the resistance of the various copper-manganese-nickel alloys, which had been recently introduced by Dr. FEUSZNER of Berlin. Some of these alloys were made for us by Messrs. JOHNSON and MATTHEY, and several specimens were given to us by Dr. FEUSZNER, to whom we owe our best thanks for his kindness; others were obtained from Messrs. WOLFF, Berlin.

The behaviour of these alloys was investigated over a range of from — 20° C. to 100° C. The different wires passed down a tube about 4 feet long, together with a platinum wire, cut from the same coil as that from which one of our platinum thermometers was made. The resistance of this wire (*in situ*) was determined in ice and steam, and the value of δ being known, by its determination when in our platinum thermometer, the mean temperature of the wires surrounding it could be deduced with great accuracy.

We hoped to proceed with an analysis of the composition of all the wires used but time has not been sufficient.

The results are summarized in the following table.

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* Had we felt certain that the insulation of the wire was perfect, we should have had no hesitation in increasing the potential difference, in which case we could have worked with larger masses of water. The uncertainty as to the temperature of the wire when thickly coated would, however, more than counterbalance any advantage gained by the use of larger masses.

Date—July 24 to September 18, 1890.

A = Platinum wire, diameter ·005, same as thermometer H (previously annealed).
B = Copper wire (commercial).
C = Copper-manganese alloy made by Messrs. Johnson and Matthey—approximate composition—·4 per cent. manganese (previously annealed in carbon).*
D = Copper and manganese (J and M), 16 per cent. manganese (previously annealed in carbon).
E = Copper and manganese alloy presented to us by Dr. Feuszner (Mn 10 per cent. and Cu 90 per cent.).
F = Copper-manganese-nickel alloy from Dr. Feuszner (diameter ·17 millim.), rather more Mn than G and H.
G = Copper-manganese-nickel alloy from Dr. Feuszner (diameter ·14 millim.), Mn 12 per cent., Ni 3 per cent., Cu 85 per cent.
H = Copper-manganese-nickel alloy from Dr. Feuszner (diameter ·10 millim.), Mn 12 per cent., Ni 3 per cent., Cu 85 per cent.
J = Copper-manganese-nickel alloy supplied by Wolff (diameter ·24 millim.), Mn, Ni, and Cu, but composition uncertain.
K = Copper-manganese-nickel alloy supplied by Wolff (diameter ·17 millim.) Mn, Ni, and Cu, but composition uncertain.
L = Copper-manganese-nickel alloy supplied by Wolff (diameter ·10 millim.), Mn, Ni, and Cu, but composition uncertain.
M = German silver.

Table II.—(Temperatures by Air Thermometer.)

	– 20°	0	10	15	20	25	50	75	100	
A (Pt)	·9288	1	1·0352	1·0525	1·0700	1·0874	1·1745	1·2608	1·3468	
B (Cu)	·9382	1	1·0306	1·0468	1·0627	1·0790	..	..	1·3054	
C	·9805	1	1·0097	1·0147	1·0195	1·0244	1·0485	1·0733	1·0977	
D	·9968	1	1·0014	1·0021	1·0027	1·0033	1·0058	1·0078	1·0096	
E	·9976	1	1·0009	1·0013	1·0017	1·0020	1·0032	1·0036	1·0031	Max. at 75° C.
F	·9983	1	1·0008	1·0012	1·0015	1·0019	1·0032	1·0042	1·0052	
G	·9966	1	1·0014	1·0022	1·0030	1·0037	1·0084	1·0104	1·0134	
H	·9999	1	1·0000	·9999	·9999	·9998	·9991	·9979	·9967	Max. at – 5° C.
J	.9987	1	1·0005	1·0007	1·0009	1·0011	1·0015	1·0014	1·0004	„ 55° C.
K	·9989	1	1·0006	1·0009	1·0011	1·0014	1·0023	1·0026	1·0027	„ 90° C.
L	·9986	1	1·0005	1·0007	1·0008	1·0010	1·0014	1·0014	1·0011	„ 65° C.
M (calculated)	·9920	1	1·0040	1·0060	1·0080	1·0100	1·0200	1·0300	1·0400	

The above resistances were taken with an E.M.F. of four Leclanché cells, and the arms of the bridge were 1000/10. A double set of electrodes was used, and the external resistance determined for each observation. Also the resistance of the platinum wire was observed before and after each determination of the other wires. The current was reversed on each occasion to eliminate thermal effects. The total swing of the galvanometer averaged about 150 scale divisions for a difference of $\cdot01\,\omega$, and thus the next figure could be determined with accuracy. The actual resistance of the wires used varied from 1 to 50 ω; but in the above table the resistance of each at 0° C. is assumed as 1 ω. It must be remembered that the temperatures are expressed in the *air* (not mercury) scale.

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* We have to thank Mr. C. T. Heycock for his kindness in making an analysis of this alloy.

According to the analysis given by Dr. FEUSZNER, G and H have the same composition, and differ only in their diameters, yet the difference in their behaviour is very marked. It is possible that this difference is due to the effect of the "drawing."

The *calculated* value of German silver alloy is given for the sake of comparison. We regret that we did not include a wire of that substance. It is, of course, improbable that its increase is proportional to its temperature—from 0° to 100° C.

Our observations on these wires* extended from July 24 to September 18, 1890, and we were unable to detect any change in their resistances during that time.

The behaviour of the alloy C is curious. Our figures, which were carried to a place beyond that given in the above table, prove that, over the above range the increase in resistance varied strictly as θ. In order to test the straightness of its line beyond the above range, we constructed a thermometer which had this wire for its coil, but which was, in all other respects, constructed in the same manner as a platinum thermometer with double electrodes.

This thermometer gave $R_0 = 12\cdot972$, $R_1 = 14\cdot255$, and R_s (*i.e.*, resistance in sulphur vapour at 760 millims.) $= 18\cdot677$. Assuming the straight line we have

$$\theta_s = (R_s - R_0)/(R_1 - R_0) \times 100 = 444\cdot6.$$

According to an investigation subsequently conducted as to the boiling point of sulphur† the true air temperature ought to be $444^\circ\cdot53$ C. It is difficult to imagine a closer approximation. A thermometer thus constructed is, therefore, a very simple and accurate instrument. It would, however, be exceedingly difficult to make another specimen of the alloy of a precisely similar composition, and the above table shows that the effect of small changes may be considerable.

A consideration of the results led to the conclusion that the wire marked H was the one most suitable for our purpose. Its change in resistance from 10° to 25° C. was only 1 in 5000, and since the resistance diminished as θ increased, it was easy to compensate by the addition of a little of the C wire. All difficulties connected with change of resistance consequent on the rise of temperature of the water, or of the difference in temperature of the wire and the water, appeared, therefore, to have been overcome. This conclusion, however, led to the loss of many weeks' work, and was the cause of much disappointment and delay. In the summer of 1891 we commenced a series of direct determinations of J, the arrangement of the apparatus being very nearly the same as that of 1892. Our coil consisted of about 30 inches of the above wire, H, which was first annealed at a temperature of about 200° C. in an atmosphere of CO_2. The fine white silk covering was allowed to remain, and it was painted with Mr. LAURIE'S amber varnish.*

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* All these wires had been previously heated to a temperature of about 150° C.

† It must be borne in mind that the date of this determination preceded the work of CALLENDAR and GRIFFITHS on this subject, and at this time (July 27, 1890) the boiling point of sulphur was accepted as $448^\circ\cdot4$ C. (REGNAULT).

The resistance of this coil, and of the leads from it to the calorimeter lid when *in situ* and immersed in rapidly stirred water, is shown in the following table.

TABLE III.

Temperatures . . .	14°	16°	18°	20°	22°	24°
August 21, 1891 . .	R=10·5214	10·5220	10·5223	10·5225	10·5226	10·5225

RESISTANCE of the coil at temperature 14° C., on different dates.

Aug. 22	10·5228		
„ 23, 11 A.M. . .	10·5275		
„ 23, 11 P.M. . .	10·5299		
„ 24	10·5342		
„ 25	10·5398		
„ 27, before exp. J_3	10·5559;	after exp. J_3	10·5600
„ 28, „ „ J_4	10·5570;	„ „ J_4	10·5632
„ 29, „ „ J_5	10·5618;	„ „ J_5	10·5696
„ 30, „ „ J_{6a}	10·5760;	„ „ J_{6a}	10·5860
„ 31, „ „ J†	10·5867;	„ „ J†	10·5962
„ 31, „ „ J_{6b}	10·5953;	„ „ J_{6b}	10·5977
Sept 1, 10 A.M. . .	10·5968;	(some current passed) 11 P.M.	10·5982

The resistance of the wire at different temperatures was also ascertained in the observations taken August 21 to 25.

It is unnecessary to give the resistance at temperatures other than 14° C., since the resulting curves are all parallel to the one on August 21. The alteration caused by a rise of 10° C. being only 1 in 10,000, the steady diurnal rise in the resistance (at the same temperature) amounts to nearly 1 in 500 in the above interval of time. We trusted that this might be a result of the annealing, and that the wire would assume a stable condition in course of time.

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* Mr. LAURIE was so kind as to allow us to have a bottle of this varnish prepared by himself; for particulars concerning this varnish, see 'Proceedings Cambridge Phil. Soc.,' vol. 7, part 2, p. 52.

It should be noticed that on the above dates we were constantly passing currents through the coil in order to adjust the temperature of the calorimeter during our stirring experiments. As, however, there was but little current when ascertaining the resistance of the wire at different temperatures, the curves were not appreciably altered, and we did not recognize the fact that the rise was a consequence of the current.

On August 27, we performed one of our "J" experiments. We then had a current of about half ampère passing through the coil for about 1½ hours. We had determined the resistance at the commencement of the experiment, and, on repeating our observations at its close, we found that the value of R had risen from 10·556 to 10·560. It would thus appear that the rise was a function of the current rather than of the time. We continued the series of experiments until September 1, when we decided to abandon the use of this alloy.* The labour thus expended was not altogether lost, for, as will be explained, in the section on the measurement of temperature, the results were of considerable value to us.

Had the rise been a function of the time it would not have been so disastrous, since we could have deduced the resistance during each experiment. As, however, the change was caused by the current, it was impossible to determine (with sufficient accuracy) the actual value of R at each stage of the experiment. It appeared probable that electrolysis of some kind took place. If, after a current had been passing for some time, the ends of the wires from the coil were connected with the galvanometer, a considerable deflection was produced, and it was impossible to determine the resistance of the coil for about half an hour after the completion of an experiment, owing to the disturbance of the galvanometer thus caused. The effect was apparently due to polarization rather than to thermal, or "Thomson" effects, and decreased regularly until it might be neglected, although the differences of temperature in the circuit were maintained unaltered. Much time and energy having already been expended in experimenting with these various alloys, we decided to retrace our steps and revert to a platinum wire coil. Another consideration which influenced us, was that we had, during the past few years, given considerable time and attention to the behaviour of platinum wires when used in platinum thermometers, and we considered that the experience thus gained of the changes in the resistance of such wires under varied conditions, might enable us to proceed with more confidence.

The smaller the diameter of the wire the less its capacity for heat and, in comparison, the greater its cooling surface. The wire selected for this year's experiments was supplied by Messrs. JOHNSON and MATTHEY. It had a diameter of ·004 in., and was of as pure platinum as could be obtained. The length used was about 13 in., and the wire was so disposed that parts near to each other should have only a small difference of potential. This wire was covered with a coating of the amber varnish before referred to. Several coatings form an excellent insulating covering for such a wire, but we contented ourselves with a thin skin. The better the insulation between the wire and the water the slower the flow of heat across the insulator, and the greater the difference in temperature between the surfaces which it separates.

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* Mr. SKINNER has recently conducted some experiments with the same coil immersed in paraffin instead of in water; and although he has passed far greater currents than any used by us, he was unable to trace any signs of consequent increase in R. The increase shown during our experiments was, therefore, probably due to some action between the wire (when carrying a current) and the water, this is not unlikely when we remember that one of the metals present was Mn.

If the insulation was not perfect we have little doubt but that it was sufficient. In order to throw more light on this point we decided to make a careful determination of the value of R, with the coil immersed in a pure paraffin (pentane), but otherwise under the same conditions as those of the experiment. We selected pentane (1) because it would be difficult to find a more perfect non-conductor; (2) if pure, it would be easy to free the calorimeter from all traces by passing through it a current of warm, dry air. The latter was an important consideration, for any substance such as ordinary impure paraffin might so alter the interior surfaces of the calorimeter as to change the stirring coefficient.

On December 8, 1892, about 250 cub. centims. of pentane* were introduced into the calorimeter, which had previously been dried with extreme care, and the resistance of the coil was taken at twelve different temperatures. A summary of the results is given at the end of this section (see p. 413), and a very slight increase in the resistance is observable, amounting to one part in 22,000. It is satisfactory to find that, whatever may be the effect of conduction through the water, it is, in this case, so small as to be unimportant.

The measurements of R were made when the coil was immersed and the water stirred in the same manner as during the experiments, hence it follows that if the water acted merely as a conductor it would in no way affect our results.

Even if its conductivity changed with the temperature our values of R include the effects of such change, and the manner in which R is built up at each temperature is of no consequence.

Again, if electrolysis takes place (of which, however, we have had no sign), polarization must ensue. The best proof of the absence of polarization is the fact that when the storage and Clark cell circuits were reversed (and this was invariably done more than once during each J experiment) no readjustment of the rheochord was required when using the platinum coil, although when using the manganine coil a considerable readjustment was necessary.

Had electrolysis occurred, it is almost certain that HNO_3 would have been formed. In our earlier experiments (both in the platinum and silver calorimeters), although the same water was never used for two experiments, we found that it became acid—in some cases strongly so.

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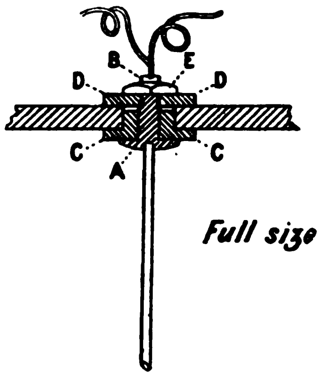
* Great difficulty was experienced in obtaining this specimen of pentane, and it was only through the kindness of Mr. VERNON HARCOURT that we were able to procure it. It was prepared under his own direction and was free from all impurities, except some traces of tetrane and hexane.

This year the water remained unchanged for twenty consecutive experiments and we were unable to detect any acidity, although all water withdrawn from the calorimeter was invariably subjected to a careful examination.

The insulation of the leads passing through the steel cover and the lid of the calorimeter was a troublesome matter, as there was great difficulty in making the insulating junctions absolutely air-tight.

Fig. 3.



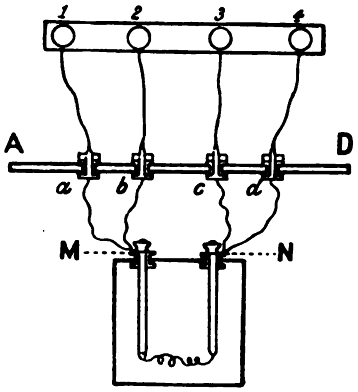
A, *B* was a brass pin, which had a screw thread on the end *B* and terminated in a nut at *A*. *C* was a ring of ebonite fitting tightly into the hole in the metal plate and having a projecting flange at the lower end.

The surface of the ebonite was moistened with gutta-percha solution (made by dissolving gutta-percha in chloroform); the ring, *C*, was then forced into the hole and the brass pin passed through it. A ring of ebonite, *D*, was then threaded on to the pin and a brass nut, *E*, firmly screwed down—the ends of the wire being soldered on at *A* and *B*.

This form of joint proved to be an excellent one, and when once adopted (after the failure of many other contrivances) gave no further trouble. The steel lid was pierced by such junctions in four places, and the calorimeter lid in two. As tap-water was continually flowing over the top of the steel lid it was necessary to insulate the exposed parts at *B*. A piece of wide india-rubber tubing enveloped the wire, the lower end fitted over the ebonite ring *D*, and was wired firmly on to it. The upper end of this tubing projected above the surface of the water in the tank. The tubes themselves were filled with distilled water to prevent any rise in temperature of the leads. The general arrangement is shown in Plate 2, fig. 2.

Leads 1 and 3 (fig. 4, p. 396) were connected with the Clark cells, and thus no current passed along them. 2 and 4 carried the current. Any heat developed in the parts above *AD* passed into the water of the outside tank, and might be disregarded. A portion of the heat developed in the wires *bM* and *dN* would pass to the water in the calorimeter. By sufficiently diminishing the resistance of *bM* and *dN* the heat thus developed could be diminished indefinitely; on the other hand, the conduction of heat between the calorimeter and the steel chamber would be increased. The wire selected was of pure copper (No. 21, B.W.G.), and each arm was about 10 centims. in length. The resistance of $bM + dN$ was carefully determined, and found to be ·0068ω. It is probable that half the heat generated in these wires would find its way to the inner, and half to the outer, vessel. We may, therefore, in order to determine the heat thus added to the calorimeter, assume this external resistance (r) as ·0034ω.

Fig. 4.



Let R be the coil resistance. Then in any time T, the quantity of heat developed in the calorimeter, will be $J.H = C^2(R + r)T$. Now $C = E/R$, and since M and N are maintained at a constant difference of potential, therefore

$$J.H = \frac{E^2}{R}\left(1 + \frac{r}{R}\right)T,$$

therefore the effective value of the resistance $= R - r$ (neglecting terms involving r^2/R and higher powers of r). Since the mean value of R is about 8·7, the correction is, in any case, a small one.

In order to test the insulation an arm of the bridge was constructed, which included wire No. 4, the insulators, and the steel casings—good connection with the steel being insured by passing a wire into the mercury contained in the thermometer-hole. High battery power was used, and the resistance was found to be greater than we could measure—that is, greater than 10 megohms. Since No. 4 was in connection with all the junctions in the plate AD, the insulation of all the junctions through the steel was thus established.

The arm was then made to include No. 4, the insulators in the lid of the calorimeter, and a copper wire, which was passed down the glass tubes until in contact with the floor of the calorimeter. The resistance of the two calorimeter junctions was, in the same way, found to be greater than 10 megohms.

Method of finding R.

P_1 and P_2 (see fig. 2, p. 382) were blocks of paraffin. The wires leading from the calorimeter to the junctions on the ebonite rod H passed through the block P_2, down which holes were then bored until the wires were exposed. These exposed portions were carefully amalgamated and the holes filled with mercury. Two thick pieces of copper were so bent that communication could be established between any of the holes in P_1 and the two troughs in P_2. When placed in position, the ends of these connecting strips rested on the conductors at the bottom of the holes. Two thick wires led from the troughs in P_1 to the resistance box, and, in order to bring their resistance up to a convenient quantity, a coil was introduced into one of these leads.

Denoting the total resistance of the conductors from the box to the lid of the calorimeter by r_1, r_2, r_3, r_4, it is obvious that all the following resistances could be taken by a movement of the connectors between P_1 and P_2, viz., $r_1 + r_2$, $r_1 + \mathrm{R} + r_3$, $r_2 + \mathrm{R} + r_4$, $r_3 + r_4$. The determinations were always made in the order thus given.

If N_1, N_2, N_3, and N_4 are the resulting numbers, we have

$$\mathrm{R} = \{(\mathrm{N}_2 + \mathrm{N}_3) - (\mathrm{N}_1 + \mathrm{N}_4)\}/2.$$

The galvanometer used had a resistance of about 8ω, and was fitted with a microscope containing a micrometer eyepiece by ZEISS. The image of a black line placed on a slip of paper in front of the mirror was thrown on to the micrometer scale.

A galvanometer fitted in this way is a very convenient instrument, for the swings can, with practice, be read with greater ease and accuracy than when the ordinary lamp and screen arrangement are used.

When a resistance was taken (whether of the coil or of a platinum thermometer) the order of the observations was as follows:—

Having determined between which plugs the resistance lay, the swing corresponding to the lower one was taken, the battery inverted and the swing in the opposite direction read. This process was repeated when the higher plug was in place. In this way all thermal or other effects are practically eliminated, and the method is independent of small changes in the zero position of the spot.

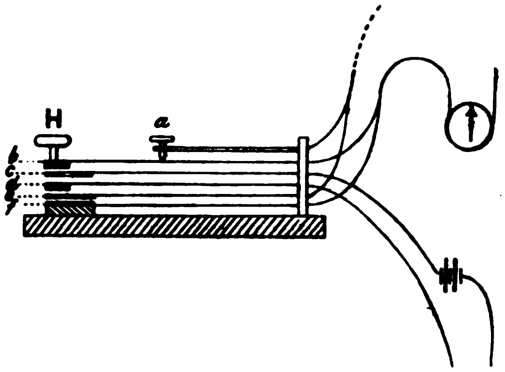
In some cases, as when working with platinum thermometers at high temperatures, the thermal effects are considerable, and the position of the zero point becomes so changed that the spot is thrown off the scale when connection with the galvanometer is made, although the battery circuit is not connected. To meet this difficulty a special key was constructed.

The ends of a and b (fig. 5) and also of e and f are connected with the galvanometer circuit, the ends of c and d with the battery. The screw at a is so arranged that, when the key is untouched, the galvanometer circuit from the box is completed.

The position of the spot at any time, therefore, is that due to the influence of any

E.M.F. in the circuit exterior to the battery circuit. Thus the spot can be placed in a convenient position by the control magnet, although a perceptible current is passing through the galvanometer. When *H* is sharply pressed down the connection at *a* is first broken; then the battery circuit is established by *c* and *d* and the galvanometer circuit by *e* and *f* (the projecting blocks on *b* and *d* being insulators). By proper adjustment of the key, the whole operation, performed by one movement, is accomplished in a small fraction of time. Thus the observed swing is due to the difference of potential caused by the battery and bridge, and is independent of any electrothermal effects.

Fig. 5.



We have described this form of key at some length because, simple as it is, we have not seen any similar arrangement, and we have found it a great convenience, especially when working with platinum thermometers.

As an example of our method of observation, we give one of our determinations of the value of R.* It should be remembered that, although the exterior leads were made as equal as possible, the measurement of R does not depend on their equality and N_1 and N_4; N_2 and N_3 are not necessarily equal. Arms of bridge 1000/10. E.M.F. that of 2 Leclanché cells. When measuring N_1 and N_4 a resistance of 1400ω was thrown into battery circuit, in order to make the total swing a convenient size.

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* A summary of our determinations of R is given at the end of this section (see Table XVI., p. 413); the value of R in the last column but one of Table IV. has yet to be corrected for box errors, &c.

TABLE IV.

	Galv. swings.		Obs. R.	Temp. E_m.*	Temp. box.	R.	R corrected for temp. of box.	Temp. C.
N_1	272	53	$59\frac{45}{390}$	311·2	16·65	8·73205	8·73058	19·560
	227	398						
	45	345						
N_2	333	110	$932\frac{150}{452}$					
	183	411						
	150	301						
N_3	348	124	$932\frac{148}{450}$					
	200	426						
	148	302						
N_4	330	138	$59\frac{54}{393}$					
	276	477						
	54	339						

$$R = \{(N_2 + N_3) - (N_1 + N_4)\}/2.$$

A large number of similar observations were taken at temperatures ranging from 13° to 26° C. Observations at certain temperatures were repeated at regular intervals, and no change of any kind could be detected. The resulting resistance curve was the characteristic platinum one, and in no way differed from those given by our platinum thermometers. This in itself seems to us to be an adequate proof that the insulation of the coil was sufficient.

On September 4, the calorimeter was taken to pieces, and the method of fastening the wire to the glass rods within the calorimeter was altered. The wire was a little strained during the operation; but the only effect was to slightly increase its resistance at all temperatures. A new resistance curve, however, had to be drawn, and new values of R used after that date.

It may be mentioned that this coil was often used by us as a platinum thermometer to determine the temperature of the calorimeter when below, or above, the range of E_m.

The reversing keys in the storage and Clark cell circuit (see fig. 2, p. 332) were of the ordinary "rocker" form, but so constructed that as the wires on one side passed out of the mercury cups, the wires on the other almost immediately made contact; the interval of time between the break and make being less than we could measure.

---

* See Section X. The mercury thermometer E_m was used throughout all observations on the temperature of the calorimeter.

If a current be passed through a naked wire, immersed in water, it is certain that the temperature of the wire will be greater than that of the water, even when the latter is rapidly stirred.

Had we been able to use the alloy wire previously referred to this would have been a matter of small importance. Since, however, the value of R, when a platinum wire is used, varies greatly with the temperature of the wire, we felt it necessary to commence a series of investigations into the extent of the change. Hence we proceeded to determine the resistance of the coil when there was a considerable difference of potential at its ends. Our first trial was a modification of POGGENDORFF's method of comparing E.M.F.'s. The storage cells were put in circuit through the coil, and a potentiometer arranged in series. One end of the Clark cell circuit was connected to the storage end of the coil, the other forming a sliding contact on the potentiometer; thus when the galvanometer in this circuit showed no deflection,

$$\frac{\text{E.M.F. of Clark cells}}{\text{E.M.F. of storages}} = \frac{R_1}{R_1 + \rho},$$

when R_1 is the resistance of the coil and part of the potentiometer, and ρ the resistance of the rest of the storage circuit. We had supposed that the internal resistance of the storages might be neglected so that ρ could be determined directly after adjustment by throwing the storages out and completing the circuit by a short length of wire. The ratio of the E.M.F.'s could also be obtained by the usual method through the potentiometer, and thus, by noting the temperature of the coil when adjustment was perfect, we had sufficient data from which to calculate the resistance of the coil. The results obtained by this method were very unsatisfactory, owing probably to the following causes :—

(1.) The heating up of the circuit ρ in the same manner as the coil itself.

(2.) The internal resistance of the storages, although almost zero when no current is passing, increases when the circuit is established, and hence

(3.) The ratio of the E.M.F.'s is not the true one.

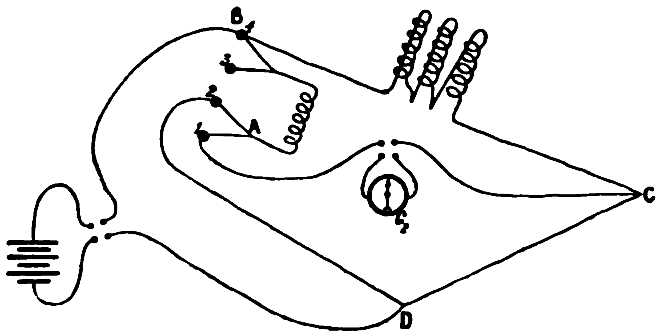
Of these, the first was probably the greatest source of error, although this circuit was entirely composed of naked wires, which had a large cooling surface.

We therefore relinquished the attempt to find the whole resistance of the coil, and devised a method whereby we could find the increment of the resistance due to increasing electromotive force; the resistance when $E = {\cdot}004$ volt having been found at all points of our range as described above (Table IV.). For this purpose a temporary WHEATSTONE's bridge was arranged, constructed as follows :—

The arm *AB* (fig. 6) was formed by the coil alone. *BC* contained three large coils of uncovered wire, the first of which (Cavendish Laboratory coil) was of German silver and had a resistance of 5ω; the second, also of German silver (belonging to the

British Association), had a resistance of $2 \cdot 5\omega$; and the third, of copper, had a resistance of $\cdot 5\omega$.*

Fig. 6.



The arms *AD*, *DC* (about 6 feet each) were made of uncovered copper wire. A preliminary rough adjustment could be made at *D* and the final adjustment of the whole bridge was made by altering the temperature of the coil in the arm *AB*. This alteration in temperature was produced by the heat developed in the wire itself, and thus the whole calorimeter was under the same conditions as those prevailing during our determination of J.

The temperature on thermometer E_m was read at the moment when the galvanometer (G_2) showed no deflection. The behaviour of the galvanometer indicated that the flow of heat from the coil was not quite steady, but took place in gushes, owing probably to the coating of insulating varnish on the wire; for when the key was depressed, the mirror did not swing steadily back to its zero, but oscillated about it, and the moment had to be estimated when the mean of these oscillations coincided with the zero. That this could be done with considerable accuracy is shown by the fact that the individual observations agreed well amongst themselves.

The general conduct of a series of these experiments was as follows.

We first found the temperature of the coil when the bridge was in adjustment, the E.M.F. at the ends of the coil being that used when taking a resistance in the ordinary way. When the calorimeter was cooled down, the E.M.F. was raised and the coil allowed to warm up until a new balance was obtained and the temperature at that moment noted.

As the temperature and consequently the resistance of the arm *BC* remains constant, and since the arms *AD*, *DC* are practically equal, their ratio remains unchanged. The resistance of the arm *AB* (the coil) must also be the same as before, hence the difference in the two observed temperatures gives us the number of degrees that the coil is hotter than the surrounding water; or we can express the result in terms of increase in resistance, as the resistance (when $E = \cdot 004$) at all temperatures is known.

---

* The total mass of metal in this arm of the bridge amounted to several pounds.

TABLE V.

The following Table gives a series of observations taken on August 22, 1892 :—

Time.	Battery in circuit.	Temp. E_m.	Temp. C.	R when E = ·004.	δR.
		At which bridge balances.			
5.35	1 Leclanché	344·7			
6.22	„	343·9	20·397	8·7519	
6.47	1 Storage	337·8	20·240	8·7471	·0048
6.55	2 Storages	316·5	19·698	8·7311	·0208
6.58	„	316·7			
7.15	1 Leclanché	343·7	20·395	8·7518	
7.40	3 Storages	277·0	18·684	8·7010	·0507
7.45*	„	279·0 (?)			
7.55	„	276·8			
8.17	1 Leclanché	344·4	20·409	8·7021	
9.9	„	342·5	20·361	8·7508	
9.40	4 Storages	221	17·286	8·6600	·0899
9.50	„	221·2			
10.13	5 Storages	137	15·231	8·5995	·1485
10.22	„	139			
10.44	1 Leclanché	337·9	20·243	8·7472	

When the readings obtained by using 1 Leclanché differ (the change being probably due to changes in the temperature of the bridge), intermediate values are deduced from the times.

It seemed absolutely immaterial whether the current was on for only a few seconds or indefinitely. In the above set, when three storage cells were being used, the current was not broken at all, the calorimeter being cooled down by the ether apparatus† in defiance of the heat developed in the coil. Although a quarter of an hour elapsed between the observations, they were in very good agreement.‡

To determine the E.M.F. at the ends of the coil, the various batteries used were compared with the Clark cells. This method was therefore open to the sources of error mentioned on p. 400, Nos. (2) and (3). The resistance of the other parts of the bridge were then determined and were as follows :—

Resistance of whole bridge 4·30ω.
„ large coils 7·69ω.
„ coil and its ratio arm 9·17ω. Hence coil = 8·75, arm ·42.
„ large coils and ratio arm 8·06ω. Hence coil = 8·75, arm ·369.
„ wires forming battery circuit ·030ω.

The E.M.F.'s were determined by the ordinary POGGENDORFF's method in terms of the Clark cell at temperature 16°·8 C., hence E.M.F. = 1·434 volts; another set of storage cells being used to give the current through the potentiometer.

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* This observation was uncertain; the observer at the galvanometer remarking so, before he knew that the reading of the temperature was different.

† See p. 419, *infra*.

‡ This would show that the increase in resistance of the other arms of the bridge was not appreciable.

TABLE VI.

Battery.	Potentiometer readings.		E.M.F.
	For battery.	For Clark cells.	
		No.	
1 Leclanché	32·232 32·225	1 . 33·832	1·366
1 Storage	23·915 23·936	1 . 17·188	1·994
2 „	47·735 47·734	2 . 34·195	4·004
3 „	72·088 72·083	3 . 51·347	6·039
4 „	64·105	4 . 45·602	8·064
5 „	79·844 79·857	5 . 57 029	10·040

To determine the internal resistance of the batteries, the current from them was sent through the coil and balanced in our usual manner with the mercury rheostat (see Section VI.). The temperature being noted, when the galvanometer was steady, the resistance of the whole battery circuit external to the coil, was immediately taken; then if x is the internal resistance of the battery, R the resistance of the coil approximately corrected for the rise due to the current, then,

$$\frac{\text{E.M.F. of Clark cell}}{\text{E.M.F. of battery}} = \frac{R}{R + \rho + x},$$

but this ratio was given by the potentiometer comparison, hence we found x, the internal resistance of the storages, = ·018ω, and of the Leclanchés 2·24ω.

Hence we have the following results :—

TABLE VII.

Battery.	E*. At end of coil.	δR. 1 Leclanché being called 0.	δR. E = ·004.
1 Leclanché . .	1·035	0	·0016
1 Storage . . .	1·90	·0048	·0064
2 „	3·81	·0208	·0224
3 „	5·74	·0507	·0523
4 „	7·67	·0899	·0915
5 „	9·55	·1485	·1501

---

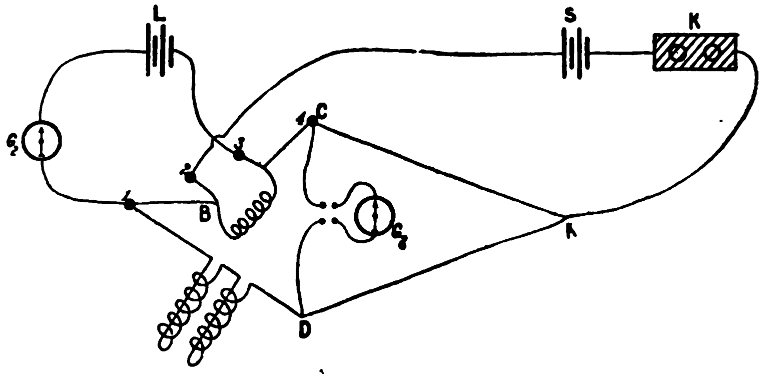
[*] It is probable that the values of E are too high, for (as before pointed out) these are the values of

These observations will be seen to agree fairly well with the parabola

$$\delta R = \cdot 00160\ E^2.$$

This method, although the results were concordant, was unsatisfactory in the manner in which the difference of potential at the ends of the coil had to be calculated. We therefore adopted the following arrangement (fig. 7), which possessed the great advantage of at once giving us the difference of potential at the ends of the coil in terms of our standards. The Clark cell circuit was connected to the ends 1 and 3 of the coil, and the mercury rheostat introduced into the storage circuit; thus, by adjustment of the rheostat the E.M.F. at the ends of the coil could be made equal to that of 1, 2, 3, &c., Clark cells. No difficulty was experienced in working the two galvanometers together, the observer at the low resistance one being entirely unaware whether the high resistance circuit was made or broken. As it was felt that the whole success of the method depended on the non-heating of the large coils, the two smaller were replaced by one large one of German silver, wound double, so as to expose a very great surface to the air. The resistance of this coil was selected so that when placed in series with the Cavendish large coil,* their united resistance was very nearly equal to that of the calorimeter coil; thus the remaining arms of the bridge were more nearly equal than before, and were made of stout German silver wire, so that each arm was about 5 feet in length and had a resistance of ·3ω. We believe that these results were quite satisfactory, since (1) there was no difficulty in the measurement of E; (2) the mass of metal in the large coils was very great, their cooling surface large (a thermometer placed on the coils never showed any tendency to rise) and their temperature coefficient small; (3) the other arms of the

Fig. 7.



E when the exterior resistance is infinitely great, hence the value of α (in the equation $\delta R = \alpha E^2$) is too low.

[*] The Cavendish coil was a triple strand one, containing nearly 1400 feet of single German silver wire, the other coil had a double strand and contained more than 400 feet of a stouter wire.

bridge were so nearly equal, and were so arranged, that they would be affected to the same degree by any temperature changes.

The figure shows the connections—

BC is the arm containing the coil.

BD contains the two large German silver coils.

AC, *AD* the two German silver wires forming the remaining arms of the bridge.

G_1, G_2 the high and low resistance galvanometers.

L, *S* the Clark cells and storages.

K the mercury rheostat.

TABLE VIII.—The following Table gives the observations taken September 20, 1892.

Time.	No. of Clark cells.	Temperature at which bridge is balanced.		R at C° (in Legal ohms).	δR.
		E_m.	C°.		
3.40	1	376·3	21·226	8·7790	x
3.46		376·3			
3.54	2	360·7	20·818	8·7671	$x+·0119$
3.56		360·2			
4. 3		360·4			
4.17	3	332·5	20·100	8·7458	$x+·0332$
4.21		332			
4.25		332·1			
4.33	4	292·5	19·080	8·7157	$x+·0633$
4.36		292·8			
4.38		292·3			
4.50	5	240	17·760	8·6771	$x+·1019$
4.59		240·5			
5.50	6	177·5	16·202	8·6316	$x+·1474$
6. 0		176·8			
7.12	1	379·4	21·306	8·7814	x
7.18	2	363·4	20·895	8·7692	$x+·0122$
7.26	3	333·4	20·130	8·7468	$x+·0346$
7.30		333·6			
7.45	4	292	19·067	8·7156	$x+·0658$
8. 0	5	240·4	17·765	8·6772	$x+·1042$

Of the second series of experiments, the first was taken immediately after entering the laboratory, when the arms *AC*, *AD* (being very near one observer, and the gas having been lighted) had not probably reached a state of thermal equilibrium, but they evidently did so during the series.

TABLE IX.*

The following numbers were plotted:—		δR (Legal ohms.)	δR deduced from $\delta R = \cdot 00422 n^2$.
No. of cells.	Increase.		
0	$-x$	0	0
1	0	·0042	·0042
2	·0120	·0163	·0168
3	·0333	·0376	·0378
4	·0638	·0681	·0675
5	·1023	·1066	·1055
6	·1478	·1516	·1519

Hence $x = \cdot 0042$,

and the above agree within the limits of experimental error with the parabola

$$\delta R = \cdot 00422 \times n^2;$$ where n is number of CLARK'S cells.

In this case there can be no doubt about the actual D.P. at the ends of the coil. The results of the previous investigation (p. 404) are of great importance, in so far as they confirm the conclusion that $\delta R = \alpha E^2$, but are of little use for determining the value of α.

In order to complete these experiments variations in the rate of stirring were tried. It was found, as would be expected, that the more perfect the stirring, the less the wire became heated; but, within the limits of our rate of stirring, the change was so slight that no correction was thought necessary.

The observer at the high-resistance galvanometer could, however, always detect minute changes in the rate of stirring by the irregular behaviour of the spot. These changes, although thus rendered very evident, had an exceedingly small effect and thus they served as a proof that the oscillations observed during our J experiments also indicated variations too minute to affect our measurements.

[NOTE.—As illustrating the importance, as also the accuracy, of the correction rendered necessary by the difference between the temperature of the wire and of the water, we here give a summary of the results of J 9 and J 34. These two experiments were performed in order to subject the corrections (both for radiation, &c., and for the difference in the temperature of the wire) to a severe test, the heat developed per second in the wire during J 34 being nine times as great as that developed per second throughout J 9.

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* In comparing the last two columns, it must be remembered that a difference of ·0010 corresponds to a difference of but 1 in 8600 in R.

	J 9.	J 34.
E.M.F. (expressed in terms of a Clark cell at 15°)	2	6
Weight of water + water equivalent = (M) = ...	273·77 grms.	345·20 grms.
Mean rise per 1″ (after correction for stirring, radiation, &c.) = γ = ...	$8225° \times 10^{-7}$	$5778° \times 10^{-6}$
Mean temperature	19°·540 C.	19°·540 C.
Hence, thermal grms. generated per 1″, when E is that due to 1 Clark cell = $\left(\gamma \times \frac{M}{E^2}\right)$ = ...	56295×10^{-6}	55404×10^{-6}
Now, resistance of wire at 19·540, as determined by box in the ordinary manner = (R_0) = ...	8·7084	8·7122*
Hence, thermal grms. per 1″ if R = 1ω; $\left\{\left(\gamma \times \frac{M}{E^2}\right)R_0\right\}$ = .	49026×10^{-5}	48270×10^{-5}
These quantities (which ought to be equal, and from which we deduce the value of J) here differ by nearly 1 part in 60.		
If, however, we deduce the values of R_{2E} and R_{6E} from Table IX., we get 8·7084 + ·0168 and 8·7122 + ·1519	8·7252	8·8641
Hence $\left(\gamma \times \frac{M}{E^2}\right)R_1$ = ...	49111×10^{-5}	49110×10^{-5}

We did not see how to apply a more severe test, and we, therefore, considered it unnecessary to carry the investigation further, although we had supposed that a repetition would be necessary.]

Comparison of our Resistance Coils with the Standards.

The resistance box was taken to the Cavendish Laboratory on October 31, 1892, to be compared with the standard coils in possession of the British Association.

In order to be able to standardize each coil separately, two plugs were made, half ivory and half brass, so that good contact was secured with the outer circle of the dials through the brass, but no contact was possible with the inner circle through the ivory. The insulation of these plugs was, however, defective, the shunt so formed being sufficient to bring the 10 legal ohms down to 10 B.A. ohms.

Two brass bridges were then made which rested over the outer circles of the dials, one end of each bridge fitted into the plug-holes, but was cut away so as not to touch the central circle. By tightening up a screw in the other end of the bridge a good and regular contact was secured. To the centre of each bridge a binding screw was fixed, and connection was made by thick wire leads from these to the Carey-Foster bridge.

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* The resistance of the coil had been altered between these two experiments (see p. 411).

In the comparison of the units and tens, the Carey-Foster bridge belonging to the B.A. was used, and the difference between the standard and the box-unit was found in terms of a length of bridge-wire. In the comparison of the 100's and 1000's, the bridge described by GLAZEBROOK ('B.A. Report,' 1883) was always used.[*]

The standard coils used, and their values, are given in the following Table:—

TABLE X.

Coil.		Value.
ELLIOTT	269	1 + ·000282 (*t* − 15·1)
„	270	1 + ·000286 (*t* − 15·3)
„	288	10 + ·00265 (*t* − 16·2)
„	289	10 + ·00260 (*t* − 15·4)
NALDER	3634	100 + ·0308 (*t* − 17·2)
„	3637	100 + ·0308 (*t* − 17·05)
1000 *b.w.d.*		·05000 + ·0000715 (*t* − 18·2)

The same thermometer (HICKS, No. 352,601) that had been employed in all our determinations was again used in the standardization, and, since its bore was very nearly uniform, no correction was necessary. The B.A. thermometer (K.O., 75,149) was used for the temperatures of the standards; its correction was − ·10 at the temperatures of our observations.

The resistance of the connections between the box and bridge was taken in terms of the bridge-wire of the Carey-Foster bridge.

The results of observations on different days agree well, the average departure from the mean being less than $\frac{2}{1000000}$.

The differences could, in most cases, be easily accounted for by small temperature changes in the interior of the coils, which could not be indicated by the thermometer.

The accompanying Table gives the results.

(Although the coils in the 1000 dials were not used during the determinations of the resistance of the calorimeter coil, we have included them in this Table, since many of them were required when ascertaining the platinum temperatures, and also because a complete standardization of this kind is interesting, as it shows the order of accuracy of boxes of this pattern.)

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[*] In comparing the 100 and 1000 ohms coils, two Leclanché cells were used. The currents in the battery circuit were ·090 and ·023 ampère respectively.

TABLE XI.

Coil.	Unit dial.			Ten dial.			Hundred dial.					Thousand dial.				
	Nov. 16.	Nov. 17.	Mean.	Nov. 7.	Nov. 9.	Mean.	Nov. 10.	Nov. 10.	Nov. 11.	Nov. 14.	Mean.	Nov. 12.	Nov. 13.	Nov. 13.	Nov. 14.	Mean.
1	..	·99523	·99523	9·9764	9·9759	9·9762	99·796	99·793	99·801	..	99·797	998·31	..	998·40	..	998·35
2	·99888	·99894	·99891	9·9826	9·9826	9·9826	99·779	99·779	99·784	..	99·780	997·71	..	997·80	..	997·75
3	·99687	·99693	99690	9·9708	9·9709	9·9708	99·798	99·797	99·802	..	99·799	997·96	..	998 07	..	998·01
4	·99595	·99600	·99597	9·9792	9·9789	9·9791	99·784	99·783	99·790	99·789	99·787	997·69	..	997·78	..	997·75
5	·99793	·99808	·99800	9 9748	9·9747	9·9747	99·808	99·805	99·811	99·811	99·809	997·97	..	998·10	..	998·04
6	·99654	·99661	·99658	9·9774	9·9772	9·9773	99·725	99·721	99·736	..	99·727	997·94	..	998·01	..	997·98
7	·99820	·99822	·99821	8·9738	9·9735	9·9736	99·862	99·859	99·870	..	99·864	..	998·12	998·10	..	998·11
8	·99952	·99959	·99956	9·9717	9·9711	9·9714	99·823	99·823	99·830	..	99·825	..	998·38	998·38	..	998·38
9	·99714	·99723	·99719	9·9713	9·9711	9·9712	99·780	90·780	99·792	99·786	99·784	..	997·91	997·90	..	997·90
Bridge R	..	..	..	9·9811	9·9808	9·9810	..	..	99·841	..	99·841	997·79	997·95	997·92	997·82	997·87
„ L	..	..	..	9·9733	9·9735	9·9734	..	..	92·815	..	99·815	997·81	997·91	997·89	997·78	997·85

NOTE.—The box is marked by Messrs. ELLIOTT as "Legal ohms at 17° C.," hence the value of each coil, in terms of a True ohm, should be ·99757. The mean value of all the coils is ·99801, at a true temperature of 16°·92 C., or 17° by its own thermometer.

As the terminals were used for testing at the time the box was constructed, the resistance of the connections through the dials was then taken. This is sufficient to account for the apparently low value of the first unit, as the resistance of the box with all the plugs at 0 is ·00334. In testing the arms of the bridge, the centre and end terminals were used in the same manner as when the box itself formed the bridge. so that no plug resistance correction had to be applied.

A table was constructed, giving the total difference between the reading and the real value (in terms of legal ohms) for every position of the plugs in each dial. This difference we termed the "plug correction."[*] Having made this correction, we then had to correct for any inaccuracy in the ratio of the arms of the bridge, and as all determinations of the calorimeter coil resistance were made with the 1000 (right) and 10 (left) as the arms, it is here only necessary to give the correction for these coils.

Now $10L/1000R = 10·0077/1000·30 = ·0100047$. This we termed the "bridge correction."

The resulting values are expressed in legal ohms, and true ohms = reading in legal ohms $(1 - ·0024275)$.

By applying the corrections in the above form it was possible to perform all the operations by means of the slide rule,[†] as it was only necessary to determine the quantity to be added or subtracted in one case to three, and in the other to four figures.

(When working with platinum thermometers the last two corrections are omitted, as it is only necessary that the resistances should be expressed in terms of the same unit, the absolute value of that unit being unimportant).

Table XII. gives the values of R, when the calorimeter coil was in water, stirred in the same way as during a "J" determination. Each number in the column R was obtained in the manner given in full on p. 399, and includes the correction for temperature. The "plug correction" cannot be deduced from column R, because the difference between the correction on $R + r$ and r has to be used in each case. The plug correction however is, in every case, about the same as that required by the values of $R + r$ and r as given in Table IV.

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[*] The correction for the temperature of the box was made before applying the "plug correction." The values of the temperature coefficient for each dial had been previously determined by Messrs. ELLIOTT, and the accuracy of the coefficients has been exposed to severe tests.

As the temperature of the box was controlled by a special regulator, it never differed greatly from 17° C. (See 'Phil. Trans.,' vol. 182, A, p. 45.)

[†] FULLER's spiral slide rule gives results correct to 1 in 20,000 or 30,000.

TABLE XII.

Date.	C°.	R (at 17°).	R corrected for "plugs."	R corrected for bridge.	True ohms.
Aug. 6 . . .	18·701	8·70152	8·70514	8·70923	8·68809
„ 6 . . .	13·133	8·53685	8·54041	8·54442	8·52368
„ 10 . . .	25·217	8·89343	8·89701	8·90119	8·87958
„ 17 . . .	19·231	8·71690	8·72046	8·72456	8·70338
„ 24 . . .	20·337	8·74906	8·75260	8·75671	8·73545
„ 31 . . .	17·066	8·65267	8·65627	8·66034	8·63932
Sept. 4 . . .	15·325	8·60104	8·60459	8·60863	8·58773

Between September 4 and 11 the coil was readjusted, and was probably somewhat strained in the process. The value of R was slightly increased.

TABLE XIII.

Date.	C°.	R (at 17°).	R corrected for "plugs."	R corrected for bridge.	True ohms.
Sept. 11 . . .	19·560	8·73058	8·73416	8·73827	8·71706
„ 11 . . .	25·522	8·90563	8·90923	8·91342	8·89178
„ 11 . . .	13·280	8·54471	8·54826	8·55228	8·53152
„ 11 . . .	15·086	8·59816	8·60171	8·60575	8·58486
„ 11 . . .	22·154	8·80618	8·80975	8·81389	8·79250
„ 11 . . .	23·274	8·83966	8·84326	8·84742	8·82594
„ 14 . . .	16·952	8·65268	8·65628	8·66035	8·63933
„ 19 . . .	18·697	8·70415	8·70777	8·71186	8·69072

The sixth figure is, of course, unnecessary. The numbers in the first and last columns were plotted on a very large scale, so that the fifth figure could be read distinctly. The resulting curves differ in no respect from those given by platinum thermometers. The difference between the curve reading (at *any* of the above temperatures) and the corresponding value in the table in no case exceeds ·0005 (*i.e.*, 1 in 17,000).

The values of R after September 4 exceed the values before that date (when the temperature is the same) by ·0038.

The values of R (true ohms) at other temperatures than those in the above tables were obtained from the curves.*

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* In order to trace the changes in the specific heat of water throughout our range, we had decided to find the time of rising 1° at each of the following temperatures, viz., 14°, 15°, 16°, 18°, 20°, 22°, 24°, and 25° C. It would have been difficult, if not impossible, to arrange our actual observations throughout our experiments so that they were exactly grouped about these points. The value of each millim. of our thermometer stem varied as the temperature changed, and the above temperatures in no case corresponded with any integral number. The values of our variables at these temperatures had to be obtained by

As mentioned (p. 394) the resistance of the coil when immersed in pentane was ascertained. The observations were taken in the same manner and with the same rate of stirring as those whose results are given in the above tables.

TABLE XIV.

C°.	Determination of R in pentane.	
	R at (17°).	True ohms (after "plug" and bridge corrections).
13·632	8·55513	8·54195
14·010	8·56516	8·55197
14·982	8·59544	8·58223
16·202	8·62941	8·61612
17·161	8·65892	8·64557
18·214	8·69058	8·67699
19·164	8·71857	8·70506
20·502	8·75805	8·74443
21·702	8·79354	8·77989
23·001	8·83186	8·81812
24·699	8·88218	8·86836
24·699	8·88204	8·86822
24·713	8·88275	8·86891

The results must be compared with those in Table XIII., since the observations were made after the change in the coil.

In order to find what difference, if any, was caused by the change from water to pentane, the values of R in water at the above temperatures were taken from the curve plotted from Table XIII.

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large scale curves. No purpose would be served by reproducing any of these curves in this paper, because if drawn on a small scale they would be useless and misleading; for they would give an exaggerated idea of the accuracy of our results—the discrepancies being imperceptible. In every case we give data by which the curves could be reproduced, and where any marked departure is shown by an individual observation we mention it.

TABLE XV.

C°.	R (pentane).	R (water).	Difference.*
13·632	8·5420	8·5419	+·0001
14·010	8·5520 ?	8·5528	−·0008 ?
14·982	8·5822	8·5818	+·0002
16·202	8·6161	8·6148	+·0013
17·161	8·6456	8·6456	+·0000
18·214	8·6770	8·6766	+·0004
19·164	8·7051	8·7046	+·0005
20·502	8·7444	8·7444	+·0000
21·702	8·7799	8·7794	+·0005
23·001	8·8181	8·8176	+·0005
24·699	8·8683	8·8676	+·0007
24·713	8·8689	8·8680	+·0009

Mean difference = ·0004,

showing a rise of about $\frac{1}{22000}$ when in pentane.

The specific heat of pentane is so small that it is possible that the wire would be at a slightly higher temperature than when in water, as it would not be robbed so rapidly of the heat supplied by the current, and thus some portion of the small rise observable might be due to this cause.

The following tables give the resistances used by us for the reduction of our "J" experiments:—

R_0 denotes the resistance when the current is so small as to produce no visible change in R.

(R_0 = R, as deduced from Table XIII., diminished by ·0034, see p. 396.)

R_E, R_{2E}, &c., the resistance when the difference of potential at the ends of the coil is that due to 1, 2, &c., Clark cells.

($R_{NE} = R_0 + \delta R_0$, where $\delta R_0 = \cdot 00421 \times n^2$ true ohms, see p. 406.)

TABLE XVI.—Series II. Values of R. For Series I. subtract ·0038.

C°.	R_0.	R_{2E}.	R_{3E}.	R_{4E}.	R_{5E}.	R_{6E}.
14·477	8·5626	8·5794	8·6012	8·6307	8·6686	8·7149
15·581	8·5959	8·6127	8·6337	8·6632	8·7011	8·7474
16·682	8·6282	8·6450	8·6660	8·6955	8·7334	8·7797
17·683	8·6578	8·6746	8·6956	8·7251	8·7630	8·8093
18·688	8·6876	8·7044	8·7254	8·7549	8·7928	8·8391
19·835	8·7214	8·7382	8·7592	8·7887	8·8266	8·8729
21·115	8·7592	8·7760	8·7970	8·8265	8·8644	8·9107
22·409	8·7972	8·8140	8·8350	8·8645	8·9024	8·9487
23·862	8·8399	8·8567	8·8777	8·9072	8·9451	8·9914
25·006	8·8735	8·8903	8·9113	8·9408	8·9788	9·0250

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* The differences in the last column show a slight tendency to increase as the temperature rises, indicating a decrease in the resistance of water.

SECTION VIII.—THE MEASUREMENT OF TIME.

As previously mentioned, an electrical clock (made by Mr. THOMAS) was used as our standard. The energy required was supplied by two Leclanché cells. The pendulum (a seconds one) was very heavy, weighing about $\frac{1}{4}$ cwt. Its rod was of wood, and we were unable to detect that any appreciable change in rate was caused by the small changes of temperature to which it was exposed. The working parts of an electrical clock of this kind are so simple that, when once standardized, it is a very reliable instrument. The pendulum was fitted with a rough and a fine adjustment, and comparisons were made at regular intervals with a ("rated") DENT's chronometer. Until August 21, 1892, the rate of gain was $-$ 1 in 2424; after that date the rate of gain was certainly less than 1 in 25,000, and such variations as were observed may have been due to temperature changes, as they were irregular and, in any case, so small as to render corrections unnecessary. All observations of time made before the above date (August 21) were corrected for clock error, before they were used in the calculations. The actual uncorrected observation is, however, the one given in all columns headed "time."

The chronograph, which was worked by a separate battery, was of a somewhat peculiar construction. It was contrived so that on "making" the circuit connecting it with the clock, the tape was at once started, and, on breaking that circuit, the tape came to rest. Thus the observer could, without moving his position, start or stop the chronograph by pressing a key.*

We rarely, however, used this contrivance, because, as our observations lasted over considerable intervals of time, the instrument had to be continuously at work. The regulator was so adjusted that the tape passed through about $\frac{3}{4}$ of an inch per second —the fractions of a second being rapidly read off by means of glass scales. The labour of numbering the seconds was great, for the instrument was not so contrived that it could make a distinguishing mark every tenth second.†

Immediately before the observer were placed two keys connected with the chronograph, one of which made permanent connection with the clock pendulum, the other was a spring key for recording the times of passing certain temperatures, or the time of each 1000 revolutions of the stirrer.

Some preliminary observations indicated that the personal error (*i.e.*, the difference between the actual time and those read off from the tape) amounted to about $\frac{3}{10}$ of a second. As, however, all times were recorded by the same observer, this constant "lag" was of no consequence.

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* It was found necessary to place the chronograph at a considerable distance from the galvanometer.

† The Table XIX., given in Section XII., shows that one experiment alone lasted more than 10,000 seconds.

SECTION IX.—THE MEASUREMENT OF MASS.

ROWLAND, in the paper to which reference has already been made, takes objection to WEBER's determination of "J," on the ground that he used only 250 grms. of water. Also M. ARNOUX in the discussion which followed D'ARSONVAL's paper said, "Its determination has always been carried out in diminutive apparatus . . . in which the losses by radiation attained a high figure. With large apparatus the loss by radiation becomes negligible since the volume increases as the cube and its radiating surface as the square only of the linear dimensions. With appliances capable of transforming into heat a few hundred kilogrammetres per second, we ought to succeed in determining the mechanical equivalent to three or four (*sic*) places."* Now as the quantity of water used by us was small, it will be well to point out the considerations by which we were influenced.

Besides the mass of water, which is used in the calorimeter, account must be taken of what is usually termed the "water equivalent" of the calorimeter. [We take this opportunity of remarking that when we, in subsequent statements, use the word "equivalent" we refer to the water equivalent of our calorimeter, and do not use the word in the sense in which it occurs in ROWLAND's paper as meaning "the mechanical equivalent;" this we have designated by the letter "J."] Now it is the very difficulty of satisfactorily determining this equivalent, rather than the estimation of loss by radiation, which has led experimenters to prefer a large mass of water, when practicable, in order to make any error in the equivalent one of the second order.

Up to the time of writing, only two methods seem to have been generally adopted for determining the equivalent, (1) the method of mixtures, (2) the method by calculations from the specific heats and masses of the component parts of the calorimeter. Of the first of these we need say but little, for JOULE himself obtained values which varied amongst themselves by 12 per cent.† We did not, however, dismiss this method without trial, for having applied it, with every conceivable precaution, to one of our calorimeters, we obtained numbers whose differences were so great as to render them useless for the purposes of this investigation.

The second method is attractive on account of its simplicity, but since the specific heats have probably been determined by the application of the first method,‡ the accuracy is not of a superior order. Of greater weight is the consideration that a calorimeter has no definite limits, it being impossible to foretell to what extent and in what manner the necessary supports, connections, and surrounding envelopes have to be introduced. The method which we have adopted of making the experiments themselves give us the value of the equivalent appears, to us, to be the only way in which this difficulty can be successfully overcome, and, as our results will show, the equivalent was determined with the same accuracy as the other quantities.*

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* ROWLAND's calorimeter, when full, weighed over 12 kilos., JOULE's about 6 kilos.

† See Table, p. 639, 'JOULE's Scientific Papers,' vol. 1.

‡ We may quote from Prof. EVERETT's 'Units and Physical Constants,' "Specific Heat of Liquids." "We have omitted decimal figures after the fourth, as even the second figure is different with different observers." To this we may add that the specific heat of solids will bear no closer scrutiny.

There does not appear to be any argument in favour of using a small mass of water when the value of "J" is to be determined according to JOULE's original method by the friction of water; but the complexion of the case is completely changed if the mode of investigation is altered, for in each different method the accuracy of the result will be some function of the variables at our disposal, and, since we cannot apply the calculus, these ought to be so selected from general considerations, as to make that accuracy a maximum. In the method which we have adopted the variables are the mass of water, the resistance of the coil, the difference of potential at its ends, and, dependent upon these, is the rate of rise in temperature.

Now the most satisfactory method of ascertaining the rate of rise in temperature, consists in timing the transit of the mercury column across the spider-wire of a reading telescope. This wire, being set along the graduations of the thermometer, several successive transits are recorded at different parts of the range and the mean time is taken at each place. Although the rate of rise ought to be fairly great, in order to diminish the importance of radiation losses, if the rate is too rapid the observer has not sufficient time to set his cross-wire to the successive graduations, and only one observation can be recorded at each part of the range. The confusion which would be introduced by attempting to observe every other scale division need scarcely be mentioned. It was only by constant practice that one of the observers was able to take observations with accuracy at the rate we adopted.

If, on the other hand, the advance of the column is too slow, the time of apparent contact with the telescope wire is uncertain and, at the same time, the phenomenon of "sticking" is rendered more probable. As during these experiments the strain on the observer was very great, mere personal fatigue would be sufficient to prevent accurate observations being taken if the duration of the experiment was extended beyond an hour. In the case of the experiments on the heat developed by the stirrer, it will be seen by reference to the tables that these sometimes lasted three or four hours; but the nature of these experiments not being so complicated, the strain was less and the observer had short intervals of rest.

The rate of rise in temperature thus practically determines the values which we must assign to the other quantities at our disposal, for to make the rate great we must either diminish the mass or resistance, or else increase the difference of potential. The considerations which led us to give to E and R the limiting values assigned to except that every withdrawal involved four weighings, so that, after a short time, the mass of water in the calorimeter was dependent upon a very great number of weighings, and small errors in each, if cumulative, might have become serious. We, therefore, soon abandoned this as a general method, and only resorted to it occasionally. Since then we have always used the following method of adding a known mass of water to the calorimeter :—

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* The value of J can be obtained without ascertaining the value of the water equivalent (see p. 481), and the determination of the changes in the capacity for heat of water is entirely independent of it. Our conclusions are based on observations of differences in time caused by differences in the mass of water, and we see no reason why such differences could be ascertained with greater accuracy if the water equivalent was diminished. The total times would be decreased, but the differences would remain unaltered.

Pairs of glass flasks (with india-rubber stoppers), of 100, 200, and 300 cub. centims. capacity, were selected and carefully cleaned. One of each pair was kept as a tare for its fellow. In introducing the water into the calorimeter it was necessary to avoid loss either by accident or by evaporation, and to guard against losses of this kind the following device was employed :—A thin glass tube, 30 centims. in length, had a wider piece, about 7 centims. long and 2 centims. in diameter, fused to its upper end, thus forming a funnel. Up the centre of this wide portion a piece of glass rod was fixed, its upper extremity projecting above the top of the wider tube, so that the lip of the flask, from which water was poured, could rest against it, and thus prevent any drops running down the outside of the flask or tube. By forming a slight support, it also somewhat relieved the difficulty of the operation, which, even with its aid, was a trying one. A cork was fixed to the stem of the funnel, in such a manner as to permit its lower end to dip into the calorimeter. This funnel was provided with a case, consisting of a length of glass tube closed permanently at the lower, and temporarily at the upper, end by an india-rubber stopper. A piece of similar tubing, closed at both ends, was used as a tare. The full flask and funnel having been weighed, the funnel was withdrawn from its case, placed in one of the tubes leading to the calorimeter, and the water poured down it from the flask. This is the only part of the process during which evaporation could possibly take place, and any such loss, during the few seconds the operation lasted, would be a matter of no consequence. The empty flask was immediately re-stoppered, the funnel withdrawn and returned to its case, and the weight ascertained at any convenient time. This method also enabled us to add water below the dew-point of the room without any error due to moisture condensed on the flask and funnel, for it was only necessary to first weigh the flask, then cool it, and pour its contents into the calorimeter. The exterior of the funnel was dried before returning it to its case, and the stoppered flask placed in a dessicator for some time before the final weighing. This method appeared to be satisfactory, and, although it was impossible to add exactly the same amount on different occasions, still, as the difference was only the fraction of a gramme, it was easy to apply any necessary correction to the resulting numbers. A glass tube, reaching to the bottom of the calorimeter, was connected with an exhaust pump, which enabled us to withdraw nearly all the water, and the remaining moisture was removed by passing a current of hot dry air through the calorimeter for several hours.

Owing to some difficulty with regard to the air-tight joints, we were obliged to take the calorimeter to pieces at the beginning of July, 1892, and to our astonishment we found, although we had, as we thought, dried completely, there still remained several cub. centims. of water. We subsequently ascertained that the calorimeter could be properly dried in the above manner, provided that sufficient time was allowed; but the hot air had to pass for several days before a drying tube, suspended within the calorimeter for 12 hours, showed no increase in weight. Now, as the first method involved too many weighings, and the second was unsafe, unless two or three days elapsed between each experiment, we were compelled to seek a different mode of procedure. Thus, as we could neither withdraw any portion of the water, nor cool the whole calorimeter down without great labour, means had to be devised for cooling it from within. A tube, through which a freezing mixture could be forced, was evidently objectionable, as there would always be uncertainty as to whether all the freezing mixture had been removed. We found the following arrangement worked admirably, for the temperature of the calorimeter fell as rapidly when cooling, as it rose when the current was passing. We were thus able to use the same water again and so save a numerical reduction.

A very thin glass tube, closed at its lower end, passed down the support tube *F* (Plate 2, fig. 1) of the calorimeter, just clearing the base, and the upper end, which was open, projected about ½ inch above the top of the tube *F*. Care was taken to so fix this tube that no water could find its way up the annular space between the two tubes. When not in use, the upper end was closed by a cork, so as to prevent air-currents circulating down to the calorimeter. When used for cooling, this tube was partially filled with ether, through which a rapid current of *dried* air was sucked by means of the arrangement shown in Plate 2, fig. 1. The fine capillary point reached just to the bottom of the closed tube, so that, when the cooling had proceeded far enough, the remaining ether could be withdrawn by reversing the connections to the exhaust pump and the H_2SO_4 drying bottles. Any ether or impurity left at the bottom of this tube would have caused great irregularities in the experiments, so, as a precaution, only freshly distilled ether was used,* and when the temperature had been sufficiently lowered, *dry* air was passed for a considerable time and, finally, the cooling tube thoroughly cleaned by means of an absorbent mop. The joint at *G* was made by slipping a small ring of india-rubber tubing round the projecting end of the cooling tube, and, by slightly splaying out the mouth of the aspirating portion, a sufficiently good union was effected, the extreme thinness of the walls rendering a ground glass joint impracticable. The entrance tube *K* was used for adding fresh ether as occasion required.

As soon as we perceived that this method of cooling was a success, it was at once evident that it could be applied to finding the latent heat of evaporation of liquids, for by regulating the rate of evaporation we could make the cooling effect such as to counterbalance the heat developed by a known current. We have already made a few rough determinations, but prefer to postpone their publication until we have applied the method with greater precautions.

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* The ether was always redistilled by us immediately before use—at a temperature under 40° C.

SECTION X.—THE MEASUREMENT OF TEMPERATURE.

When commencing this account of our experiments, we proposed to give full details of the calibration and standardization of the mercury thermometer E_m, upon which all our measurements depend. Nearly half of our time and attention has been devoted to this part of the subject, and there are but two courses open to us, viz., to give a full account of our method of standardization, with the tables upon which our conclusions are based; or to reserve a detailed description for a further communication, and give a short summary of the results in the present paper. We have decided to adopt the latter course, for the following reasons:—

(1.) We are (with the help of Mr. CALLENDAR) now entering on a careful direct comparison of thermometer E_m with a new form of air thermometer, which, there is every reason to believe, will give very accurate results; but we are unable to assign any definite limit to the time that this investigation may take.

(2.) Dr. GUILLAUME is now* engaged in the comparison with the Paris standard of a thermometer specially constructed for us by M. TONNELOT, and some time must elapse before this comparison can be completed.

(3.) The dimensions of this communication are already very great, and if, in addition, we gave a full account of our thermometric comparisons, its length would become immoderate.

(4.) We consider that the success of the method of exact determination of temperature by platinum thermometers may now be regarded as established, and that the accuracy of the results obtained by platinum thermometers in our possession is proved by communications already published.†

(5.) The results of our J experiments may themselves be taken as a proof of the accuracy of the method of time-calibration adopted by us, and described on pp. 423 and 424, *infra*.

Assuming the accuracy of the calibration it is possible we are in error—

(1.) In the actual elevation;

(2.) In the value of our total range.

No. (1) may be disregarded, for it is only the changes in the specific heat of water which render the actual elevation of any consequence whatever. Now it is impossible that we are in error by 0°·2 C., for the comparisons we have made with the B.A. thermometers, which have been carefully standardized at Kew, would alone have been sufficient to detect any error of that kind, and an error of 0°·2 C. in elevation would mean an alteration of only 1 in 20,000 in the value of J.

No. (2) is, of course, of vital importance.

In 'Phil. Trans.,' vol. 182, A, p. 155, will be found the results of a comparison of one of our platinum thermometers (H) and the air thermometer over a range of 0° to 100°, and the numbers there given show that the probable limit of error at *any* point in the above range is $0^{\circ}\cdot01$ C. This limit may be further diminished by the "plug corrections" (see p. 410), then neglected by us. It must be remembered that this limit of error refers to the actual elevation, and is probably decreased when we come to consider differences over a range of 11° C. Further, the table there given is a record of only a single series of observations, and must include experimental errors, which would tend to mean out if sufficient observations were taken.

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* December, 1892.

† 'B.A. Report,' 1890; 'Phil. Trans.,' vol. 182, A, pp. 43–72 and 119–157; 'Phil. Mag.,' December, 1891; 'Chem. Soc. Journ.,' 1890, p. 656.

Again as regards *differences*, the quantity which chiefly affects us (apart from experimental errors, etc.) is the value of δ in the equation

$$d = \theta - pt = \delta \{ \overline{\theta/100}|^{2} - \theta/100 \} ;$$

and it is therefore important to point out to what extent the variations we have observed in the value of δ may influence our results.

We have standardized our mercury thermometer E_{m} by means of the platinum thermometer N.* (For particulars see 'Phil. Trans.,' 1891, A, p. 151.) This thermometer contained two wires, and by placing these in series a third resistance could be determined; it thus practically contained three coils, which we denote by A + B, A, and B.

The fixed points (R_{s}, R_{1}, and R_{0}) of this thermometer have been re-determined at regular intervals during the past two years, and no change of any consequence has been observed.

The values of its constants are as follows† :—

	A + B.	A.	B.
R_{0}	9·8636	5·9881	3·8762
R_{1}	13·2987	8·0743	5·2252
R_{s}	24·2703	14·7631	9·5343 hence
δ	1·641	1·645	1·640

Now, the value of d at 14° C. and 25° C. obtained by assuming the above values of δ are $-\cdot198$ and $-\cdot308$, and it is on the *difference* between these numbers (·110) that possible error in the value of our range depends. Assume it as possible that the true value of δ is as high as 1·70 or as low as 1·60 (our greatest differences in any determinations of this constant have been ·009) we should then get $-\cdot205$, $-\cdot319$, and — ·193 and — ·300 as the differences, thus causing changes of ·004 and ·003, respectively, in our range, although causing differences of ·010 and ·008 in the elevation.

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* The coil of thermometer H was so long that it was barely contained in the calorimeter, and, as it required complete immersion, it was not suitable for use in this case.

† The constants differ slightly from those previously published ('Phil. Mag.,' December, 1892), as the "plug correction" (see p. 410), has, for the first time, been applied to them. By assuming that $R = 0$ we get for the absolute zero $-270\cdot66$, $-270\cdot55$, and $-270\cdot86$ instead of the numbers there given.

Apart from errors of observation, we may, therefore, regard the limit of our range error as ± ·003 in 11° C., *i.e.*, 1 in 4000; and we believe it to be much less.

As to experimental errors, it must be remembered that we have made the comparison of the platinum and mercury thermometers under the most favourable circumstances. Both were immersed in the calorimeter, the cooling tube (Plate 2, fig. 2), having been removed to allow of the insertion of the platinum thermometer. The observer, with his eye at the telescope and his hand on the key could, by means of the current, set the temperature with extreme exactitude to any required reading E_m, stirring being maintained at the same rate as during an experiment. Observations could thus be repeated at the same reading as often as necessary.

Again, the observations taken by the observer at the galvanometer were entirely independent of the readings of the observer at the telescope, yet a difference of 0°·004 C., on the repetition of an observation, was very unusual.

Many hundreds of comparisons were made, the particulars of which we reserve for our second communication.

We may mention that the comparisons made in 1892 differ from those of 1891 by ·003 in range, but show a rise of about ·010 in the actual elevation of E_m.

E_m was one of three thermometers specially constructed for this work by HICKS. We set him a somewhat difficult task, as the dimensions of the bulb were strictly limited both in length and diameter, and the weight of mercury was also specified.

The stem was divided into millimetres, about 40 millims. to 1° C., and the lower 18 centims. were out of sight within the calorimeter tube, the graduation marked 0 being within this tube and the 26th millimetre just appearing above it.

The reading 87·5 = 13°·990 C.; the reading 537·5 = 25°·471 C.

The graduations continued to 580 millims.

Our method of observation obviated the necessity of attempting to divide the millimetres by eye. The thermometer was fixed in such a position that the view of the bore was unimpeded by any of the shorter graduations. When timing changes of temperature the spider-wire of the telescope was placed on these graduations and the time of contact with the spider-wire observed. No attempt was made to take any reading terminating in 0 or 5, as these graduations extended across the bore and obscured the edge of the mercury column for about 1/20th millim. Thus to obtain the time of passing a given point we were limited to the observation of four consecutive transits, and for this reason, our "fixed points" were determined at 87·5, 127·5, &c., and the time of passing these points deduced from the times at 86, 87, 88, and 89, &c.

A consideration of the behaviour of this thermometer (as well as of other narrow bore ones) rendered it evident that the ordinary method of calibration was insufficient if the thermometer was to be used for the determination of changing temperatures. We believe that no calibration can be considered satisfactory unless made under the *same circumstances* as those in which the thermometeris to be used. As an illustration of our meaning we may mention that the range 87·5 to 127·5 $E_m = 0°·975$ C. when the thermometer was stationary, but $= 0°·973$ C. when rising at the normal rate of our experiments, *i.e.*, in addition to calibration for inequalities in the bore, a thermometer used under circumstances similar to ours must be calibrated for "sticking" and for certain periodic alterations in the rate of advance which are probably due to the behaviour of the glass bulb. Whatever may be the case with other thermometers, a close and prolonged study of the thermometer E_m has rendered it certain that the rate of advance of the column (when the rise in temperature is uniform) is influenced by other conditions than mere irregularities in the bore, but that these changes in rate will, under the same conditions, almost always recur at the same places.

During 1891 the times of rising every 10 millims.[*] throughout the whole range were observed and the observations repeated on more than thirty occasions with varying rates of rise. Nearly half of these observations were taken with the alloy wire, whose resistance did not increase as the temperature rose, the remainder with a platinum wire. During this series of observations the weights of water were different, the supply of heat was trebled, the exterior temperature was shifted from the bottom to the top of our range, the pressure changed from 1 to 760 millims., and, in fact, all the conditions were altered. The times of rising 10 millims. under all these different circumstances were thus determined at 42 points in our range, and the results plotted; the mean path in each case was, of course, different, but the deviations from it were regular in their occurrence and similar in their nature. Such deviations as invariably occurred throughout all the altered conditions must have had their *origin in the thermometer itself.*

We hope to publish some of these curves in our further communication for, although the changes in the curves themselves are great, the steady repetition of lag and acceleration is both curious and interesting. This year these differences recurred with another coil and a different method of stirring.

The mean values of these deviations, which were repeated throughout the whole of the experiments, were expressed in terms of the curve-reading at each point and plotted as ordinates on a straight line,[†] and the remainder of the calibration completed in the usual manner. The effect of the application of the resulting corrections to the numbers obtained from our J experiments is a sufficient proof of the accuracy of the method. An error of 0°·001 C. in the comparative value of any of our ranges would produce a difference of from ·4 to ·7 second between the "mean times" given in columns 5, 9, 14, 18, and 22, Table XL., and the curve numbers in the succeeding columns. Further, this difference would *recur in each experiment.* No persistent discrepancy of any consequence is observable. In our own work we have expressed the value of these ranges to another figure, but we refrain from printing it until we produce more evidence. One possible cause of error in our total range yet remains, viz., the difference caused by unequal lag at the beginning and end of that range, and this difficulty cannot be surmounted by means of the same method as that used for determining the comparative values of the smaller ranges.

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[*] 10 millims. on thermometer $E_m = 0°·25$ C. approximately.

[†] *E.g.*, if t'' was observed time over a small range $\theta_2 - \theta_1$, and if t' was the value over that range as given by the smooth curve, then t''/t' was plotted as ordinate and $\frac{1}{2}(\theta_2 - \theta_1)$ as abcissa, and the areas thus obtained were integrated, &c.

Our experiments on this point were of the following nature :—

A platinum thermometer E (Pt) of exceedingly small capacity for heat (the external diameter was only 3/16 inch) was placed in the calorimeter. The coil of this thermometer was pressed tightly against its glass envelope, which was of egg-shell thinness, and thus the thermometer was rendered extremely sensitive. Its stem passed down a condenser-tube, through which the tank water was continually drawn by means of the water pump. The stem-resistance, therefore, remained constant, and thus only *one* observation was required, whereas with N three observations had to be taken to accurately determine a temperature. The value of R when the temperature of the calorimeter was steady was determined at the initial and final points of our range. The temperature was then lowered to our usual starting-point (10° C.), and raised in the same manner as during an experiment. The value of R was ascertained when the reading of E_m (mercury thermometer) again indicated the initial and final points. No doubt the platinum thermometer also lagged, but we proved by independent experiments that it did so to an extremely small extent, and, if uniform, the lag was of no consequence, as it would only affect the elevation and not the range. Now in the platinum thermometer we have no "sticking," &c., and there is every probability that the lag *is* regular. The observer at the galvanometer called at the moment when the "spot" was seen to pass the zero point, the other observer taking the mercury readings at the same instant. This was done over four or five consecutive plugs, and the resistance deduced at the moment of passing the required points. The results indicated that the lag of the mercury thermometer was greater at the commencement than at the end of our range by (at our normal rate of rise) about 0°·008 C.* The "stationary range" would, therefore, require diminishing by about ·008 in 11° C., or by 1 in 1400.

In order to investigate this point more fully another platinum thermometer (Q), having a naked wire, was placed in the calorimeter. The coil of this thermometer had fifty turns to the inch, and thus the different parts of the wire were very close together. The results were not satisfactory, the conduction across the water, under such circumstances, being sufficient to render the galvanometer uncertain and to produce irregularities in our results. We propose to repeat these "lag" determinations, using the naked wire, but placing pentane instead of water within the calorimeter. In the meantime we have left our total range uncorrected for the difference in the initial and final lag, as our own data are not at present sufficiently accurate for us to apply the correction, which is certainly less than 1 in 1400. Its effect would be to increase the value of J,[*] but it would not affect the changes in the specific heat of water or the calorimeter.

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* The average lag of the mercury thermometer proved to be less than we anticipated. It, of course, depends, in a great measure, on the thoroughness of the stirring; but when the mass of mercury that has to be heated is considered, the extreme lag, observed at our rate of rise, viz., ·025, is surprisingly small.

The correction will only have to be applied to the final result.

The exterior temperature was determined by a mercury thermometer A (by HICKS), which was similar in most respects to E_m, but its range was not so open—about 27 millims. = 1° C. It was placed in the calorimeter with E_m and the readings compared at every 5 millims. of E_m. When conducting our experiments we only required to know accurately the difference between the internal and external temperature, and, for such a purpose, this comparison was sufficient. As A was to be used for the reading of stationary temperatures, a careful calibration of the usual kind was made by means of the Cavendish dividing microscopes.[†]

Our method of experimenting eliminated several of the errors which are associated with the use of mercury thermometers.

(1.) Changes of temperature of stem.

The exposed portion of the stems of A and E_m was always immediately above the surface of the water in our twenty-gallon tank. The water was always maintained at a constant temperature (see p. 454) and (as we ascertained by direct measurements) the temperature of the air above it varied but slightly. The lower portion of the stem passed through a tube immersed in this tank water, and thus the stem temperature remained steady when the temperature of the bulb was rising. The change in the temperature of even the upper portion of the stem never exceeded three or four degrees throughout the year, and the change in the average stem temperature must have been much smaller. At the time of determination of the values of the ranges, the stem was, therefore, under exactly the same conditions as those prevalent during an experiment.

(2.) The effect of changes of pressure on the bulb.

The only changes to which it was exposed were atmospheric ones, which, even if they affected the elevation, would not affect the range. Again, A and E_m were so similar in construction and dimensions that the result of such changes would be nearly the same in both. Thus the differences of temperature deduced from the readings would not be altered to any appreciable extent, and it was only with such differences that we had to deal.

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[*] ROWLAND appears to have paid no attention to the effect of this difference in initial and final lag. It is probable, therefore, that the correction, if it was possible to obtain it, would cause a greater increase in his value than it is likely to effect in ours, since his average rate of rise was three times as great.

[†] This calibration is in practical agreement with another calibration of A conducted by platinum thermometers; the inequalities as determined by the latter method are, however, always greater than those shown by the former. This is to be expected when the *length* of the mercury thread used in the ordinary method of calibration is considered.

(3.) The changes in zero caused by the sudden changes in temperature.

The history of thermometer E_m* was much the same from day to day. During the night it remained about the temperature of the tank. Before an experiment, its temperature was lowered to 10° C. and, during an experiment, it was steadily raised from 10° C. to 25° C., no readings being taken until it reached 13°·9 C. When determining the fixed points, the same routine was, as nearly as possible, observed; thus the conditions were, in all cases, so alike that the effects were probably similar. In any case, we have been unable to detect any irregularities arising from this cause.

TABLE XVII.—Values of "Fixed Points" on Thermometer E_m when the Temperature is Steady.

E_m.	C.
87·5	13·990
127·5	14·965
177·5	16·198
217·5	17·187
257·5	18·180
297·5	19·196
347·5	20·474
397·5	21·755
447·5	23·061
507·5	24·662
537·5	25·471

The following Table shows the difference between the comparative values of the ranges when the thermometer is steady and when rising.

Range E_m.	Range C. Thermometer steady.	Range C. Thermometer rising.
87·5–127·5	·975	·973
127·5–177·5	1·233	1·235
177·5–217·5	·989	·987
217·5–257 5	·993	·995
257·5–297·5	1·016	1·016
297·5–347·5	1·278	1·278
347·5–397·5	1·281	1·282
397·5–447·5	1·306	1·307
447·5–507·5	1·601	1·598
507·5–537·5	·809	·811

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* E_m remained in the calorimeter from the commencement to the close of our work except on the few occasions when water was added or withdrawn.

The calibration of those portions of the thermometer between our "fixed points" was conducted in the same manner as the calibration over the whole range.

[Note by E. H. GRIFFITHS, added March 31st, 1893.

On February 14 I received the thermometer by M. TONNELOT (No. 11048), which had been standardized both by Dr. GUILLAUME, and under his direction, at the Bureau International des Poids et Mesures.

This thermometer is of hard glass, with transparent stem, and is divided into tenths of a degree Centigrade. The scale is continuous from − 3° to 27° C. when there occurs a bulb, then a narrow portion giving readings 48° to 52° C., then a second bulb, and, finally, readings from 98°·5 to 101°·5 C. The length of each degree is about 10 millims.

The study of this thermometer has been a most exhaustive one; two separate calibrations were performed for the most useful part (10° to 26°·5 C.)—the *greatest* difference between any two values given by the separate determinations being 0°·0028 C. The fundamental interval and the coefficients of external and internal pressure were repeatedly determined. Several hours must have been occupied in the mere copying of the observations forwarded to me—although the record consists, almost entirely, of numerals. I mention this as indicating the labour which has been devoted to this standardization, and I owe my warmest thanks to the Bureau International for the care and attention bestowed upon it.

The papers include printed tables for the reduction of the corrected mercury readings both to the hydrogen and the nitrogen scale as resulting from the comparisons of Dr. CHAPPUIS.

The comparison between this thermometer (denoted hereafter by P) and the thermometer E_m was conducted in the outer tank, for had the comparison been made in the calorimeter, the lower readings of P would have been invisible. E_m was, therefore, under the same conditions as during our J experiments, except in the following particulars:—

(1.) The external pressure was increased by about 110 millims. of water. A series of observations proved that the resulting correction was − ·3 millim. (rather less than 0°·001 C.) and this correction was always made before entering the reading of E_m.

(2.) When in the calorimeter, the lower portion of the stem passed through tubes washed by the tank water. About 60 millims. were thus maintained at a constant temperature, whatever the extent of alteration in the bulb temperature. About 50 millims. intervened between the top of the calorimeter and the lower end of this constant temperature portion. We may consider that this 50 millims. assumed the mean temperature between the tank and the calorimeter. Thus a length of (60 + 25) millims. of the stem, which during our previous experiments was at the tank temperature 19°·26, assumed, during this comparison, the temperature of the bulb θ_1.

The correction in millimeters is given with sufficient accuracy by the formula

$$85\ (\theta_1 - 19{\cdot}3) \times {\cdot}00016,$$

and as the mean value of 1 millim. is 0°·025 C., the resulting correction is

$$85\ (\theta_1 - 19{\cdot}3) \times {\cdot}00016 \times 0^{\circ}{\cdot}025\ \text{C.} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (\alpha).$$

This amounts in the greatest case to 0°·002 C.

The readings of P were taken by a microscope (re-levelled before each observation) which was fitted with a micrometer scale. The one hundredth of a degree Centigrade was thus read directly and no difficulty was experienced in estimating the one thousandth, the scale being an unusually clear one, made by the Cambridge Scientific Instrument Company.

The thermometer was placed in a tube, open at the lower extremity, and an opening at the upper end was connected with a rubber tube leading to the waste pipe. This arrangement was used as a siphon and the thermometer stem was thus maintained at the bulb temperature, and, therefore, as the mercury column rose, any alteration in the stem immersion was rendered unnecessary.

In order that the readings might be obtained under the exact conditions observed by Dr. GUILLAUME, I implicitly followed the instructions given in a printed note added to the certificate. After the tank had arrived at the required temperature, at least half-an-hour was allowed to elapse before taking a reading, and immediately after the observation the thermometer P was transferred to ice, and a series of readings taken to obtain the greatest zero depression. The admirable methods adopted by the Bureau International are less known, or, if known, are less practised in this country than they deserve to be. I therefore give one example in full showing the various corrections.

COMPARISON of the Reading 87·5 on E_m with P (March 15, 1893).

	Temperature reading by P. millims.	Zero point. millims.
Barometer (corrected)	754·1	754·1
H_2O pressure (expressed in terms of Hg)	12·1	4·4
Total external pressure	766·2	758·5
Observed reading	14·025	— ·036
Calibration correction	— ·026	0
External pressure correction	— ·001	0
Internal ,, ,,	+ ·026	+ ·008
Zero	+ ·028	— ·028
Fundamental error correction	— ·006	
Sum corrections	+ ·021	
Correction for stem E_m [see (α), *supra*]	— ·002.	

Hence reading on the mercury scale = 14·044.
Correction to H scale — ·067; to N scale — ·059.

Hence

Reading E_m.	Hydrogen scale.	Nitrogen scale.
87·5	13·977	13·985

I confess that I did not know until I adopted in detail the precautions advised by Dr. GUILLAUME in his "Traité de Thermométrie," that it was possible to regard the mercury thermometer as an instrument of precision, and I had considered such corrections as too refined for so rough an instrument. I am, however, now convinced that, given the *right conditions*, it is possible to ascertain temperatures by a mercury thermometer with a limit of error of 0°·002 or 0°·003. It is rarely, however, that such conditions prevail. The labour involved in the previous standardization of the thermometer is enormous, and there are but few men capable of constructing thermometers on which such labour may be profitably expended. The actual operation of observing and reducing the results is also considerably greater than is the case when platinum thermometers are used.

The following table gives the results of all observations made with the micrometer eye-piece. I had previously made several comparisons whose mean result is in fair agreement with those here given; but, as the last two figures of the readings had to be estimated, the discrepancies are greater, and the results of less value.

COMPARISON of E_m with the French Standards.

Reading E_m.	87·5	127·5	177·5	217·5	257·5	297·5	347·5	397·5	447·5	507·5	537·5
	θ_1 on the corrected mercury scale.										
March 14	—	—	—	—	—	—	20·541	21·828	—	—	—
„ 15	14·044	15·022	16·260	17·249	—	—	—	—	—	—	—
„ 16	14·044	15·020	—	—	—	—	—	—	—	—	—
„ 17	14·045	15·023	16·259	17·248	18·245	19·261	20·544	—	—	—	—
„ 18	—	—	—	—	—	—	20·545	21·832	23·140	24·740	25·552
„ 19	—	—	—	—	—	—	—	—	23·138	24·738	25·550
Mean . .	14·044	15·022	16·260	17·249	18·245	19·261	20·543	21·830	23·139	24·739	25·551
	Hence θ_1 on the hydrogen scale.										
H	13·977	14·952	16·186	17·172	18·165	19·178	20·457	21·741	23·048	24·645	25·455
	And θ_1 on the nitrogen scale.										
N	13·985	14·960	16·193	17·181	18·174	19·188	20·467	21·751	23·058	24·655	25·466
	θ_1 as determined from platinum thermometers (see Table XVII.).										
Table XVII.	13·990	14·965	16·198	17·187	18·180	19·196	20·474	21·755	23·061	24·662	25·471
Difference between N and Table XVII.	·005	·005	·005	·006	·006	·008	·007	·004	·003	·007	·005

Mean difference N—XVII = — ·0055.
Mean discrepancy from the mean ± ·0011.
Difference in range between 14° and 25° : ·001.

1°. *Étude du thermomètre* P.

a. Calibrage (y compris l'interpolation dans la supposition que la division est équi distante);

b. Détermination du coefficient de pression;

c. Détermination de l'intervalle fondamental.

Les corrections résultant de ces deux dernières études n'agissent que sur le *range;* et sont de peu d'importance (moins de ·001 degré) sur un intervalle de 10 degrés. Le calibrage, au contraire, peut introduire de point en point des erreurs irrégulières. J'ai montré autrefois* que, dans un calibrage fait avec tous les soins possibles par un observateur très-exercé, l'extrême limite d'erreur, pour un thermomètre divisé en dixièmes de degré est de ·001; le calibrage du thermomètre P a été fait avec soin, mais par une méthode abrégée et l'on peut s'attendre à des erreurs de ·002 environ sur les points déterminés directement; le tube étant assez irrégulier les erreurs des points interpolés peuvent atteindre ·003.

2°. *Réduction au thermomètre à azote.*

Cette réduction se fait au moyen des tables générales déterminées par la comparaison, faite une fois pour toutes, d'un groupe de thermomètres en verre dur avec le thermomètre à gaz. Dans l'application de cette table au thermomètre P, on suppose que les corrections sont les mêmes, c'est-à-dire que le verre et le mercure de ce thermomètre sont identiques à ceux des thermomètres-étalons. Cette identité thermique a été vérifiée approximativement par des comparaisons précises faites sur un grand nombre de thermomètres; mais si l'on songe: (*a*) que le verre est un corps chimiquement compliqué, et dont la dilatation est soumise à des lois singulières; (*b*) qu'il suffit d'une différence relative de $\frac{1}{400}$ ($2 \cdot 10^{-11}$ en valeur absolue) dans le coefficient de θ^2 dans la formule de la dilatation linéaire du verre pour provoquer une différence de ·001 degré dans les indications du thermomètre à 50°, on ne peut être que fort étonné d'une telle concordance, qui résout la question vitale du thermomètre à mercure, et détruit tous les préjugés que l'on avait contre cet instrument, aussi bien que sur la dilatation du verre.

3°. *Comparaisons.*

Celles-ci comprennent:—

(*a.*) les observations à diverses températures;

(*b.*) les déterminations du zéro.

On peut vraisemblablement s'attendre à ce que les erreurs de température, les défauts de l'éclairage, les irrégularités du ménisque provenant en partie des irrégu-

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* "Etudes thermométriques," 'Travaux et Mémoires du Bureau International des Poids et Mesures,' tome 5, p. 23 (1886).

larités du tube, enfin les erreurs de l'estimation produisent au total des discordances de ·001 degré au minimum dans (*a*) et dans (*b*).

Toutes les causes mentionnées ci-dessus et ne concernant que le thermomètre P auraient pu, sans aucune doute, conduire dans la réduction au thermomètre à azote à des erreurs irrégulières atteignant ·005.

La *différence systématique* de θ_1 ·005 en moyenne s'explique suffisamment par deux causes :—

(*a*.) Pression capillaire moindre dans la colonne toujours descendante à zéro que dans la colonne à ménisque le plus souvent ascendant ou en équilibre moyen aux autres températures.

Dans les tubes de thermomètres l'angle de raccordement varie entre 46° et 75° environ, ce qui, dans les tubes de ·1 millim. de diamètre, correspond à une variation de pression de 50 millims. de mercure au maximum.

(*b*.) Le zéro se relève dès l'instant où le thermomètre est plongé dans la glace ; ce relèvement au moment initial est à peu près indépendant de la dépression, et dépasse ·001 degré par minute.

Ces deux causes réunies suffiraient à expliquer des différences constantes plus fortes même que celle qu'a trouvée M. GRIFFITHS ; pour la plupart des recherches, comme dans le cas actuel, elles sont sans nulle conséquence. Il me sera permis en terminant d'insister sur la sécurité que donne aux mesures de températures le contrôle si indirect et en même temps si parfait qui résulte des comparaisons de M. GRIFFITHS, et d'exprimer le vœu que la vérification soit bientôt étendue à un intervalle de température aussi large que possible. Jusqu'à ce jour, l'étude du thermomètre à mercure au Bureau International a été faite entre − 25° et + 100° ; l'échelle sera prochainement étendue jusqu'à 200°. Le thermomètre à toluène a été parfaitement étudié et comparé au thermomètre à hydrogène jusqu'à − 75° ; à cette température la discordance entre divers instruments étudiés individuellement n'atteint pas ·05 degré."]

SECTION XI.—THE HEAT DEVELOPED BY THE STIRRING.

During our earlier experiments (1887) we stirred by means of a screw which revolved about 200 times per minute. Throughout 1888 and 1889 we continued to use the same form of stirrer, but endeavoured to make the mixing more efficient by increasing the rate of revolution. As our experience increased we became strongly impressed with the necessity of *thorough* mixing. Any slackness in this respect makes itself evident (*a*) in changes in resistance of the wire, owing to changes in its rate of cooling; (*b*) irregularities in the radiation and conduction, probably owing to varying temperatures of the lid; (*c*) irregularities in the thermometer readings; (*d*) in the water equivalent, which then is found to alter to some extent with the mass of the contained water.

One assumption underlies all the calculations, viz., that whatever the mass of the contained water, the water equivalent of the calorimeter remains constant.* We find this assumption is not correct, unless the water is thrown against every part of the calorimeter throughout each experiment. At first sight it would appear that, whatever might be the "lag" of any unwetted portion of the calorimeter when the flow of heat had become steady, the quantity of heat absorbed by it for each rise of 1° C. would be the same, and that it would only be at the commencement of each experiment that differences would be observable. It must, however, be remembered that the radiation coefficient of the whole calorimeter would be altered by the existence of this colder portion; and, as this cause of error would continue to exist throughout the whole experiment, the effect would be appreciable.

In 1891 we adopted a new form of stirrer. An ebonite shaft, passed down the central tube, and terminated at its lower extremity in an agate point, which rested on a metal bearing on the base of the calorimeter. Two gold tubes joined together in the form of a V were fastened to this shaft, the apex of the V being downwards and immediately above the agate point. This stirrer acted as a centrifugal pump, and, when revolving at a rate of about 1200 per minute, threw the water upwards and outwards with great rapidity—the whole contents of the calorimeter being removed in a few seconds when the lid was off. There could be no doubt that, provided there was sufficient water to at all times cover the lower openings of the tubes, the stirring was sufficient, and that no portion of the calorimeter could remain for any appreciable time at a different temperature to the water. This form of stirrer appeared to us so satisfactory that we did not attempt to improve upon it during our experiments in 1891; and it was this misplaced confidence which was the chief cause of our failure, in that year, to obtain satisfactory results.

Its defects may shortly be summed up as follows:—

(1.) As the depth of water in the calorimeter increased, the work expended in forcing the tubes through the water also increased; and the rate of rise in temperature (which ought to have diminished) was much greater when the mass of the water was large.

(2.) This change in the quantity of heat developed would not have been fatal, since it could have been determined separately for each mass of water used, and we persevered in our attempts to thus determine it. Unfortunately, however, it varied capriciously, even when the mass of water and rate of revolution were constant; and we have never arrived at any satisfactory explanation of these variations. It appears probable, however, that their origin was due to differences in pressure between the agate point and its bearing, the slightest variation in the position of the shaft of the stirrer causing very different quantities of heat to be generated. The quantity of heat generated by the stirrer varied from about one-sixth to one-fifteenth of the heat generated by the current, according to the electromotive force used; and as the capricious changes above referred to amounted, in some cases, to 5 or 6 per cent. of the heat due to the stirring, they might affect our ultimate result by as much as a half and, in some cases, nearly 1 per cent. The quantity of heat appeared to vary with the rate of revolution in such a manner that $tr^{5/2}$ was a constant, where t was the time of rising 1° C. and r the rate of revolution.

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* If the water equivalent varies, the fact is rendered very evident when the values of J deduced from different masses of water are compared. Until, in 1891, we adopted forms of stirrers which threw the water against the roof of the calorimeter, our experiments were rendered useless by this constant source of error. The agreement between the values obtained in 1892 is a satisfactory proof that this difficulty had been overcome.

Our experience of this stirrer led to the following conclusions:—

(1.) The ideal stirrer must be able to throw the water from the bottom to the top of the calorimeter whatever the mass of the water.

(2.) The "work done" by the stirrer should be the same, whatever the depth of the water.

(3.) The "work done" by the stirrer should be, as far as possible, employed in throwing the water upwards to the lid, rather than in simply causing a rotation of the water near the bottom of the calorimeter.

(4.) The whole of the energy should, if possible, be employed in lifting the water and not in overcoming the friction between solid surfaces, *i.e.*, the bearings should be outside the calorimeter.

Although we found it impracticable to devise a form which would entirely fulfil the above requirements, the one adopted by us during our 1892 experiments was satisfactory.

During the spring of 1892, we made a large number of trial experiments with different forms of stirrers, and it should be borne in mind the form it ultimately assumed was an example of the "survival of the fittest." A section will be found in Plate 2, fig. 2.

A cylindrical tube *AB*, 1 inch in diameter, closed at the lower end, rested on four little feet about 1/16 of an inch in length, which were fixed to the base of the calorimeter. The plate at the lower end was perforated in the centre, so as to allow the end of the revolving shaft to pass through without touching it. Round this centre hole were four slits through which the water in the calorimeter passed into the cylinder. The end of the revolving shaft was fitted with a small cylinder of agate, which hung loosely within a ring fixed to the base of the calorimeter at *C*. The end of the agate was not in contact with the base of the calorimeter. The bearing by which the stirrer was supported was fixed at the top of the central glass tube, and thus any increased downward thrust of the stirrer did not alter the friction within the calorimeter. The stirrer consisted of an inverted cone with nearly vertical paddles at its extremities (a section is shown at *D*). When the stirrer revolved, the water was drawn in through the base of the cylinder, thrown outwards by the paddles, and, being unable to escape at the lower end, mounted rapidly to *A*, whence it was ejected with considerable violence against the roof of the calorimeter. If the rate exceeded about twenty-six revolutions per second, it was found that when the lid was removed, the water was thrown many inches above the top of the cylinder, while the stirring-shaft remained dry. A small window at *E* caused a stream of water to be continually thrown on to the top of the thermometer bulb.

The bearings at the top of this shaft ultimately assumed the form shown at *H*. A small block of steel was fastened by a screw to the shaft, and the lower end of this steel terminated in a projecting ring, which revolved in a brass bearing of the form shown at *H*. This brass bearing was, as already mentioned, fixed to the top of the central glass tube. A drop of oil placed in the channel, on the top of the brass, rendered this bearing practically air-tight, at all events, sufficiently so to prevent any possibility of evaporation.* The shaft from *H* to the top of the calorimeter was about 8 inches long, and made of ebonite; the conduction of heat along this shaft was, therefore, very small. The heat generated in the bearing at *H* passed almost entirely into the water of the outer tank, for the brass tube which terminated there was continued downwards to the steel lid, and thus the lower three inches were immersed in the water.

The portion at *D* was at first shaped like a horizontal paddle-wheel. It was, however, found that although it would work excellently for a time, a gradual accumulation of air at the centre of the paddle-wheel greatly diminished its efficiency. Many attempts were made to surmount this difficulty, and it was only by the conversion of the upper portion into the form of an inverted cone that it was finally overcome.

To adjust the stirrer, the screw in *S* was loosened and the shaft lowered until the end of the agate touched the base of the calorimeter. The stirrer was then raised about 1 millim., and fixed in that position by means of the screw at *S*. When investigating the action of these stirrers, the lid of the calorimeter was replaced by a sheet of glass which prevented the contents from being ejected. As far as it was possible to see the interior through the shower of water that poured against the glass, the water without the cylinder assumed a slow circular movement only. At the same time, there can be but little doubt that the entire contents passed from the bottom to the top within the space of a second or two. As long as the volume of the contained water exceeded 130 cub. centims., the throw appeared to be the same, whatever the quantity, and, as the figures will show, the work done altered only slightly.

Our first series of experiments was performed with the object of ascertaining (1) If, under the same conditions, the work done by the stirrer was constant; (2) In what respect the work done varied, if at all, with the mass of water; (3) If there was any constant relation between the heat developed and the rate of revolution.

The results of our preliminary series are given in the following table, the rise in temperature being expressed in millimeters of thermometer E_m, since our object at this time was not to determine the actual quantity of heat developed, but to investigate the points mentioned in the above headings.

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* It was found that this "join" would support a pressure of 7 or 8 inches of water without leakage.

TABLE XVIII.—Experiments indicating that tr^2 = constant.

(t = time of rising 1 millim. r = rate of revolution in seconds.)

Date, [illegible]	Weight.	Exp.	Range in millims.	Time in seconds.	r	t	[illegible]	Remarks.
July [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Preliminary		Outer temp. [illegible]
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	Thermometer not in position		
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
							[illegible]	
July [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Outer temp. [illegible]
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
							[illegible]	
July [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Outer temp. [illegible]
July [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Outer temp. [illegible]
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
							[illegible]	
July [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
July [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	210	Outer temp. [illegible]
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	71·[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
							[illegible]	
July [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Outer temp. [illegible]
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
[illegible]	—	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	
							[illegible]	

From the above results we drew the following conclusions :—

(1.) That under the same conditions the quantity of heat developed was the same.

(2.) That the work done increased slightly with the mass of water.

(3.) That tr^3 was a constant quantity.*

If Nos. 1 and 3 were established, No. 2 was of little consequence, as we proposed to determine the heat developed for each mass of water.

When it is remembered that our "rate" varied in the above experiments from 26 to 34 revolutions per second, the close approximation of tr^3 to a constant is remarkable, and, as we proposed to maintain a rate of, as nearly as possible, 30 revolutions per second throughout our experiments, it was evident that any assumption based on this conclusion, could lead to no appreciable error. Again, a considerable variation in the value of tr^3 would have but little effect on the value of t, *i.e*, the time of rising 1° C.

These experiments were regarded as only preliminary, and no special care was taken with regard to external temperature or pressure, and, it is probable, that many of the apparent discrepancies arose from such causes. Again, when the movement of the mercury column was so slow that from 50 to 100 seconds were taken in rising 1 millim., it was difficult to estimate with great precision the time of apparent contact of the mercury column and the spider wire of the telescope. Personal errors of this description must, however, in the long run, tend to "mean out" and give a true average.

Had we been able to bring our mechanical arrangements to such perfection that the rate of revolution was absolutely constant, it would have rendered the above investigation unnecessary, for the stirring correction would have been constant whenever the mass of water was the same.

Although by the pressure regulator, described p. 378, the regularity of the motor was increased to an extent which those accustomed to the behaviour of these instruments hardly anticipated, we were unable to ignore the residual variations. It was, therefore, necessary to record the revolutions (the time per 1000 being taken in each case) throughout all our experiments. The labour, both of observing and calculating, was thereby greatly increased; but as we had no better means at our disposal, we had to submit to the inevitable.

The establishment of the law that (with this stirrer) tr^3 might be regarded as constant, enabled us to apply with certainty the necessary corrections.

As previously stated, our object was to deduce the rise in temperature per 1 second, at rate 30. If t is time of rise of 1° in any scale, then $1/t = \mathrm{K}r^3$, therefore, $\delta \,.\, 1/t = 3\mathrm{K}r^2 \,.\, \delta r$. Knowing $1/t_2$ (the rise per 1 second at any observed rate), and $1/t_1$ (the rise at rate 30), we know $1/t_1 - 1/t_2$, that is the term required to reduce the observation to the rate 30.

The accuracy of the correction can be estimated by means of the above table. For example, the mean value of tr^3 for the weight 194·20 is 1600, the greatest departure from the mean is 1611 in Experiment 13. By assuming $tr^3 = 1600$, we find the rise per 1 second at rate 30 = ·001687 (expressed in millims. of E_m), whereas if we assume that $tr^3 = 1611$ and correct Experiment 13 (rate 26·33) to rate 30, we get ·001676.* Again, the mean value of tr^3 for weight 285·37 is 2119, which gives the rise at rate 30 as ·001274. Experiment 22 (rate 26·60) is an extreme case; this would give ·001265 at rate 30.

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* t = time rising 1° (in any scale), in seconds.
r = rate of revolution, *i.e.*, number of revolutions per second.

Experiment 7 (weight 104·02) shows a greater divergence from the mean value, since it would give a rise of ·00242 per 1 second as against ·00238, the value as given by the mean tr^3. As, however, we discarded all J experiments where the weight of water was less than 130 grms., this extreme case did not affect our results.

During our J experiments our rate was always between 29 and 31 (generally between 29·8 and 30·2), hence the divergences in the cases, above selected, are far greater than any which occurred during our experiments.

The value of tr^3 appeared to increase very regularly as the quantity of water increased, and thus changes in its value caused by small changes in the mass could be deduced. Thus experiment 30 (239·27 grms.) gave a value of 1839. Hence we deduced that, if the weight of water was 239·73 (as in Nos. 17 to 19), the value of $tr^3 = 1843$ as against 1842, the mean of the actual experiments with the latter weight.

Had the work done by the stirrer been independent of the quantity of water, we could have deduced the water equivalent of the calorimeter from the values of tr^3, for, if K_1, K_2, etc., are the values of tr^3 when W_1, W_2 are the weights, we have $\frac{W_2K_1 - W_1K_2}{K_2 - K_1}$ as the value of the water equivalent. Using the values of K deduced from Table XVIII., we get

W.	K.	Water equivalent.	M/K.
104·02	1134	115·5	·1418
194·20	1600	105·7	·1740
239·73	1843	73·5	·1769
285·37	2119		·1753

We ultimately ascertained that the water equivalent was as nearly as possible 86·0. Assuming this value and denoting by M the mass of contained water + water equivalent, then, if the work done by the stirrer was constant, M/K would be constant The numbers in the last column show that the quantity of heat developed increases rapidly at first and arrives at a maximum.

This maximum, as well as we can estimate, occurs when the quantity of water is sufficient to fill the calorimeter to about the top of the tube up which the water is thrown, and it is possible that the conditions would, at this depth, change slightly.

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* We required but *three* significant figures when we applied a similar correction to our J experiments.

In any case, the work done by the stirrer alters but very little as the height of water changes, provided that 130 cub. centims. is regarded as the minimum quantity.*

The examples given above show that the limit of error introduced by applying this method of reduction to variations far exceeding any occurring during our experiments is less than 1 per cent., and as the heat generated by the stirrer during the majority of our 1892 experiments varied from about $\frac{1}{8}$ to $\frac{1}{16}$ of the heat due to the current, it is evident that any error must be less than $\frac{1}{1000}$ of the whole supply. Bearing in mind the small variations which actually occurred, we may safely regard the limits of error as a small fraction of this $\frac{1}{1000}$.

As will be seen later on, we performed an experiment in which the current was so diminished that the stirring supply was half that due to the current (see J 9), and another in which the current was so increased that the proportions were as 1 : 20 (see J 34). Any inaccuracy in the determination of the stirring supply would have caused considerable divergence in the corrected results; but the numbers obtained are themselves a proof of the accuracy of the method adopted. (See note, p. 406.)

It appeared possible that changes in the viscosity of water at different temperatures would have an influence on the results.

This may be the case, although our experiments do not lead to that conclusion (see p. 448). If so, however, it would not affect us, for the total heat lost or gained by the calorimeter at different temperatures (independently of the current supply) was directly determined for each weight of water, and therefore although our values of the radiation coefficient, &c., might be modified, the expression for the total loss or gain would remain unchanged.

Want of time alone prevented our investigating this matter more fully, but the consideration that neither the value of J, nor the changes in the specific heat of water would be affected thereby, justified its postponement to another season.

No use was made of the numbers obtained in Table XVIII.; our only object being to establish the law $tr^3 = K$. This was fortunate, for on afterwards examining the water used for this series we had reason to believe that it had become slightly soiled. We found that the lead ring used in the calorimeter lid had been insufficiently gilded, and also we had not allowed the gutta-percha varnish, which we used as a "size" for the gold leaf, to harden sufficiently before introducing the water.

The results of nearly ninety additional experiments of a similar kind indicate that the irregularities of the numbers in the above table are unusual, and that the actual values of tr^3, obtained from them, are unreliable, but that the conclusion drawn (viz., that under the same circumstances tr^3 is a constant) is correct.

Very small changes in the adjustment caused considerable changes in K, but its value remained constant so long as the adjustment was unaltered. Any alteration in the fixing of the stirring shaft necessitated a re-determination of K.

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* This minimum quantity, 130 cub. centims., is deduced from the results of later experiments.

After experiment J 19, the whole apparatus was taken to pieces. It was then found that small fragments of gutta-percha had, by the continued impact of the water, been detached from the ring of gutta-percha which had been placed between the cooling tube and the tube down which it passed. Many of these fragments had collected beneath the base of the cylinder at B* and must have considerably impeded the flow of water. In consequence, therefore, before putting the apparatus together again, we removed all remaining gutta-percha surfaces which were exposed to the action of the water. The values obtained for K were, from this time, very different from preceding ones, and, once obtained, remained constant as long as the same mass of water was used.

The experiments J 20 to 34, are in better agreement amongst themselves than the earlier ones, and the cause no doubt is that just indicated.

The mean result of the earlier experiments is satisfactory as likewise the mean time of each experiment, but the times over small ranges (especially in experiments Nos. J 5 to 12) leave much to be desired. The fact that the quantity of heat developed by the stirrer was increased by the removal of the gutta-percha fragments is a proof that the flow of water was appreciably diminished by the obstruction.

SECTION XII.—THE GAIN OR LOSS BY RADIATION, &c.

It has already been pointed out that the method of observation adopted enabled us to determine, with sufficient accuracy, the total loss or gain of heat per second at any temperature, apart from the supply due to current. We propose to use the phrase "non-electrical supply" to denote heat due to all other sources than the current.

By the method described in Section XI., we were enabled to deduce the rise per second when the rate of revolution of the stirrer was 30, although the actual rate differed slightly from our standard one. The rate of rise, when radiation, &c., was eliminated, was first ascertained by noting the time of a small rise across the outside temperature. The temperature of the calorimeter was then gradually raised through the whole range by means of stirring only, and the times ascertained of small changes at the different temperatures. Assuming the "stirring heat" to be the same at all temperatures (when the rate is the same), we can deduce the radiation coefficient. (By radiation coefficient we denote the loss or gain in temperature per second, due to the combined effects of radiation, conduction, and convection, for a difference in temperature of 1° C).

One advantage of this method is, that all observations of temperature are taken on a rising thermometer, for "sticking" is far less likely to occur when the mercury is advancing than when receding.

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* Plate 2, fig. 2.

We have found that all observations of slowly falling temperatures (when taken with a mercury thermometer) are unsatisfactory for the above reason. Again, the conditions resemble more nearly, in every respect, those prevalent during a "J" experiment. On this account, we did not desire to reduce the "stirring heat," even if we had been able to do so, since the loss, due to radiation, &c., would, at the higher portions of the range, have mastered the stirring supply, and the observations would have had to be taken on a falling thermometer, a mode of observation which, guided by the results obtained in previous years, we had decided to reject.

It appeared unlikely that NEWTON'S Law of Radiation would hold over our range, and we anticipated a decided curvature in the line showing our "non-electrical supply."

We first proceeded to determine isolated points on this curve (ordinate = rise in temperature per second, abscissa = difference in temperature between the calorimeter and the outside temperature), but these appeared to fall so nearly on a straight line, that we decided to follow its course over our range of temperature, using a small mass of water so that the changes in the value of the rise should be as marked as possible. A very large number of experiments were made in order to decide this point—for, if the exponential formula given by DULONG and PETIT holds, the curvature ought to have been marked. Although changes in the specific heat of water and of the calorimeter would influence the inclination of the line, it appeared unlikely that they would affect it in such a manner as to straighten it; and further, the effect of such changes would alter as the mass of water changed.*

The numbers alone resulting from these observations half fill a large note-book, and to give them in full would occupy too much space. As an example, we venture to give experiments 51 to 54, exactly as they appeared in our notes. As the method employed never varied, we will, in other cases, simply give results, and we select this series because it was the first done by us with the object of tracing the "non-electrical supply" curve throughout the greater part of our range.

The numbers in the columns marked "time," are those given by the chronograph tape. New tapes were used after the readings at 343, 407, and 458, and the times of rising between these temperatures were not noted; therefore, the experiments at those ranges are numbered as separate experiments although taken consecutively.

No temperatures were taken whose readings ended in 0 or 5 because, as these were denoted by longer lines on the thermometer scale, the time of contact with the "spider wire" was not sufficiently definite, and the thickness of the graduation would have made a considerable difference in the value of *t*.

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* We may state at once that (although some slight signs of curvature are visible when the mass of water is great) by assuming the non-electrical supply to be a linear function of the temperature (when the rate of stirring is constant), we introduce no error sufficient to affect our results, and this conclusion is based on the data supplied by more than 100 "stirring and radiation" experiments. If the results were expressed in terms of the mercury thermometer scale, the curvature would, of course, be marked.

We did not attempt to record the time for every individual 1000 revolutions, but sufficient observations were taken to enable us to detect any change in rate.

The pressure in the annular space round the calorimeter was high (·98 millim.) during experiments 51 to 54, as we had only recently commenced re-exhausting.

TABLE XIX.—Experiment LI. Aug. 11, 1892.

McLeod gauge = 62 millims. Outside temp. = 241·3 A, = 300E_m.
Pressure = ·98 millim. Mass of water (*in vacuo*) = 103·01 grm.
Clock error = + ·0004125.

Experiment.	Temperature E_m.	Time.	Revolutions.	Time.
		seconds.	thousands.	seconds.
LI.	86	111·0	6	0·8
	87	137·9	8	67·6
	88	164·8	10	134·0
	89	193·0	1	167·4
	91	248·4	2	200·7
	92	275·9	3	234·3
	93	302·5	4	267·7
	94	330·2	5	300·9
	96	386·0	6	334·0
	97	412·5	7	367·3
	98	..	9	433·5
	99	469·3	21	501·0
	101	524·4	2	533·9
	102	552·8	3	567·2
	103	579·8	4	601·0
	104	609·8		
	..	..	38	1065·2
	121	1095·6	9	1098·4
	122	1124·2	40	1131·6
	123	1154·4	1	1164·8
	124	1183·3	2	1197·9
	126	1244·3	4	1264·6
	127	1274·0		
	128	1304·0	7	1364·7
	129	1334·2	8	1397·8
	131	1394·6	50	1463·9
	132	1425·4	1	1497·3
	133	1455·3	2	1530·5
	134	1485·0	53	1563·5
	136	1546·0		
	..	..	76	2332·4
	161	2345·9	7	2365·8
	162	2379·4	8	2399·3
	163	2412·0	9	2432·6
	164	2444·9	80	2466·0
	166	2511·2	1	2499·3
	167	2544·0	3	2565·8
	168	2578·0	4	2599·4
	169	2611·7	5	2632·5
	..	..	109	3435·0

Table XIX.—continued.

Experiment.	Temperature E_m.	Time.	Revolutions.	Time.
		seconds.	thousands.	seconds.
LI.	193	3440·9	110	3468·3
	194	3477·5	1	3501·6
	196	3548·0	3	3568·2
	197	3583·8	4	3601·8
	198	3619·6	5	3635·0
	199	3656·6	6	3668·5
	201	3727·2	7	3701·7
	..	..	8	3735·0
	226	4665·0	145	4633·2
	227	4704·5	7	4699·9
	228	4743·0	8	4733·2
	229	4782·0	9	4766·6
	231	4859·9	150	4800·0
	232	4900·0	1	4833·3
	233	4939·2	2	4866·6
	234	4978·8	4	4932·9
	..	..	155	4966·2
	252	5719·8	178	5732·0
	253	5762·9	180	5798·9
	254	5806·7	2	5865·9
	256	5892·1	4	5932·4
	257	5935·8	6	5999·3
	259	6023·0	8	6066·0
	261	6110·6	190	6132·8
	262	6154·6	2	6199·3
	276	6787·7	210	6797·6
	277	6834·6	2	6864·3
	278	6881·9	5	6964·4
	279	6929·4	6	6997·5
	281	7021·3	8	7064·0
	282	7071·5	9	7097·3
	296	7763·0	240	7797·7
	297	7813·7	1	7831·0
	298	7864·7	3	7897·8
	299	7917·1	245	7964·5
	301	8020·9	6	7998·0
	302	8073·4	8	8064·9
	303	8127·0	250	8131·9
	304	8179·4	2	8198·6
	306	8285·9	4	8265·6
	322	9163·7	280	9133·5
	323	9220·8	3	9234·2
	324	9277·9	5	9301·4
	326	9393·5	7	9368·0
	327	9452·2	9	9435·4
	328	9510·6	291	9502·2
	337	10044·7	307	10038·0
	338	10104·8	310	10138·9
	339	10167·3	2	10206·0
	341	10290·8	4	10272·5
	342	10350·2	6	10339·2
	343	10411·5	8	10406·0

Table XX.—Experiment LII.–LIV. Aug. 11, 1892. The constants are the same as for LI.

Experiment.	Temperature E_m.	Time.	Revolutions.	Time.
		seconds.	thousands.	seconds.
LII.	403	1·0	0	15·0
	404	93·7	3	116·4
	406	281·6	6	217·1
	407	378·5	10	352·3
	408	472·0	3	453·4
	409	465·5	6	554·2
	411	757·4	9	655·0
			21	722·0
LIII.	456	1·0	3	36·8
	457	149·6	6	137·4
	458	297·4	10	271·6
	459	438·5	4	405·5
	461	732·0	22	673·6
	462	883·7	8	875·6
LIV.	507	1·1	3	13·8
	508	290·5	10	247·8
	509	560·2	2	314·4
	511	1179·9	20	580·3
			39	1215·2

Method of reducing the results in the above Table.

The time over any range is the mean time obtained by taking the observations in pairs—thus the time over the range 87·5–97·5 E_m is the mean of the times taken in rising from 86–96, 87–97, 88–98, and 89–99. Thus each number in column 4 of the following Table is the mean of, at least, four intervals of time.

TABLE XXI.

Range E_m.	Range C.	Mean Temperature C.	Time.	Rate of stirring.	$\frac{\text{Range}}{\text{Time}} \times 10^6$	(Correction to rate 30) $\times 10^6$.	(Rise, in degrees, per second at rate 30) $\times 10^6$.	Calculated rise $[\sigma \pm \rho(\theta_1 - \theta)] \times 10^6$.	Difference between last two columns.
87·5– 97·5	·242	14·110	274·4	30·01	883	0	883	880	+ 3
92·5–102·5	·242	14·232	277·6	30·00	873	0	873	871	+ 2
102·5–122·5	·487	14·597	572·9	30·14	851	− 7	844	844	0
122·5–132·5	·246	14·963	300·8	30·07	816	− 3	813	817	− 4
127·5–162·5	·864	15·395	1106·9	29·96	780	+ 2	782	784	− 2
167·5–197·5	·740	16·321	1041·2	29·93	711	+ 4	715	716	− 1
197·5–227·5	·742	17·063	1122·1	30·03	661	− 1	660	661	− 1
227·5–232·5	·124	17·495	195·9	30·02	632	− 1	631	629	+ 2
232·5–252·5	·498	17·805	822·1	30·02	606	− 1	605	605	0
257·5–277·5	·504	18·432	901·2	30·02	560	− 1	559	559	0
277·5–297·5	·510	18·940	981·6	29·98	520	+ 1	521	521	0
297·5–302·5	·128	19·260	259·7	29·92	493	+ 4	497	497	0
302·5–322·5	·512	19·580	1092·7	29·92	469	+ 4	473	473	0
322·5–337 5	·382	20·028	883·2	29·88	433	+ 8	441	440	+ 1
327·5–342·5	·383	20·156	900·2	29·86	425	+ 7	432	431	+ 1
404·5–409·5	·130	22·002	473·3	29·69	275	+ 16	291	293	− 2
456·5–461·5	·133	23·367	732·9	29·74	181	+ 10	191	192	− 1
507·5–510·5	·081	24·701	879·5	29·96	92	+ 2	94	93	+ 1

Mean rise due to stirring alone = ·0004971.

Hence $tr^3 = 5420 \times 10^4$. (The "Correction to rate 30" is deduced from this value of tr^3.)

The outside temperature throughout the above experiments was 19°·260 C. The actual observation across that temperature was over a range of 0°·128 C. only, and an error of 0°·001 C. in the range would make a difference of 4 in the number in the last column. As the rise at this point was one of the most important of all, a straight line was drawn through the numbers obtained from the pairs of observations on each side of it, the resulting value being ·000497. Assuming this to be the rise due to stirring only (at rate 30), we get radiation coefficient $(\rho) = (\gamma - \cdot000497)(19\cdot260 - t)$, where γ is rise (gain) per second due to all non-electrical sources.

The mean temperature of the first four observations is 14°·475 C., and the mean value of $\gamma = \cdot853 \times 10^{-4}$, hence

$$\rho = \cdot000744.$$

The straightness of this "non-electrical supply line" is a matter of extreme importance, as if there were any marked curvature it would seriously affect our conclusions as to the specific heat of water. In the last columns of the above Table we give the numbers obtained by direct calculation, assuming $\rho = \cdot000744$ and $\sigma = \cdot000497$, also the difference between the calculated and experimental results. The irregularities are obviously experimental ones, and are to be expected when it is remembered that an error of 0°·001 C. in the "range" would account for nearly all of them. Again, we are here dealing with our smallest mass of water, and the individual experiments are in better agreement when the depth of water is greater. The even distribution of the numbers about a straight line is rendered more evident to the eye when plotted on a large scale than when presented in rows of figures. The Table also illustrates the accuracy of the correction for differences in rate of stirring.*

Assuming the value of the water equivalent of the calorimeter as 85·70, the value of M (mass of water + water equivalent) = 103·01 + 85·70 = 188·71, we get ·0140 as the number of thermal grammes lost or gained by radiation, &c., per second for a difference in temperature of 1° C.

In order to test to what extent any change in the viscosity of the water caused a change in the stirring heat, we performed a few experiments with different external temperatures.

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* This "non-electrical supply curve" being a straight line, we are enabled to deduce the values of both ρ and σ by two experiments only, conducted at any two different temperatures.

TABLE XXII.

Date.	Exp.	Range E_m.	Range C.	Mean temperature and temperature of outer bath.	Time.	Rate.	$\frac{\text{Range}}{\text{Time}} \times 10^6$.	(Rate correction) $\times 10^6$.	(Rise per second at rate 30) $\times 10^6$.
Aug. 12, 1892	LV.	87·5–108·5	·509	14·251	1009·2	29·99	504	+ 1	505
"	LVI.	394·5–409·5	·389	21·873	802·7	29·80	485	+10	495
"	LVII.	494·5–503·5	·241	24·433	470·9	30·21	512	−11	501

As we regarded these experiments only as preliminary, they were somewhat hurriedly performed, and we proposed investigating this point more fully later on. It has been previously pointed out (p. 440) that, in any case, the conclusions arrived at would not affect our results, and we only give them to show that no marked change is indicated. Some similar experiments performed in 1891 had led to the same conclusion.

During this year's (1892) experiments we were not troubled by any air, or water, leakages into the annular space between the calorimeter and the steel chamber. A great portion of our time and attention in previous years having been devoted to leak-hunting, we had considerable (unwished for) opportunities of investigating the effects of changes of pressure, or alterations in the dryness of the surrounding medium.

The smallest trace of moisture has a most astonishing effect on the radiation coefficient. The change in the rate of rise is most marked, and if the moisture present is sufficient to saturate the space when the calorimeter is cooled down, all attempts to obtain a radiation coefficient are hopeless. Any trace of aqueous vapour could immediately be detected by its effect on the rate of rise, and on one or two occasions was sufficient to prevent any rise whatever.

Considerable effect was also caused by changes in pressure, and when comparing the results obtained from different masses it must be remembered that the pressure was frequently changed. Although we were then not conscious of the full importance of this point, we endeavoured to maintain the pressure unaltered from the time that any set of stirring determinations were made until the corresponding J experiments were completed.

Results of the Stirring and Radiation Experiments

Throughout the remainder of this section the following notation is adopted

W = weight of water (*in vacuo*).
M = W + water equivalent of calorimeter.
p = pressure in annular space.
t = time of rising 1° C.
r = rate of revolution of stirrer (revolutions per 1 second).
K = value of tr^3.
σ = rise per 1 second (in degrees C.) due to stirring only.
ρ = gain or loss (in degrees C.) per 1 second due to radiation, conduction, and convection, when the difference of temperatures = 1° C.
Q = thermal [illegible] lost or gained per 1 second by radiation, conduction, and convection = $M\rho$.

Experiments 1 to 31 (see Table XVIII.). [illegible] the outside temperature only and were [illegible] — K.

Experiments 32 to 41.—[illegible] the conclusion that [illegible] = K.

As the calorimeter was [illegible] J experiments were commenced [illegible]

Experiments 42 to 51.—

W = [illegible]
r = [illegible]
p = [illegible]
t = [illegible] p = [illegible] millions

Experiments [illegible] in the reduction [illegible]

Experiments [illegible] alteration. [illegible] [illegible] regular [illegible] the pressure [illegible] water. [illegible] determinations [illegible]

Experiments [illegible]

[illegible]

[illegible]

Experiments 60 and 61 :—

$$\left.\begin{aligned} \sigma &= \cdot 000358 \\ \rho &= \cdot 000478 \\ Q &= \cdot 0131. \quad p = \cdot 37 \text{ millim.} \end{aligned}\right\} \text{J 8 and 9.}$$

Experiments 62 to 65 :—

$$\left.\begin{aligned} \sigma &= \cdot 000360 \\ \rho &= \cdot 000483 \\ Q &= \cdot 0132. \quad p = \cdot 37 \text{ millim.} \end{aligned}\right\} \text{J 10 to 12.}$$

Experiments 66 to 80.—The cause of irregularity had not yet been ascertained, and therefore not removed. During this series we made several changes in the connections between the stirring rod and the revolving shaft above it, and also in the pressure around the calorimeter.

$$W = 277 \cdot 971 \text{ throughout this series.}$$

Experiments 66 to 71 :—

$$\left.\begin{aligned} \sigma &= \cdot 000276 \\ \rho &= \cdot 000361 \\ Q &= \cdot 0131. \quad p = \cdot 37 \text{ millim.} \end{aligned}\right\} \text{J 13 and 14.}$$

Experiments 72 and 73.—The position of the revolving shaft and counter was slightly altered, and the whole of this portion of the apparatus re adjusted.

$$\left.\begin{aligned} \sigma &= \cdot 000272 \\ \rho &= \cdot 000374 \\ Q &= \cdot 0136. \quad p = \cdot 38 \text{ millim. ?*} \end{aligned}\right\} \text{J 15 and 16.}$$

Experiments 74 to 78.—The double HOOKE's joint was now removed and replaced by a single one in the hope of diminishing the vibration of the stirring rod, the pressure was at the same time somewhat reduced.

The value of σ showed considerable alteration.

$$\left.\begin{aligned} \sigma &= \cdot 000267 \\ \rho &= \cdot 000363 \\ Q &= \cdot 0132. \quad p = \cdot 37 \text{ millim.} \end{aligned}\right\} \text{J 17 and 18.}$$

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[*] There is probably some error here. We have only a single note of the pressure during this day (August 28), and we have no means of checking it since Experiment 74 was not performed until August 30. Now (as we find recorded in our notes), we gave twenty-three strokes to the mercury pumps on August 29, the resulting pressure on August 30 being ·37 millim.; it appears almost certain, therefore, that the pressure on August 28 must have exceeded ·38 millim.

Experiments 79 and 80.—The simple Hooke's joint was now replaced by a light ring of wire, which passed through holes in the stirring rod and the revolving shaft.

$$\left.\begin{array}{l}\sigma = \cdot 000272 \\ \rho = \cdot 000374 \\ Q = \cdot 0136. \quad p = \cdot 44 \text{ millim.}\end{array}\right\} \text{J 19.}$$

It thus appeared that both σ and ρ had resumed their former values.

Experiments 81 and 82.—These two experiments were intended to be the first pair of a series. On commencing J 20, the same evening (September 4), it was discovered that the coil in the calorimeter had become loose. The calorimeter had, in consequence, to be taken to pieces, and a week elapsed before we were able to resume our J determination. These two experiments are consequently of no use, but they are interesting as showing the kind of accuracy obtainable.

$$\begin{array}{l}W = 140 \cdot 270. \\ \sigma = \cdot 000439. \\ \rho = \cdot 000578. \\ Q = \cdot 0130. \qquad p = \cdot 36 \text{ millim.}\end{array}$$

Experiments 83 to 93.—After Experiment 82 a marked difference is observable in the value of σ. Between September 4 and 10 the calorimeter had been thoroughly cleaned, and all gutta-percha surfaces removed from its interior. The "throw" of the stirrer was evidently improved, and as a consequence the value of σ was raised.

The series of J Experiments (Series II., J 20 to 34), performed after this change, are so much superior to their predecessors that we base our conclusions almost entirely upon them. We feel it right, therefore, to give more particulars concerning their "non-electrical supply" determinations than we have considered necessary in the case of Series I. (J 1 to 19). The following table is a summary of the "stirring experiments" used in the reduction of J 20 to 25.

Table XXIII.—W = 139·776.

Date.	Range E_m.	Range C.	Time.	Rate.	$\frac{\text{Range}}{\text{Time}}$.	Rate correction × 10^6.	Rise at rate 30.	Mean temperature C.
Sept. 10 .	292·5–307·5	·384	752·05	29·93	·000510	+ 3	·000513	19·260
„ 10 .	87·5–127·5	·973	1173·43	30·35	·000829	− 18	·000811	14·476
„ 10 .	292·5–307·5	·384	746·75	30·07	·000514	− 4	·000510	19·260
„ 11 .	292·5–307·5	·384	733·6	30·18	·000523	− 9	·000514	19·260
„ 11 .	292·5–307·5	·384	753·22	29·96	·000510	+ 2	·000512	19·260
„ 11 .	442·5–452·5	·263	899·23	30·31	·000293	− 13	·000280	23·064
„ 13 .	292·5–307·5	·384	723·05	30·20	·000522	− 10	·000512	19·260
„ 13 .	87·5–127·5	·973	1168·85	30·39	·000832	− 20	·000812	14·476
„ 13 .	442·5–452·5	·263	890·65	30·30	·000296	− 15	·000281	23·064

Hence

$$\left.\begin{array}{l} {}^{*}\sigma = \cdot000512, \text{ therefore } tr^3 = 5273 \times 10^4 \\ \rho = \cdot000620 \\ Q = \cdot0140. \quad p = 1\cdot15 \text{ millims.} \end{array}\right\} \text{J 20 to 25.}$$

Experiments 94 to 102.

TABLE XXIV.—W = 199·674.

Date.	Range E_m.	Range C.	Time.	Rate.	$\frac{\text{Range}}{\text{Time}}$.	Rate correction $\times 10^6$.	Rise at rate 30.	Mean temperature C.
Sept. 14 .	292·5–307·5	·384	898·68	30·10	·000428	− 4	·000424	19·260
„ 14 .	442·5–452·5	·263	1161·05	29·81	·000227	+ 8	·000235	23·064
„ 14 .	442·5–452·5	·263	1110·62	30·04	·000237	− 2	·000235	23·064
„ 14 .	87·5–127·5	·973	1481·7	30·16	·000657	− 7	·000650	14·476
„ 15 .	292·5–307·5	·384	856·65	30·52	·000448	− 22	·000426	19·260
„ 15 .	87·5–127·5	·973	1519·8	29·77	·000640	+ 10	·000650	14·476
„ 16 .	292·5–307·5	·384	880·98	30·25	·000437	− 11	·000426	19·260

Hence

$$\left.\begin{array}{l} \sigma = \cdot000426, \text{ therefore } tr^3 = 6353 \times 10^4 \\ \rho = \cdot000485 \\ Q = \cdot0138. \quad p = \cdot48 \text{ millim.} \end{array}\right\} \text{J 26 to 29.}$$

Experiments 103 to 110.

TABLE XXV.—W = 259·500.

Date.	Range E_m.	Range C.	Time.	Rate.	$\frac{\text{Range}}{\text{Time}}$.	Rate correction $\times 10^6$.	Rise at rate 30.	Mean temperature C.
Sept. 16 .	292·5–307·5	·384	1081·6	30·03	·000355	− 1	·000354	19·260
„ 17 .	292·5–307·5	·384	1025·7	30·50	·000374	− 18	·000356	19·260
„ 17 .	87·5–127·5	·973	1816·5	29·78	·000536	+ 8	·000544	14·476
„ 18 .	87·5–127·5	·973	1784·2	30·05	·000545	− 2	·000543	14·476
„ 18 .	292·5–307·5	·384	1081·8	30·03	·000355	− 1	·000354	19·260
„ 18 .	292·5–307·5	·384	1078·7	30·03	·000356	− 1	·000355	19·260
„ 18 .	442·5–452·5	·263	1335·7	29·88	·000197	+ 4	·000201	23·064
„ 18 .	442·5–452·5	·263	1284·1	30·10	·000205	− 4	·000201	23·064
„ 18 .	452·5–462·5	·265	1278·6	29·78	·000207	− 14	·000193	23·327

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[*] The values of σ and ρ given at the end of the Tables are derived from large scale curves, and thus give the mean of all the observations.

Hence

$$\sigma = {\cdot}000355 \qquad tr^3 = 7606 \times 10^4$$
$$\rho = {\cdot}000399$$
$$Q = {\cdot}0138 \qquad p = {\cdot}48 \text{ millims.}$$

The reduction of our observations involved so much arithmetic that the whole series of experiments was concluded before we were able to get the results in the form now given.

Although we were prepared to find a rapid reduction in the value of Q as the pressure diminished, we had not realized the importance of small changes in the pressure. This was unfortunate, for, as the figures show, our average pressure was just that at which small alterations produced the greatest effect. We usually considered it sufficient to record the pressure each day, whereas it appears that a careful observation of the pressure during each stirring experiment ought to have been made. It is evident that we should have done better to select pressures of not less than 1 millim., and thus small variations would have had but little effect.

At the time of the observations we did not know the real value of the water equivalent. It was thus impossible for us to obtain Q with any accuracy, and it is only by the changes in the value of Q that the effects of changes in pressure are rendered evident.

The results compare very favourably with the conclusions arrived at by BOTTOMLEY.*

Collecting the numbers given in this section (with the exception of Experiments 72 and 73, see note, p. 450), we get

TABLE XXVI.

Date.	Experiment.	Mass of water.	Pressure in millims.	Thermal grms. per second.
Sept. 10–13 . . .	83– 93	139·78	1·15	·0140
Aug. 8–10 . .	41– 50	103·01	1·15	·0140
„ 11	51– 54	103·01	·98	·0140
Sept. 14–16 . . .	94–102	199·67	·48	·0138
„ 16–18 . . .	103–110	259·50	·48	·0138
Aug. 26	79– 80	277·93	·44	·0136
„ 14–26 . . .	58– 59	188·07	·40	·0134
„ 24, 25 . . .	62– 65	188·07	·37	·0132
„ 30, 31 . . .	74– 78	277·93	·37	·0132
„ 26, 27 . . .	66– 71	277·93	·37	·0131
„ 17, 18 . . .	60– 61	188·07	·37	·0131
Sept. 4	81– 82	140·27	·36	·0130

These numbers are plotted in fig. 8 the abscissæ denoting pressures and the ordinates the number of the thermal grms. gained per second. When it is remembered that adjacent points are obtained from very different masses of water, the close agreement is a proof of the accuracy of the observations. The sudden change in the value of Q when the pressure falls below ·50 millim. is remarkable; but a reference to **Bottomley's** paper will show that the curve given by him is of a similar type, although the critical point is at a higher pressure and the bend in the curve is less sharp.

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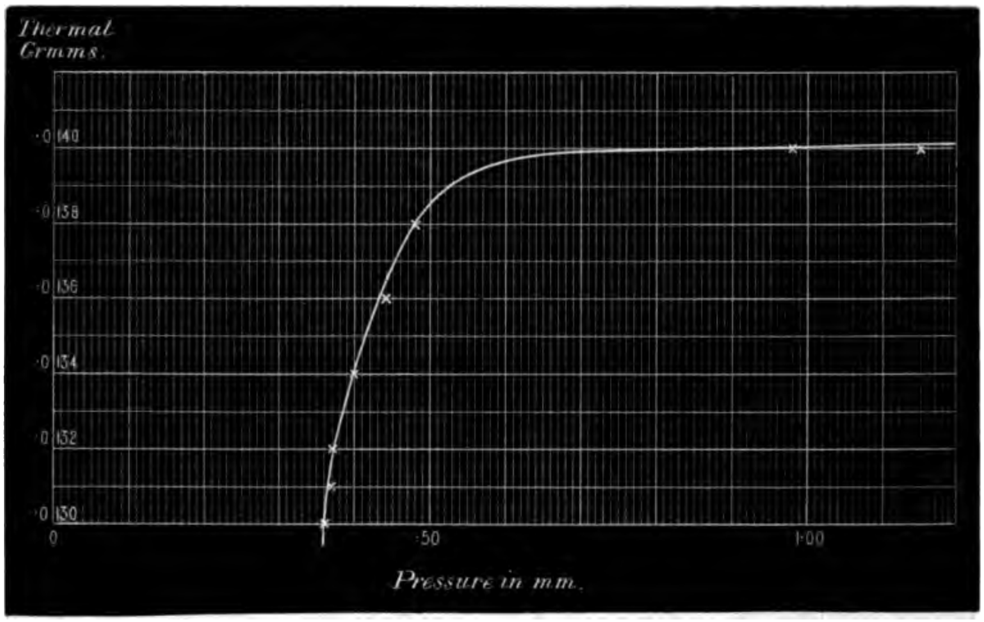
* 'Phil. Trans.,' 1887, A.

Fig. 8.



Although all the observations on the "non-electrical supply" have been brought together in this section for the purposes of comparison, they were not taken continuously but in groups. Each group of observations was used for the reduction of J experiments performed at the same time and on the same weight. The stirring experiments were usually performed during the daytime, the corresponding J experiments in the evening, and it was rarely that two sets of J observations were taken on the same day. Throughout the whole of the experiments J 1 to 34, as also their corresponding stirring experiments, the surrounding temperature was maintained at 300 E = 19·260 C.*

If θ_1 = temperature of calorimeter, then $(\theta_1 - 19{\cdot}260)\,\rho$ gives the rise or fall in temperature per second due to radiation, &c., hence $\sigma + (\theta_1 - 19{\cdot}260)\,\rho$, gives the rise in temperature per second due to the whole "non-electrical" supply. As previously pointed out (see note, p. 411) we proposed to reduce the numbers obtained from our J experiments at certain fixed temperatures and the following Table gives the changes in temperature per 1 second, due to the "non-electrical" supply at these points. The numbers were obtained from large-scale curves, but can be calculated from the values of σ and ρ given in this section.

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* In previous years we had performed several pairs of experiments in which the outside temperature was, in one case at the bottom, and in the other at the top, of our range. By "meaning" the reciprocals of the times of rising 1°, the effect of radiation, &c., could be eliminated. We hoped to repeat these experiments this year, but want of time prevented our doing so.

TABLE XXVII.—Rise per second due to non-electrical supply, at temperatures required for the reduction of the J experiments.

J (expts.) =	1–4.	5–7.	8–9.	10–12.	13–14.	15–16 and 19.	17–18.	20–25.	26–29.	30–34.
W =	103·010	188·065	188·065	188·065	277·931	277·931	277·931	139·776	199·674	259·500
$\sigma \times 10^6$ =	498	354	358	360	276	272	267	512	426	355
$\rho \times 10^6$ =	743	490	478	483	361	374	363	621	483	399
	VALUE of $\{\sigma + \rho\,(19{\cdot}260 - \theta_1)\} \times 10^6$.									
Temp. C.										
14·477	853	587	587	591	449	451	441	809	658	546
15·581	771	534	534	538	409	410	401	741	604	502
16·682	690	480	481	485	369	368	361	672	551	458
17·683	615	431	433	436	333	331	324	610	502	418
18·688	541	382	385	388	297	293	288	548	454	378
19·835	455	326	331	332	255	250	246	476	397	332
21·115	360	263	269	270	209	203	200	397	336	281
22·409	264	200	207	208	162	154	153	316	273	229
23·862	156	128	138	138	110	100	100	226	203	171
25·006	71	78	83	82	69	57	58	155	148	125

The above Table is deduced from "stirring" experiments Nos. 41 to 110. Th values of ρ and σ, obtained by repeating an experiment under the same conditions, rarely show any difference as great as ·000004, and thus a difference from the mean value of ·000002 may be regarded as our limit of experimental error. Now a change of two in the last digit would, when 5 Clark cells were used, cause a difference of less than 1 in 4000 in the resulting value of J; but as the numbers in each column are obtained from independent experiments, and as all the columns are used in the final determination of J, it is probable that the mean resulting error is of very small dimensions.

[Note by E. H. G. Added April, 1893. The statement that the loss or gain by radiation, convection, and conduction is, in our apparatus, a linear function of the difference in temperature may at first sight appear to differ from the conclusions arrived at by DULONG and PETIT, MACFARLANE, and other observers, but I think that the contradiction is more apparent than real. The curve which shows the absolute loss or gain (fig. 8, p. 454) indicates that, if we had reduced the pressure to something under one-thousandth of a millimetre, the loss or gain by radiation, convection, and conduction might almost have been disregarded over our range, and hence it is evident that convection plays by far the greatest part at such pressures as those with which we were working; and the wonderful experiments of Professor DEWAR (the account of which had not been published when the preceding section was written) appear to place this conclusion on a firm foundation. Now DULONG and PETIT make the loss or gain by convection vary directly as the difference in temperature when the density of the gas remains constant. In our apparatus the density of the air surrounding the calorimeter must have remained nearly constant throughout each experiment, and therefore our observations appear to corroborate, rather than to contradict, the conclusions arrived at by DULONG and PETIT.]

SECTION XIII.—THE METHOD OF PERFORMING A J EXPERIMENT.

The general method of procedure was as follows.

Freshly distilled ether having been placed in the cooling tube, the temperature of the calorimeter was lowered until the resistance of the coil indicated a temperature of 10° C., any remaining ether was then withdrawn and dried air passed through the cooling apparatus for about ten minutes. The aspirating portion of the apparatus was removed and the cooling tube cleaned by a small mop of absorbent cotton wool. The open end was closed by a long cork, and a metal tube placed over the whole of the projecting portion, the lower end of this tube being washed by the tank water.

The storage circuit was then completed and the rheostat adjusted until the balance with the Clark cells was obtained. About twenty minutes elapsed before the temperature rose to 86 E_{n}—the point at which we commenced our observations. During this interval the chronograph circuit was tested, and any final adjustment of the motor and external circuit completed.

The task of one observer was invariably that of maintaining the potential balance by means of the rheostat, and also inverting the currents at regular intervals. The attention of the second observer was directed to recording the group of transits about each fixed point, the chronograph key being placed at the base of the telescope stand. In the intervals he recorded the times of the revolutions of the stirrer, which were distinguished on the tapes from the temperature records by double marks. He also occasionally recorded the readings of thermometer A (the external temperature), but these were as a rule so regular as to render this observation a matter of form.

Thus, during the progress of an experiment, no notes had to be taken unless some exceptional incident occurred. We believe this to be an important matter during observations of this description, for the mechanical operation of note-taking greatly distracts the attention.

The chronograph required rewinding about every twenty minutes, and in the press of the other observations this was sometimes forgotten. Some of the omissions in the tables are due to this cause. The duration of a whole experiment varied from 40 to 80 minutes, according to the weight of water present and the E.M.F. used.

At the end of an experiment, the chronograph tape was counted and the results tabulated in the form shown in Tables XXVIII. to XXXIII. These six experiments are typical of the rest, and have not been in anyway selected, except that they are the first performed on each weight, in Series II., with different E.M.F. s.

TABLE XXVIII.—Series II. J 20. September 11, 1892.

Weight of water (*in vacuo*) = 139·776 grms.

Clark Cells.* $n = 4$. Nos. **(37, 38, 39), (40, 41, 42), (43, 44, 45), (46, 47, 48).**

Temperature = 15°·05 C.

External temperature throughout experiment = 241·2 A = 300 E = 19°·260 C.

Pressure = ·97 millim.

Calorimeter cooled to 10°·00 C. before commencing experiment. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	7	..	4	4	
87	11·8				
88	16·8				
89	21·8	..	5	36·8	
Mean					
126	206	199	10	200·2	
127	211	199·2			
128	216	199·2			
129	221·2	199·4	11	233·2	
Mean	..	199·2	..	..	30·54
176	464·2	258·2	17	431	
177	469·2	258·4			
178	474·7	258·7			
179	479·8	258·5	19	496·6	
†Mean	..	258·45	..	..	30·37
216	673·8	209·6			
217	679·1	209·7			
218	684·5	209·8			
219	689·9	210·1	25	693·5	
Mean	..	209·8	..	..	30·47
256	888·3	214·5	30	857·5	
257	893·8	214·7			
258	899·2	214·7			
259	904·8	214·9	32	922·8	
Mean	..	214·7	..	..	30·53

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* The cells in brackets were placed in parallel arc.

Table XXVIII.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	1111·3	223	37	1087	
297	1116·9	223·1			
298	1122·7	223·5			
299	1128·3	223·5			
Mean	..	223·38	..	..	30·15
346	1398	286·7	46	1383·5	
347	1403·9	287			
348	1409·6	286·9			
349	1415·4	287·1			
Mean	..	286·93	..	..	30·35
396	1691·5	293·5			
397	1697·4	293·5			
398	1703·4	293·8			
399	1709·5	294·1	56	1712·4	
†Mean	..	293·75	..	..	30·41
446	1997·4	305·9			
447	2003·7	306·3			
448	2010·0	306·6			
449	2016·2	306·7	66	2039·6	
Mean	..	306·38	..	..	30·56
506	2380·8	383·4	76	2366·7	
507	2387·5	383·8			
508	2394·2	384·2			
509	2400·8	384·6	78	2431·5	
Mean	..	384	..	..	30·54
536	2579·8	199	82	2563·8	
537	2586·8	199·3			
538	2593·5	199·3			
539	2600·3	199·5	84	2629·7	
Mean	..	199·28	..	..	30·42

Table XXIX.—Series II. J 23. September 12, 1892.

Weight of water (*in vacuo*) = 139·776 grms.

Clark cells. $n = 3$. Nos. **(37, 38, 39), (40, 41, 42), (43, 44, 45).** Temperature 15°·21 C.

External temperature. 241·2 A = 300 E = 19°·260 C. Pressure = ·78 millim.

Calorimeter cooled to 10°·10 C. before commencing experiment. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolution.	Rate.
	seconds.		thousands.	seconds.	
86	− 9·5 ?				
87	− 1·7				
88	6				
89	14	..	7	21	
Mean					
126	305·2	314·7 ?			
127	313·2	314·9			
128	321	315			
129	329·5	315·5	17	354·2	
Mean	..	315·03	..	..	30·01
176	714·2	409			
177	723·6	409·4			
178	731	410			
179	739·2	409·7	29	753·4	
†Mean	..	409·52	..	..	30·06
216	1050·3	336·1			
217	1058·7	336·1			
218	1067·3	336·3			
219	1076	336·8	40	1120	
Mean	..	336·33	..	..	30·01
256	1397·2	346·9			
257	1406	347·3			
258	1415	347·7			
259	1423·9	347·9	50	1453	
Mean	..	347·45	..	..	30·02

TABLE XXIX.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolution.	Rate.
	seconds.		thousands.	seconds.	
296	1760·9	363·7	59	1753·6	
297	1770·2	364·2			
298	1779·7	364·7			
299	1789	365·1	61	1820·3	
Mean	..	364·43	..	..	29·95
346	2231·5	470·6	73	2219·9	
347	2240·9	470·7			
348	2250·5	470·8			
349	2260	471	75	2286·7	
Mean	..	470·78	..	..	30·02
396	2719	487·5	87	2686·6	
397	2729	488·1			
398	2739	488·5			
399	2749·3	489·3	89	2753	
†Mean	..	488·35	..	..	30·02
446	3233·9	514·9	103	3219·3	
447	3244·8	515·8			
448	3254·9	515·9			
449	3265·8	516·5	105	3286	
Mean	..	515·78	..	..	30·02

Owing to the breakage of a wire in the clock and chronograph circuit, the records of the times from 449 to 539 are so uncertain that we have decided to omit them.

TABLE XXX.—Series II. J 26. September 14, 1892.

Weight of water (*in vacuo*) = 199·674 grms.

Clark cells. $n = 4$. **(37, 38, 39), (40, 41, 42), (43, 44, 45), (46, 47, 48).**

Temperature 15°·12 C.

External Tempereture 241·2 A = 300 E = 19·260° C. Pressure = ·48 millim.

Calorimeter cooled to 10°·00 C. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	45·2	..	0	19·6	
87	51·7				
88	57·9				
89	64·1	..	2	86·1	
126	297·7	252·5	8	285·3	
127	304	252·3			
128	310	252·1			
129	316·8	252·7			
Mean	..	252·4	..	..	30·10
176	623·8	326·1	18	617·6	
177	630	326			
178	636·7	326·7			
179	643	326·2	19	650·8	
† Mean	..	326·25	..	..	30·12
216	888·6	264·8	26	883·2	
217	895·2	265·2			
218	902	265·3			
219	908·9	265·9	27	916·4	
Mean	..	265·3	..	..	30·12
256	1160·7	272·1	34	1148·8	
257	1167·1	271·9			
258	1174	272			
259	1181	272·1			
Mean	..	272·03	..	..	30·10

TABLE XXX.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	1442·1	281·4	42	1414·6	
297	1449·4	282·3			
298	1456·5	282·5			
299	1463·8	282·8	44	1481	
Mean	..	282·25	..	..	30·12
346	1804·3	362·2	53	1780·3	
347	1811·7	362·3			
348	1818·9	362·4			
349	1826·5	362·7	55	1847·3	
Mean	..	362·4	..	..	30·02
396	2175	370·7	64	2147	
397	2182·9	371·2			
398	2190·5	371·6			
399	2198	371·5	66	2213·7	
† Mean	..	371·25	..	..	30·02
446	2562	387	76	2546·4	
447	2569·9	387			
448	2577·7	387·2			
449	2585·6	387·6	78	2613	
Mean	..	387·20	..	..	30·05
506	3046	484	90	3012	
507	3054·3	484·4			
508	3062·8	485·1			
509	3071·2	485·6	92	3078·5	
Mean	..	484·78	..	..	30·08
536	3297·4	251·4	97	3244·7	
537	3305·9	251·6			
538	3314	251·2			
539	3322·6	251·4	98	3278	
Mean	..	251·4	..	..	30·07

TABLE XXXI.—Series II. J 27. September 15, 1892.

Weight of water (*in vacuo*) = 199·674 grms.

Clark cells. $n = 5$. Nos. **(37, 38, 39), (40, 41, 42), (43, 44, 45), (46, 47, 48), (52, 53, 54)**.

Temperature 15° C.

External Temperature 241·1 A = 299·8 E = 19°·258 C. Pressure = ·48 millim.

Calorimeter cooled to 10°·01 C. Connections reversed at †.

Reading E_w.	Time.	Time over interval.	Revolutions.	Time of revolutions.	**Rate.**
	seconds.		thousands.	seconds.	
86	76·7	..	7	22·9	
87	80·9				
88	85·2				
89	89·4				
Mean					
126	249·7	173	13	223·7	
127	254	173·1			
128	258·4	173·2			
129	262·6	173·2	15	290·7	
Mean	..	173·13	..	..	29·88
176	471·8	222·1	20	457·5	
177	476	222			
178	480·6	222·2			
179	485	222·4	21	490·5	
Mean	..	222·18	..	..	30·13
†216	651·6	179·8	25	623·2	
217	656	180			
218	660·5	179·9			
219	665	180	27	689·5	
Mean	..	179·95	..	..	30·15
256	834·7	183·1	31	822·4	
257	839·5	183·5			
258	844	183·5			
259	848·8	183·8	32	855·6	
Mean	..	183·48	..	..	30·22

TABLE XXXI.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	1024·4	189·7	37	1021·6	
297	1029	189·5			
298	1034	190			
299	1038·6	189·8	38	1054·8	
Mean	..	189·75	..	..	30·12
346	1267·2	242·8	44	1254·4	
347	1271·8	242·8			
348	1276·7	242·7			
349	1281·6	243	45	1287·5	
Mean	..	242·83	..	..	30·08
396	1513·6	246·4			
397	1518·6	246·8			
398	1523·6	246·9			
†399	1528·8	247·2	53	1553·2	
Mean	..	246·83	..	..	30·11
446	1769·3	255·7	59	1752·6	
447	1774·7	256·1			
448	1779·9	256·3			
449	1785	256·2			
Mean	..	256·08	..	..	30·10
506	2087·5	318·2	68	2051·1	
507	2092·8	318·1			
508	2098·4	318·5			
509	2103·7	318·7	70	2117·3	
Mean	..	318·38	..	..	30·16
536	2251	163·5	73	2216·6	
537	2256·8	164			
538	2262·2	163·8			
539	2267·9	164·2			
Mean	..	163·88	..	..	30·22

TABLE XXXII.—Series II. J 33. September 18, 1892.
Weight of water (*in vacuo*) = 259·500 grms.
Clark cells. $n = 5$. **(37, 38, 39), (40, 41, 42), (43, 44, 45), (46, 47, 48), (52, 53, 54).**
Temperature 15°·06 C.
External temperature 241·2 A = 300 E = 19°·260 C. Pressure = ·44 millim.
Calorimeter cooled to 10°·10 C. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	42·9	..	8	18	
87	48				
88	53·1				
89	58·5	..	10	84·6	
Mean	..	..	..	..	
126	251·7	208·8			
127	257	209			
128	262·2	209·1			
129	267·7	209·2	16	283·8	
Mean	..	209·03	..	..	30·10
176	520·4	268·7			
177	525·7	268·7			
178	531·2	269			
179	536·5	268·8	24	549·2	
Mean	..	268·75	..	..	30·15
†216	737·7	217·3			
217	743	217·3			
218	748·5	217·3			
219	754·1	217·6	31	781·6	
Mean	..	217·38	..	..	30·12
256	959·3	221·6	36	947·4	
257	965	222			
258	970·6	222·1			
259	996·5	222·4	37	980·7	
Mean	..	222·03	..	..	30·14

Table XXXII—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	1188·9	229·6	43	1180·3	
297	1194·7	229·7			
298	1200·4	229·8			
299	1206·2	229·7	44	1213·6	
Mean	..	229·7	..	..	30·06
346	1482	293·1	52	1479·7	
347	1487·8	293·1			
348	1493·6	293·2			
349	1499·2	293	53	1512·9	
Mean	..	293·1	..	..	30·07
396	1780	298			
397	1786·2	298·4			
398	1792·3	298·7			
399	1798·4	299·2	62	1812	
Mean	..	298·58	..	..	30·10
†446	2088·8	308·8	70	2077·7	
447	2095	308·8			
448	2101·2	308 9			
449	2107·7	309·3	71	2110·9	
Mean	..	308·95	..	..	30·11
506	2472	383·2	81	2441·7	
507	2478·6	383·6			
508	2485·2	384			
509	2491·7	384	83	2507·6	
Mean	..	383·7	..	..	30·25
536	2669·7	197·7	87	2639·8	
537	2676·4	197·8			
538	2682·9	197·7			
539	2689·9	198·2			
Mean	..	197·85	..	..	30·26

TABLE XXXIII.—Series II. J 34. September 18, 1892.

Weight of water (*in vacuo*) = 259·500 grms.

Clark cells. $n = 6$. Nos. **(37, 38, 39)**, **(40, 41, 42)**, **(43, 44, 45)**, **(46, 47, 48)**, **(49, 50, 51)**, **(52, 53, 54)**. Temperature 15° C.

External temperature 241·2 A = 300 E = 19°·260 C. Pressure = ·48 millim.

Calorimeter cooled to 10°·01 C. Connections reversed at †.

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
86	65·9	..	0	18·5	
87	69·6				
88	73·7				
89	77·3	..	2	85·5	
126	217·4	151·5	5	184	
127	221·4	151·8			
128	225·2	151·5			
129	228·9	151·6	7	250	
Mean	..	151·6	..	..	30·23
176	411·5	194·1	11	382·5	
177	415·4	194			
178	419·3	194·1			
179	423·2	194·3	13	448·5	
†Mean	..	194·13	..	..	30·25
216	568	156·5	16	547·5	
217	572	156·6			
218	576	156·7			
219	580	156·8	18	613·5	
Mean	..	156·65	..	..	30·30
256	727·5	159·5	21	712·4	
257	731·5	159·5			
258	735·5	159·5			
259	739·6	159·6	22	745·4	
Mean	..	159·53	..	..	30·23

TABLE XXXIII.—(continued).

Reading E_m.	Time.	Time over interval.	Revolutions.	Time of revolutions.	Rate.
	seconds.		thousands.	seconds.	
296	891·7	164·2	26	877	
297	895·8	164·3			
298	900	164·5			
299	904·2	164·6	27	910	
Mean	..	164·4	..	..	30·28
346	1100·8	209·1			
347	1104·8	209			
348	1109	209			
349	1113	208·8	34	1140·3	
Mean	..	208·98	..	..	30·40
396	1312·5	211·7	39	1304·7	
397	1317	212·2			
398	1321	212			
399	1325·4	212·4	40	1337·7	
Mean	..	212·08	..	..	30·40
†446	1531·2	218·7			
447	1535·7	218·7			
448	1540	219			
449	1544·6	219·2	47	1567·7	
Mean	...	218·9	..	..	30·43
506	1802·7	271·5	54	1797·8	
507	1807·2	271·5			
508	1811·9	271·9			
509	1816·5	271·9	55	1830·6	
Mean	..	271·7	..	..	30·43
536	1942·3	139·6	58	1929·8	
537	1947	139·8			
538	1951·7	139·8			
539	1956·3	139·8			
Mean	..	139·75	..	..	30·33

For convenience of reference in the succeeding Tables, we repeat here the Table showing the values we have assigned to each range on the rising thermometer.

TABLE XXXIV.

No. of range.	E_m.	C.	Mean.	$d\theta$.
1	87·5–127·5	13·990–14·963	14·477	·973
2	127·5–177·5	14·963–16·198	15·581	1·235
3	177·5–217·5	16·198–17·185	16·682	·987
4	217·5–257·5	17·185–18·180	17·683	·995
5	257·5–297·5	18·180–19·196	18·688	1·016
6	297·5–347·5	19·196–20·474	19·835	1·278
7	347·5–397·5	20·474–21·756	21·115	1·282
8	397·5–447·5	21·756–23·063	22·409	1·307
9	447·5–507·5	23·063–24·661	23·862	1·598
10	507·5–537·5	24·661–25·471	25·006	·811

We consider it unnecessary to give more of our original observations in full, but have collected in Tables XXXV. and XXXVI. the unreduced experimental results of all the J experiments, with the exception of J 1 to 4. (See p. 475.)

TABLE XXXV.—Series I. Experiments 5–19.

Group B. W = 188·065 gr.

Date and number.	Temperature and number of Clark cells.		Ranges (see Table XXXIV.).									
			1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
Aug. 14 5	16·20 5	Time Rate	168·87 30·21	216 73 30·21	175·85 30·10	179·47 30·06	185·60 30·08	236·90 30·07	— —	250·00 30·05	311·46 30·04	160·12 30·13
Aug. 14 6	16·50 5	Time Rate	— —	— —	175·65 30·13	179·47 30·22	185·41 30·19	236·43 30·35	241·01 30·23	250·20 30·18	311·31 30·17	160·45 30·11
Aug. 16 7	16·70 5	Time Rate	168·97 30·02	217·34 30·00	175·92 30·02	179·70 29·99	186·23 29·92	237·28 30·05	241·18 29·97	250·40 29·97	311·78 30·01	160·40 30·16
Aug. 17 8	15 95 4	Time Rate	249·22 29·87	321·71 29·93	261·36 29·97	267·71 29·97	277·59 30·10	356·73 29·97	364·13 29·94	381·01 30·04	475·88 30·00	246·55 30·00
Aug. 18 9	16·30 2	Time Rate	— —	— —	755·64 29·81	792·83 29·86	844·13 29·90	1116·16 29·90	— —	— —	— —	— —
Aug. 23 10	16 55 4	Time Rate	246·20 31·20	317·62 31·16	258·03 31·14	264·75 31·11	275·23 31·18	351·64 31·22	359·96 31·22	— —	— —	— —
Aug. 24 11	16·60 4	Time Rate	249·84 29·48	322·12 29·66	262·47 29·26	269·52 29 19	279·60 29·15	357·90 29·20	368·69 29·43	378·83 30·36	471·63 30·71	245·38 30·81
Aug. 25 12	16·60 4	Time Rate	248·28 30·20	320·77 30·19	260·30 30·21	267·63 30·24	277·55 30·16	355·27 30·18	364·93 30·04	380 52 30·11	476·35 30·07	246·63 30·06

TABLE XXXV.—(continued).

Group E. W = 277·931 grms.

Date and number.	Temperature and number of Clark cells.		Ranges (see Table XXXIV.). 1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
Aug. 26 13	16·15 4	Time	330·52	426·35	346·63	355·30	368·82	472·58	484·37	504·48	633·02	327·63
		Rate	29·77	29·80	29·82	29·87	29·79	29·84	29·78	29·78	29·74	29·71
Aug. 27 14	16·15 4	Time	330·70	426·22	347·08	355·22	368·75	473·43	485·32	504·85	632·50	327·73
		Rate	29·72	29·82	29·84	29·83	29·82	29·61	29·68	29·77	29·78	29·76
Aug. 28 15	16·05 5	Time	223·60	287·18	232·62	237·23	245·22	313·25	319·28	330·47	411·63	211·52
		Rate	30·34	30·38	30·44	30·54	30·52	30·58	30·54	30·61	30·63	30·47
Aug. 28 16	16·05 5	Time	224·45	288·35	233·45	238·40	246·47	314·65	320·45	332·18	412·50	213·10
		Rate	29·87	29·88	29·89	29·90	29·91	29·89	29·85	29·81	29·79	29·77
Aug. 30 17	16·50 4	Time	331·12	426·88	346·82	355·98	370·02	473·83	485·20	505·12	633·10	327·83
		Rate	30·17	30·13	30·02	30·04	29·95	29·92	30·10	30·14	30·11	30·11
Aug. 31 18	16·10 4	Time	329·57	427·18	346·70	353·90	367·97	471·68	483·25	504·15	629·47	327·50
		Rate	30·16	29·68	30·22	30·36	30·39	30·33	30·35	30·31	30·37	30·37
Sept. 1 19	16·00 5	Time	224·52	288·33	233·50	238·80	246·80	314·42	320·60	331·33	412·60	211·07
		Rate	29·73	29·84	29·59	29·79	29·54	29·82	30·03	29·77	29·75	31·52

TABLE XXXVI.—Series II. Experiments 20–34.

Group A. W = 139·776 grms.

Date and number.	Temperature and number of Clark cells.		Ranges (see Table XXXIV.).									
			1	2	3	4	5	6	7	8	9	10
Sept. 11 20	15·05 4	Time Rate	199·20 30·54	258·45 30·37	209·80 30·47	214·70 30·53	223·38 30·45	286·93 30·35	293·75 30·41	306·38 30·56	384·00 30·54	199·38 30·42
Sept. 11 21	15·00 5	Time Rate	136·60 30·36	175·98 30·37	142·45 30·31	145·63 30·13	150·60 30·12	191·98 30·26	195·38 30·37	202·70 30·34	252·33 30·37	130·20 30·40
Sept. 12 22	15·00 4	Time Rate	202·40 29·48	260·60 29·53	211·60 29·67	217·00 29·64	225·30 29·66	289·05 29·58	296·72 29·69	308·97 29·68	387·40 29·68	201·25 29·63
Sept. 12 23	15·10 3	Time Rate	315·03 30·01	409·53 30·06	336·33 30·01	347·45 30·02	364·43 29·95	470·78 30·02	488·35 30·02	515·78 30·02	— —	— —
Sept. 13 24	15·00 5	Time Rate	136·13 30·82	175·28 30·89	142·05 30·87	144·73 30·80	149·65 30·96	190·60 31·04	194·38 30·97	201·35 31·06	250·75 31·26	— —
Sept. 13 25	15·25 4	Time Rate	199·80 30·25	258·68 30·18	210·45 30·18	215·68 30·21	223·88 30·24	287·40 30·21	294·58 30·23	307·38 30·25	384·90 30·26	199·68 30·27

TABLE XXXVI.—(continued).

Group C. W = 199·674 grms.

Date and number.	Tempe-rature and number of Clark cells.		Ranges (see Table XXXIV.).									
			1	2	3	4	5	6	7	8	9	10
Sept. 14 26	15·10 4	Time Rate	252·40 30·10	326·25 30·12	265·30 30·12	272·03 30·10	282·25 30·12	362·40 30·02	371·25 30·02	387·20 30·05	484·78 30·08	251·40 30·07
Sept. 15 27	15·00 5	Time Rate	173·13 29·88	222·18 30·13	179·95 30·15	183·48 30·22	189·75 30·12	242·83 30·08	246·83 30·11	256·08 30·10	318·38 30·16	163·88 30·22
Sept. 15 28	15·10 4	Time Rate	252·40 30·10	326·00 30·14	265·18 30·15	272·08 30·05	282·10 30·13	361·28 30·33	369·15 30·40	384·95 30·44	482·15 30·47	249·00 30·70
Sept. 16 29	15·00 5	Time Rate	172·70 30·25	221·88 30·36	179·65 30·39	183·25 30·33	189·60 30·27	241·98 30·34	246·00 30·43	255·30 30·39	317·70 30·42	163·73 30·39

Group D. W = 259·500 grms.

Sept. 16 30	15·05 5	Time Rate	209·33 29·93	269·10 29·93	217·80 29·93	222·30 29·85	230·48 29·50	294·45 29·50	300·05 29·47	310·50 29·58	386·40 29·61	199·03 29·70
Sept. 17 31	15·20 5	Time Rate	209·46 30·00	269·23 30·07	217·95 30·05	222·40 29·98	230·00 30·00	293·23 30·07	298·63 30·08	309·63 30·19	384·60 30·18	198·28 30·20
Sept. 18 32	14·80 5	Time Rate	209·13 30·00	268·97 30·11	217·78 30·08	222·38 30·04	229·68 29·92	293·80 29·83	298·95 29·95	309·38 30·15	384·58 30·14	198·25 30·19
Sept. 18 33	14·80 5	Time Rate	209·03 30·10	268·75 30·15	217·38 30·12	222·03 30·14	229·70 30·06	293·10 30·07	298·58 30·10	308·95 30·11	383·70 30·25	197·85 30·26
Sept. 18 34	15·00 6	Time Rate	151·10 30·23	194·13 30·25	156·65 30·30	159·53 30·23	164·40 30·28	208·98 30·40	212·08 30·40	218·90 30·43	271·70 30·43	139·75 30·33

Observations on Tables XXXV. and XXXVI.

Experiments 1 *to* 4.

These have been rejected and the observations are not included in the Tables. The discrepancies over the shorter ranges are absurdly great, and the mean times far too large. During these experiments we had great difficulty in maintaining the potential-balance; this ought to have been sufficient to indicate the source of error, but, at the time, we attributed it to want of practice in the observers.

It must be borne in mind that it was not until many weeks after the close of our observations that the reductions were completed, so that, at the time of experimenting, we were wholly in the dark as to the actual value of our results. Our only means of obtaining any glimpse of the value of a particular experiment was its resemblance to some other one taken under similar conditions, but, as all the conditions were never precisely the same, we could thus obtain little information.

We have now no hesitation in assigning the following cause to the failure of these four experiments. With this weight of water, the depth in the calorimeter, when the stirrer was at rest, was only 2 centims., and as, when rapid stirring took place, a great portion of the contents must have been within the cylinder and in the upper portion of the calorimeter, it is evident that the higher parts of the coil must have been at times uncovered, since it required 1·5 centims. for complete immersion. This would account (1) for the irregular behaviour of the potential-balance, and (2) for the great lengthening of the times shown in these experiments. We at no time anticipated success with so small a mass of water, but the excellent results obtained from our stirring experiments on this weight (see Table XXI.) justified the attempt.

Experiments 5 *to* 12. (*Group B.*)

These experiments are only slightly more satisfactory than those we have rejected. The cause is partly that so frequently alluded to in Section XII., viz., obstructions in the openings at the base of the internal cylinder, the resulting irregularities in the throw of the stirrer affecting not only the non-electrical supply, but also changing the coefficient of increase in the resistance of the wire. That this is the case is indicated by the results of Experiment 9, where the E.M.F. was so reduced that the current was only ·1 ampère, and the changes in the temperature of the wire were less marked. This experiment was performed in order to test the accuracy of our various corrections, rather than as a serious effort to determine the value of J. The observations in this experiment only extended over $\frac{2}{5}$ of our total range, for, had we endeavoured to carry it out through the upper ranges, its duration would have extended over some hours.

In every case (excepting as above stated, Experiments 1 to 4) we have given all the numbers recorded, because, however bad individual observations may appear, we have no sufficient grounds for their rejection. The omissions in the table represent experimental mistakes; in No. 5, Range 7, the storage circuit was broken for a second or two by an accident with the reversing key; in No. 6, the chronograph refused to work throughout the first two ranges; in No. 10, some contact must have taken place between the wires of the coil, as the potential-balance was entirely destroyed after the 7th range. No. 11 was an exceptionally bad experiment, especially in the latter half; the changes in water pressure were so rapid and violent as to pass beyond the control of the motor-regulator, the rate changing from 29·43 to 30·36 in two consecutive observations. Had it not been for our desire to present our results in full, we would certainly have rejected this experiment.

Experiments 13 to 19. (*Group E.*)

In this group, our heaviest weight, the observations although still somewhat irregular are in better agreement. As might be anticipated the throw of the stirrer was less affected by the obstructions at the base when the depth of water was greater. The alterations made in the connections of the stirring rod have been already alluded to.

Our conclusions as to the value of J and the changes in the specific heat of water are based almost entirely on the results of Series II. Although the results deduced from Nos. 13 to 19 are in almost perfect harmony with those from Series II., the agreement, for the reasons given above, must be in part fortuitous. The mean value of J deduced from Nos. 5 to 12 agrees with the most probable value, but the change in the specific heat of water alters from positive to negative according as the group is compared with the results obtained from a lighter or a heavier weight.

Series II.

This series contains but fifteen experiments, which may appear a small number on which to base our conclusions. We would, however, point out that these are only the concluding series of many hundred J experiments conducted by the same observers, who are therefore in the best position to judge of their value. Furthermore, these fifteen experiments are conducted on very different weights of water and with very different currents, and our conclusions would be but little altered if they were based on any two individual experiments rather than on the mean results of all. Want of time alone prevented our increasing their number, but we believe that such increase would have given them small additional weight. The results of Series I. are quite sufficient to corroborate the value of J, although the least trustworthy group of that series gives contradictory values for the change in specific heat of water.

Such irregularities as are observable in the members of these series, are we believe inseparable from the use of mercury thermometers. Although many irregularities which would otherwise be apparent have been eliminated by the method of calibration adopted (Section X.), there yet remain residual ones, not strictly recurrent, whose effect can only be eliminated by taking the mean path of a series of observations over consecutive ranges. The magnitude of such irregularities can be estimated by a comparison of the columns headed "mean" and "from curve" in Table XL.

None of these experiments call for special comment, with the exception of No. 34. This was performed with the object of testing the accuracy of the correction given by the formula $\delta R = \alpha C^2$ (Section VII., Table IX.), and we did not anticipate from it results of much value. The irregularities in ranges 8, 9, 10, of Group D, Column 21, Table XL., are clearly an example of the eccentric behaviour of the thermometer when rising at this rapid rate; their mean however is excellent.

SECTION XIV.—THE CALCULATION OF THE RESULTS.*

The method which we have adopted in our calculations can be put into a general form, thus:—

Suppose a to be the rate of production of heat in the calorimeter at some standard temperature θ, whilst a' denotes the value of a at any other temperature θ_1;

ρ the change in temperature per one second due to radiation, &c., when the difference between the external and internal temperature is 1° C;

M the capacity for heat of the calorimeter and its contents at the standard temperature θ, M′ its value at θ_1;

Then if θ_1 is the temperature of the calorimeter at any time t, the rate (α') of production of heat is

$$M' \frac{d\theta_1}{dt} + M'\rho\,(\theta_1 - \theta_0),$$

where θ_0 is the temperature of the surrounding envelope.

Thus the equation of condition is

$$M' \frac{d\theta_1}{dt} = a' - M'\rho\,(\theta_1 - \theta_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Now, by JOULE's law,

$$JM \frac{d\theta_1}{dt} = C^2R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

therefore

$$a_e = M \frac{d\theta_1}{dt} = \frac{C^2R}{J} = \frac{n^2E^2}{JR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where a_e is the rate of production of heat due to the electrical supply, and E is the E.M.F. of a Clark cell, and n is the number of cells used.

But since the rate of production of heat is dependent on the resistance of a platinum wire, and E is kept constant, this rate will diminish as the temperature rises, or,

$$a'_e = a_e \{1 - k\,\overline{\theta_1 - \theta + \beta}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

---

* We have changed the notation adopted in Section I. to one more suited for arithmetical operations.

where k is the resistance temperature-coefficient of the wire, and β is the number of degrees that the wire is hotter than the water in contact with it.

Similarly

$$M' = M\{1 + l\,\overline{\theta_1 - \theta}\}$$

where l is the mean coefficient of increase of specific heat of the water and calorimeter.

For the non-electrical supply, we have shown that $tr^3 = K$ (p. 438), where t is the time of rising 1° C. and r is the rate of revolution of the stirrer, and therefore the rise per second is given by

$$\sigma = r^3/K \quad \ldots\ldots\ldots\ldots \quad (5),$$

and

$$\delta\sigma = \frac{3r^3}{K}\,\delta r \quad \ldots\ldots\ldots\ldots \quad (6).$$

By (6) we can reduce all observations to the standard rate r.

Equation (1) after introducing terms due to stirring becomes

$$M'\frac{d\theta_1}{dt} = a'_{e.\rho.\sigma.} + (\sigma - \rho\,\overline{\theta_1 - \theta_0})\,M',$$

that is,

$$M[1 + l(\theta_1 - \theta)]\frac{d\theta_1}{dt} = \frac{n^2E^2}{JR}\{1 - k\,\overline{\theta_1 + \beta - \theta}\} + (\sigma - \rho\,\overline{\theta_1 - \theta_0})\,M(1 + l\,\overline{\theta_1 - \theta}) \ (7);$$

or

$$\frac{d\theta_1}{dt} = \frac{n^2E^2}{JRM}\cdot\frac{1 - k(\theta_1 + \beta - \theta)}{1 + l(\theta_1 - \theta)} + \sigma - \rho(\theta_1 - \theta_0) \quad \ldots\ldots \quad (8^*).$$

If we write

$$\left(\frac{d\theta_1}{dt}\right)_e \text{ and } \left(\frac{d\theta_1}{dt}\right)_0$$

for the electrical and non-electrical terms in this expression,

$$\frac{d\theta_1}{dt} = \left(\frac{d\theta_1}{dt}\right)_e + \left(\frac{d\theta_1}{dt}\right)_0 \quad \ldots\ldots\ldots\ldots \quad (9).$$

---

[*] Equation (8) may be expressed in the following form—

$$d\theta_1/dt = A + \lambda\theta_1,$$

where A is the sum of quantities independent of θ_1, and λ is the sum of a number of small quantities.

The range $d\theta_1$ is reckoned from slightly below θ_1, say from θ', to slightly above θ_1, say θ''; hence if t' and t'' are the times at θ' and θ'', we may assign to $d\theta_1/dt$ the value $(\theta'' - \theta')/(t'' - t')$.

Strictly speaking, the equations ought to be integrated from $t = t'$, when $\theta_1 = \theta'$, to $t = t''$, when $\theta_1 = \theta''$, but the error introduced by the above method may, provided the ranges are sufficiently small, be neglected.

We have adopted throughout our experiments ranges of from 1° to 1°·5 C. It does not appear probable that increased accuracy would have resulted from the adoption of smaller ranges, for the effect of errors in the thermometry is increased as the size of the ranges is diminished.

The experiments on the heat developed by the stirrer give us the values of

$$\left(\frac{d\theta_1}{dt}\right)_0 = \sigma - \rho(\theta_1 - \theta_0).$$

(See Table XXVII.)

1. The first step in the calculation of our observations has been to correct, where necessary, all times for the clock error, which was + ·0004125, up to J 9 inclusive. This correction has been applied in Table XXXVIII.

2. We found the value of $d\theta_1/dt$, *i.e.*, the rise per second produced by the combined effect of all sources of gain or loss (Table XXXVII., col. 4).

3. Since the rate of revolution throughout an experiment was always nearly 30 per second, we eliminated irregularities in the rate by reducing to rate 30 by means of the formula given in (6). Thus we obtained the value $d\theta_1/dt$ would have had, had the rate been uniformly 30 per second. This reduction is shown in Table XXXVII., cols. 5 and 6.

4. In order to be able to more readily compare experiments performed under almost identical circumstances, the next correction applied was that for the temperature of the Clark cells. Lord **Rayleigh** gives the formula

$$E_{15} = E_t\{1 + {\cdot}00077\,(t - 15)\}$$

hence

$$E^2_{15} = E^2_t\{1 + {\cdot}00154\,(t - 15)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where t is the temperature of the Clark cells.

This correction cannot be applied to $d\theta_1/dt$ as a whole (equation 9) but only to the portion $(d\theta_1/dt)_e$ the electrical supply. As the correction was always very small, it was sufficient to find the approximate value of $(d\theta_1/dt)_e$ by subtracting, at any temperature, the value of $(d\theta_1/dt)_0$ as given in Table XXVII., and then find the correction on the value of $(d\theta/dt)_e$ so obtained by formula (11). The result of this correction gives col. 7, Table XXXVII.

The correction was applied in this order as we could thus compare all the experiments performed with the same E.M.F.'s so long as the values of σ and ρ were the same. The results at this stage of their reduction are given in Tables XXXVIII. and XXXIX.

5. Table XXXVII., col. 8, gives the value of $(d\theta/dt)_0$ taken from Table XXVII. This subtracted from col. 7 gives the value of $(d\theta/dt)_e$, *i.e.*, the rise per second due to the current only, the difference of potential at the ends of the coil being n Clark cells (col. 9).

Hence equation (8) is reduced to

$$\left(\frac{d\theta_1}{dt}\right)_e = \frac{n^2E^2\{1 - k\,\overline{\theta_1 - \theta + \beta}\}}{\mathrm{JRM}\,(1 + l\,\overline{\theta_1 - \theta})}.$$

6. In the section on resistance in Table XVI. we have collected the values of $\frac{R}{1 - k(\theta_1 - \theta + \beta)}$ at the mean temperature of each range. The value of n, the number of Clark cells used, is given with each experiment. Substituting their values we obtain the time of rising 1° C. with unit resistance and one Clark cell, or

$$\frac{J}{E^2}M\,(1 + l\,\overline{\theta_1 - \theta}) = T \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12).$$

This gives column 10 of Table XXXVII.

7. The results of all the experiments were then meaned and plotted, the columns headed "Mean" and "From curve" in Tables XL. and XLI. giving the results. From these smoothed curves the values of the required constants were calculated thus:—

8. If W_x is the water-equivalent of the calorimeter, and f and g are the temperature coefficients of the specific heat of water and of the calorimeter respectively, and suffixes written to W, l, T, denote the values of these quantities for different weights of water, then from (12) we obtain

$$\left.\begin{aligned} &\frac{J}{E^2}(W_1 + W_x)\,(1 + l_1\overline{\theta_1 - \theta}) = T_1 \\ \text{and} \qquad & \\ &\frac{J}{E^2}(W_2 + W_x)\,(1 + l_2\overline{\theta_1 - \theta}) = T_2 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (13),$$

where

$$l_1 = \frac{fW_1 + gW_x}{W_1 + W_x} \quad \text{and} \quad l_2 = \frac{fW_2 + gW_x}{W_2 + W_x} \quad . \; . \; . \; . \; . \; . \quad (14).$$

Hence, subtracting equations (13), we get

$$\frac{J}{E^2}(W_2 - W_1)\,(1 + f\,\overline{\theta_1 - \theta}) = T_2 - T_1$$

or

$$\left.\begin{aligned} &\frac{J}{E^2}(W_2 - W_1)\,(1 + f\,\overline{\theta_0 - \theta}) = T_{2,0} - T_{1,0} \\ \text{and} \qquad & \\ &\frac{J}{E^2}(W_2 - W_1)\,(1 + f\,\overline{\theta_2 - \theta}) = T_{2,2} - T_{1,2} \end{aligned}\right\} \quad . \; . \; . \; . \quad (15),$$

where the observations are taken at two temperatures θ_0 and θ_2, the second suffix of the T's referring to these temperatures.

Hence, by division, we obtain the value of f, the temperature-coefficient of the specific heat of water.

And, if θ_0 is our standard temperature, the value of J is found without any knowledge of the water-equivalent of the calorimeter.

Now $W_1(T_{2,0} - T_{1,0})/(W_2 - W_1)$ is the time that W_1 grms. of water at temperature θ_0 would take to rise 1° C. Subtracting this from $T_{1,0}$ and dividing by $(T_{2,0} - T_{1,0})/(W_2 - W_1)$ we obtain the number of grms. of water at θ_0, to which the calorimeter also at θ_0 is equivalent . . . . (16).

Similarly the expression

$$\frac{T_{1,2} - W_1(T_{2,2} - T_{1,2})/(W_2 - W_1)}{(T_{2,0} - T_{1,0})/(W_2 - W_1)} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (17)$$

gives the number of grammes of water at θ_0 to which the calorimeter at θ_2 is equivalent. Thus we obtain both the water-equivalent of the calorimeter and also its temperature-coefficient g.

Finally J is calculated from

$$J = \frac{E^2 . T}{M(1 + l\overline{\theta_t - \theta})} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (18).$$

The following Table gives the complete working out of J 20. This experiment was selected, because it was the first of which we have given the experimental details (See Table XXVIII.).

TABLE XXXVII.—Experiment J 20.

Date, September 11, 1892. W = 139·776 grms. Clark cells, $n = 4$ at 15°·05 C.

$\sigma = {}$·000512. $\rho = {}$·000621. $tr^3 = 5232 \times 10^4$.

$d\theta_1$.	dt.	Rate.	$\frac{d\theta_1}{dt} \times 10^6$.	Rate correction × 10^6.	$\frac{d\theta_1}{dt} \times 10^6$ at rate 30.	$\frac{d\theta_1}{dt} \times 10^6$ corrected for temperature of Clark cells (+ 0·3).	$\left(\frac{d\theta_1}{dt}\right)_0 \times 10^6$.	$\left(\frac{d\theta_1}{dt}\right)_t \times 10^6$.	T.*	Mean temperature.
·973	199·20	30·54	4884	− 28	4856	4856	809	4047	458·1	14·477
1·235	258·45	30·37	4779	− 19	4760	4761	741	4020	459·4	15·581
·987	209·80	30·47	4704	− 24	4680	4681	672	4009	459·0	16·682
·995	214·70	30·53	4636	− 28	4608	4608	610	3998	458·7	17·683
1·016	223·38	30·45	4548	− 24	4524	4524	548	3976	459·7	18·688
1·278	286·93	30·35	4454	− 18	4436	4437	476	3961	459·6	19·835
1·282	293·75	30·41	4364	− 21	4343	4343	397	3946	459·4	21·115
1·307	306·38	30·56	4266	− 29	4237	4237	316	3921	460·3	22·409
1·598	384·00	30·54	4161	− 28	4133	4134	226	3908	459·7	23·862
·811	199·38	30·42	4067	− 22	4045	4045	155	3890	460·3	25·006
1	2	3	4	5	6	7	8	9	10	11
					REFERENCES.					
Table XXXIV.	Table XXVIII.	Table XXVIII.	p. 479, 2	p. 478, eq. 6, p. 479, 3.	p. 479, 3	p. 479, 4, eq. 11	Table XXVII.	p. 479, 5	p. 480, 6 Table XVI.	Table XXXIV.

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* These values can be compared with Table XL., cols. 11–15. Taking the differences between the individual results of Experiment 20, and the mean numbers of all the experiments on this weight, we get by the method of least squares, mean probable error ± ·075 second. This does not include the inevitable errors arising from the thermometry, the values of the ranges being only given to the nearest $\frac{1}{1000}$ of a degree, such errors can only be removed by the smoothed curve.

The two following Tables, XXXVIII. and XXXIX., give the values of $(d\theta_1/dt) \times 10^6$, corrected to rate 30 and for the temperature of the Clark cells. They correspond to Col. 7 of Table XXXVII. :—

TABLE XXXVIII.—Series I. Values of $(d\theta_1/dt) \times 10^6$.

Group B. W = 188·065.

No. of experiment.	No. of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
9	2	—	—	1314·6	1261·7	1208·9	1150·3	—	—	—	—
8	4	3914	3847	3782	3722	3661	3588	3526	3434	3362	3293
10	4	3916	3853	3801	3725	3655	3597	3523	—	—	—
11	4	3922	3854	3801	3727	3671	3599	3509	3444	3369	3282
12	4	3919	3851	3791	3716	3663	3598	3519	3438	3359	3294
	Mean	3919	3853	3798	3723	3663	3598	3517	3441	3364	3288
5	5	5769	5701	5622	5554	5482	5404	—	5238	5142	5071
6	5	—	—	5629	5551	5488	5406	5326	5232	5143	5064
7	5	5776	5698	5626	5554	5475	5400	5333	5236	5142	5065
	Mean	5773	5699	5626	5553	5482	5403	5329	5235	5142	5067

The following Table gives the complete working out of J 20. This experiment was selected, because it was the first of which we have given the experimental details (See Table XXVIII.).

TABLE XXXVII.—Experiment J 20.

Date, September 11, 1892. W = 139·776 grms. Clark cells, $n = 4$ at 15°·05 C.

$\sigma = \cdot000512$. $\rho = \cdot000621$. $t r^3 = 5232 \times 10^4$.

$d\theta_1$.	dt.	Rate.	$\frac{d\theta_1}{dt} \times 10^6$.	Rate correction × 10^6.	$\frac{d\theta_1}{dt} \times 10^6$ at rate 30.	$\frac{d\theta_1}{dt} \times 10^6$ corrected for temperature of Clark cells (+ 0·3).	$\left(\frac{d\theta_1}{dt}\right)_0 \times 10^6$.	$\left(\frac{d\theta_1}{dt}\right)_e \times 10^6$.	T.*	Mean temperature.
·973	199·20	30·54	4884	− 28	4856	4856	809	4047	458·1	14·477
1·235	258·45	30·37	4779	− 19	4760	4761	741	4020	459·4	15·581
·987	209·80	30·47	4704	− 24	4680	4681	672	4009	459·0	16·682
·995	214·70	30·53	4636	− 28	4608	4608	610	3998	458·7	17·683
1·016	223·38	30·45	4548	− 24	4524	4524	548	3976	459·7	18·688
1·278	286·93	30·35	4454	− 18	4436	4437	476	3961	459·6	19·835
1·282	293·75	30·41	4364	− 21	4343	4343	397	3946	459·4	21·115
1·307	306·38	30·56	4266	− 29	4237	4237	316	3921	460·3	22·409
1·598	384·00	30·54	4161	− 28	4133	4134	226	3908	459·7	23·862
·811	199·38	30·42	4067	− 22	4045	4045	155	3890	460·3	25·006
1	2	3	4	5	6	7	8	9	10	11
REFERENCES.										
Table XXXIV.	Table XXVIII.	Table XXVIII.	p. 479, 2	p. 478, eq. 6, p. 479, 3.	p. 479, 3	p. 479, 4, eq. 11	Table XXVII.	p. 479, 5	p. 480, 6 Table XVI.	Table XXXIV.

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* These values can be compared with Table XL., cols. 11–15. Taking the differences between the individual results of Experiment 20, and the mean numbers of all the experiments on this weight, we get by the method of least squares, mean probable error ± ·075 second. This does not include the inevitable errors arising from the thermometry, the values of the ranges being only given to the nearest $\frac{1}{1000}$ of a degree, such errors can only be removed by the smoothed curve.

The two following Tables, XXXVIII. and XXXIX., give the values of $(d\theta_1/dt) \times 10^6$, corrected to rate 30 and for the temperature of the Clark cells. They correspond to Col. 7 of Table XXXVII. :—

TABLE XXXVIII.—Series I. Values of $(d\theta_1/dt) \times 10^6$.

Group B. W = 188·065.

No. of experiment.	No. of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
9	2	—	—	1314·6	1261·7	1208·9	1150·3	—	—	—	—
8	4	3914	3847	3782	3722	3661	3588	3526	3434	3362	3293
10	4	3916	3853	3801	3725	3655	3597	3523	—	—	—
11	4	3922	3854	3801	3727	3671	3599	3509	3444	3369	3282
12	4	3919	3851	3791	3716	3663	3598	3519	3438	3359	3294
	Mean	3919	3853	3798	3723	3663	3598	3517	3441	3364	3288
5	5	5769	5701	5622	5554	5482	5404	—	5238	5142	5071
6	5	—	—	5629	5551	5488	5406	5326	5232	5143	5064
7	5	5776	5698	5626	5554	5475	5400	5333	5236	5142	5065
	Mean	5773	5699	5626	5553	5482	5403	5329	5235	5142	5067

The following Table gives the complete working out of J 20. This experiment was selected, because it was the first of which we have given the experimental details (See Table XXVIII.).

TABLE XXXVII.—Experiment J 20.

Date, September 11, 1892. W = 139·776 grms. Clark cells, $n = 4$ at $15^{\circ}\cdot05$ C.

$\sigma = \cdot000512$. $\rho = \cdot000621$. $t_r^3 = 5232 \times 10^4$.

$d\theta_1$.	dt.	Rate.	$\frac{d\theta_1}{dt} \times 10^6$.	Rate correction × 10^6.	$\frac{d\theta_1}{dt} \times 10^6$ at rate 30.	$\frac{d\theta_1}{dt} \times 10^6$ corrected for temperature of Clark cells (+ 0·3).	$\left(\frac{d\theta_1}{dt}\right)_0 \times 10^6$.	$\left(\frac{d\theta_1}{dt}\right)_e \times 10^6$.	T.*	Mean temperature.
·973	199·20	30·54	4884	− 28	4856	4856	809	4047	458·1	14·477
1·235	258·45	30·37	4779	− 19	4760	4761	741	4020	459·4	15·581
·987	209·80	30 47	4704	− 24	4680	4681	672	4009	459·0	16·682
·995	214·70	30·53	4636	− 28	4608	4608	610	3998	458·7	17·683
1·016	223·38	30·45	4548	− 24	4524	4524	548	3976	459·7	18·688
1·278	286·93	30·35	4454	− 18	4436	4437	476	3961	459·6	19·835
1·282	293·75	30·41	4364	− 21	4343	4343	397	3946	459·4	21·115
1·307	306·38	30·56	4266	− 29	4237	4237	316	3921	460·3	22·409
1·598	384·00	30·54	4161	− 28	4133	4134	226	3908	459·7	23·862
·811	199·38	30·42	4067	− 22	4045	4045	155	3890	460·3	25·006
1	2	3	4	5	6	7	8	9	10	11
REFERENCES.										
Table XXXIV.	Table XXVIII.	Table XXVIII.	p. 479, 2	p. 478, eq. 6, p. 479, 3.	p. 479, 3	p. 479, 4, eq. 11	Table XXVII.	p. 479, 5	p. 480, 6 Table XVI.	Table XXXIV.

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* These values can be compared with Table XL., cols. 11–15. Taking the differences between the individual results of Experiment 20, and the mean numbers of all the experiments on this weight, we get by the method of least squares, mean probable error ± ·075 second. This does not include the inevitable errors arising from the thermometry, the values of the ranges being only given to the nearest $\frac{1}{1000}$ of a degree, such errors can only be removed by the smoothed curve.

TABLE XXXIX.—Series II. Values of $(d\theta_1/dt) \times 10^6$:

Group A. W = 139·776.

Number of experiment.	Number of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
23	3	3090	3014	2935	2864	2791	2713	2624	2534	(2431)*	(2351)*
20	4	4856	4761	4681	4608	4524	4437	4343	4237	4134	4045
22	4	4834	4763	4681	4604	4526	4442	4336	4246	4141	4049
25	4	4859	4768	4683	4604	4528	4437	4342	4240	4139	4048
	Mean	4850	4764	4682	4605	4526	4439	4340	4241	4138	4047
21	5	7102	6995	6904	6825	6740	6641	6541	6428	6303	6208
24	5	7103	6996	6904	6831	6743	6650	6543	6425	6301	—
	Mean	7103	6995	6904	6828	6741	6646	6542	6426	6302	6208

* Obtained by producing the curve through the preceding numbers. (See Table XXIX.)

TABLE XXXVIII.—(continued).

Group E. $W = 277\cdot931$.

No. of experiment.	No. of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
13	4	2955	2906	2857	2809	2766	2712	2658	2603	2535	2487
14	4	2955	2907	2852	2811	2764	2714	2655	2600	2535	2484
17	4	2940	2895	2851	2801	2748	2704	2646	2591	2526	2477
18	4	2953	2905	2845	2806	2754	2705	2649	2590	2532	2470
The above four experiments cannot be meaned here, as the values of σ and ρ are different. (See p. 450).											
15	5	4349	4299	4238	4185	4135	4070	4006	3945	3889	3826
16	5	4346	4290	4238	4183	4131	4070	4010	3947	3884	3817
19	5	4347	4291	4245	4178	4135	4075	4004	3956	3884	3804
	Mean	4347	4293	4240	4182	4134	4072	4007	3949	3886	3816

TABLE XXXIX.—Series II. Values of $(d\theta_1/dt) \times 10^6$:

Group A. W = 139·776.

Number of experiment.	Number of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
23	3	3090	3014	2935	2864	2791	2713	2624	2534	(2431)*	(2351)*
20 22 25	4 4 4	4856 4834 4859	4761 4763 4768	4681 4681 4683	4608 4604 4604	4524 4526 4528	4437 4442 4437	4343 4336 4342	4237 4246 4240	4134 4141 4139	4045 4049 4048
	Mean	4850	4764	4682	4605	4526	4439	4340	4241	4138	4047
21 24	5 5	7102 7103	6995 6996	6904 6904	6825 6831	6740 6743	6641 6650	6541 6543	6428 6425	6303 6301	6208 —
	Mean	7103	6995	6904	6828	6741	6646	6542	6426	6302	6208

* Obtained by producing the curve through the preceding numbers. (See Table XXIX.)

TABLE XXXIX.—(continued).

Group C. W = 199·674.

Number of experiments.	Number of Clark cells.	Mean temperature of each range.									
		14·477	15·581	16·682	17·683	18·688	19·835	21·115	22·409	23·862	25·006
26 28	4 4	3852 3852	3781 3783	3717 3717	3652 3654	3596 3596	3526 3523	3454 3457	3375 3377	3295 3296	3224 3226
	Mean	3852	3782	3717	3653	3596	3525	3455	3376	3295	3225
27 29	5 5	5625 5623	5554 5550	5479 5478	5414 5415	5349 5347	5264 5266	5188 5192	5101 5102	5013 5012	4939 4936
	Mean	5624	5552	5478	5415	5348	5265	5190	5102	5013	4937
Group D. W = 259·500.											
30 31 32 33	5 5 5 5	4650 4646 4651 4649	4592 4585 4586 4589	4536 4529 4528 4535	4482 4476 4473 4475	4426 4419 4425 4419	4359 4357 4355 4357	4291 4290 4289 4289	4225 4222 4219 4225	4149 4151 4148 4154	4086 4082 4083 4088
	Mean	4649	4588	4532	4476	4422	4357	4290	4223	4150	4085
34	6	6431	6352	6290	6229	6171	6101	6032	5956	5867	5791

Table XL. gives the values of T (equation 12), *i.e.*, the times of rising 1° C. with unit R, and E = 1 Clark cell at 15° C.

TABLE XL.—Times of rising 1° C. = T.

Series 1.

	Group B. W = 188·065.					Group E. W = 277·931.			
No. of cells.	2	4	5			4	5		
No. of experiments performed.	1	4	3			4	3		
Temperature.				Mean.	From curve.			Mean.	From curve.
14·477	—	557·3	556·6	557·0	557·1	740·1	740·6	740·4	740·4
15·581	—	557·6	556·5	557·1	557·2	789·7	740·3	740·0	740·5
16·682	558·5	557·0	556·6	557·0	557·3	740·4	739·7	740·1	740·5
17·683	557·0	558·0	557·3	557·6	557·5	740·4	741·2	740·8	740·6
18·688	558·3	558·2	557·8	558·1	557·6	742·2	740·7	741·4	740·6
19·835	558·7	558·0	558·1	558·1	557·7	741·0	741·5	741·3	740·6
21·115	—	558·0	557·0	557·5	557·8	741·2	741·8	741·5	740·7
22·409	—	558·7	558·0	558·4	557·9	740·4	740·4	740·4	740·7
23·862	—	557·3	557·6	557·4	558·0	739·9	738·6	739·3	740·7
25·006	—	558·3	558·2	558·3	558·2	741·1	741·1	741·1	740·8
I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.

TABLE XL.—(continued).

Series II.

	Group A. W = 139·776.					Group C. W = 199·674.				Group D. W = 259·500.			
No of cells.	3	4	5			4	5			5	6		
No. of experiments performed.	1	3	2			2	2			4	1		
Temperature.				Mean.	From curve.			Mean.	From curve.			Mean.	From curve.
14·477	458·7	458·7	458·3	458·7	458·8	580·5	580·8	580·7	580·9	702·8	701·9	702·7	702·9
15·581	458·7	459·0	459·4	459·1	458·9	581·2	580·8	581·0	581·0	703·2	703·5	703·3	703·0
16·682	459·0	458·8	459·4	459·1	459·0	581·2	581·0	581·1	581·1	703·0	703·2	703·0	703·0
17·683	459·2	459·0	458·9	459·0	459·1	581·9	580·8	581·3	581·1	703·0	703·2	703·1	703·0
18·688	459·9	459·4	459·1	459·4	459·2	581·7	581·0	581·3	581·2	703·1	703·0	703·1	703·0
19·835	459·4	459·4	459·1	459·3	459·3	582·0	581·8	581·9	581·3	703·7	703·1	703·6	703·1
21·115	459·4	459·7	459·0	459·4	459·5	581·1	581·1	581·1	581·4	703·5	702·5	703·3	703·1
22·409	459·3	459·9	459·6	459·7	459·6	581·7	581·6	581·6	581·4	703·0	702·5	702·9	703·1
23·862	459·6	459·4	460·0	459·7	459·7	581·0	581·1	581·1	581·5	702·4	702·9	702·7	703·2
25·006	459·9	459·8	460·1	459·9	459·8	581·6	581·4	581·5	581·6	703·2	704·0	703·4	703·2
I.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.	XIX.	XX.	XXI.	XXII.	XXIII.

Remarks on Table XL.

The column headed "Mean" was obtained by multiplying the numbers in the preceding columns of each section by the number of experiments from which they were deduced, and dividing the sum of the products by the total number of experiments. The "means" were plotted on a scale such that $\frac{1}{10}$ inch of ordinate represented a difference of ·1 second in T, and the smooth curve in each case so drawn that the sum of the positive and negative areas included between it and the experimental curve was zero. The curvature is so slight over our range that by assuming T to be a linear function of θ_1, our "curve numbers" are not affected when written to four figures.

It will be found that our results (obtained from the "smooth curve" values of T) are in much closer agreement than might be expected from a study of Table XL. This indicates that the irregularities are due to some (or all) of the following causes.

(1.) Errors in the comparative values of the ranges. The effect of such errors would be common to all the experiments, and there is evidence that the sixth range is too small, the value of T over that range being nearly always too large; the error, however, does not amount to 0°·001. Again, the ninth range is evidently too great. These errors do not affect the result when the values of T over two or three consecutive ranges are meaned (as is done by the smooth curve), the excess in one case is then compensated by the deficiency in another.

(2.) The *irregular* (as apart from the strictly recurrent) "stickings" of the thermometer; these, as previously mentioned, must mean out when sufficient observations are taken over the same range.

(3.) Irregularities in the temperature of the water ejected through the opening in the cylinder on to the thermometer bulb, water from cooler parts of the calorimeter being followed by a gush of warmer water brought direct from the hot wire. This, no doubt, is a fruitful cause of alternate lag and acceleration, but again, it is evidently an irregularity which would but slightly affect the results of many experiments continued through a sufficient number of intervals of time. We have previously pointed out that the "throw" was irregular throughout Groups B and E, and the effect on the individual experiments is very marked; nevertheless the values of T derived from the fifteen experiments in those groups are in practical agreement with the values deduced from the experiments in Series II.

(4.) Personal errors of observation which, from their very nature, are unlikely to recur at regular intervals or over identical ranges. These, no doubt, are the origin of many discrepancies, the strain on the observer during these experiments being great.

The cumulative effect of all the above causes of irregularity may at times be large. In Series I. differences of 1 in 500 occasionally present themselves, although, in Series II., there is only one case in which the difference between the mean and the curve-numbers exceeds 1 in 1000, and only two others in which it exceeds 1 in 2000.

The values of J deduced from the different groups (using the numbers given by the respective smooth curves) are, however, in much closer agreement, the extreme divergence being less than 1 in 4000 (see Table XLIII.).

It is worth mentioning that in previous years we found the agreement between individual experiments in the same group (having the same E.M.F.) was closer than that shown in the above Table; but, on the other hand, the values of J and of the water equivalent given by different groups (or by members of the same group when the number of cells was changed), differed hopelessly, in some cases by more than 1 per cent., showing that constant causes of error had not been eliminated. We attribute the improvement (1) to the alteration in the form of stirrer, for the mechanical work done in previous years varied capriciously, although no doubt the thermometer bulb was more thoroughly "washed" than before the alteration; (2) to the correction (now introduced for the first time) for the difference between the temperature of the wire and that indicated by the thermometer.

As it is more convenient to examine the values of T for integral values of the temperature, we have read off from the smooth curves the values of T for every two degrees covered by our range, and also the values at 15° C. and 25° C. These are given in the following table.

TABLE XLI.—Values of T from Smoothed Curve.

Temperature.	Series I.		Series II.		
	Group B. W = 188·065.	Group E. W = 277·931.	Group A. W = 139·776.	Group C. W = 199·674.	Group D. W = 259·500.
14·000	557·04	740·44	458·77	580·90	702·88
15	557·14	740·46	458·87	580·95	702·91
16	557·24	740·48	458·97	581·01	702·95
18	557·43	740·55	459·16	581·13	703·00
20	557·62	740·60	459·35	581·25	703·05
22	557·80	740·66	459·53	581·37	703·11
24	557·99	740·72	459·72	581·49	703·17
25	558·09	740·75	459·81	581·55	703·20
26	558·18	740·77	459·92	581·61	703·22
1	2	3	4	5	6

Taking the data given in Columns 4 and 6 of the above table, we give an example of the final calculation of our results.

Using the notation of pp. 478–481, we have the following values

$$W_1 = 139{\cdot}776 \qquad W_2 = 259{\cdot}500 \qquad \theta_0 = 15^\circ\text{ C.}$$
$$T_{1,0} = 458{\cdot}87 \qquad T_{2,0} = 702{\cdot}91 \qquad \theta_1 = 25^\circ\text{ C.}$$
$$T_{1,2} = 459{\cdot}81 \qquad T_{2,2} = 703{\cdot}20 \qquad E = 1{\cdot}4344^{*}\text{ volts.}$$

Substituting these values in equation (15) and dividing, we obtain

$$\frac{1 + 15f}{1 + 25f} = \frac{243{\cdot}39}{244{\cdot}04} = {\cdot}99734.$$

As the difference in weight of water here amounts to nearly 120 grms., it is probable that the value obtained in this case is the most reliable.

In the same manner using the values of W and T given in Columns 4 and 5, we get

$$\frac{1 + 15f}{1 + 25f} = \frac{121{\cdot}74}{122{\cdot}08} = {\cdot}99722.$$

And again from Columns 5 and 6,

$$\frac{1 + 15f}{1 + 25f} = \frac{121{\cdot}65}{121{\cdot}96} = {\cdot}99746.$$

Hence mean value of $\frac{1 + 15f}{1 + 25f}$ deduced from Series II. = ·99734.†

Hence, adopting 15° C. as the standard temperature, the

Specific Heat of Water = 1 − ·000266 (θ_1 − 15).‡

Also by means of equation (15) we get the following values of J,

Columns 4 and 6 . . . . $J = 4{\cdot}1939 \times 10^7$
„ 4 „ 5 . . . . $J = 4{\cdot}1940 \times 10^7$
„ 5 „ 6 . . . . $J = 4{\cdot}1940 \times 10^7$
Mean . . . . **$4{\cdot}1940 \times 10^7$.**

This value of J, as pointed out on p. 481, is entirely *independent of the value assigned to the water equivalent of the calorimeter.*

Performing the operations of equation 16, we find that the water equivalent at

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* See p. 388, *supra.*

† If we mean the results from Columns 4 and 5 and 5 and 6, we must of course obtain the same result as that given by Columns 4 and 6. We have however given the numbers in the above form as they show the nature of the agreement between the different groups.

‡ Over the range 14° to 26° C.

15° C. in terms of water at 15° C. = 85·340 grms.; and from the operations indicated in equation (17), we get water equivalent at 25° C. in terms of water at 15° C. = 86·174 grms. Hence

$$\textbf{Water equivalent} = 85{\cdot}340\ \{1 + {\cdot}000977\ (\theta_1 - 15)\}.^*$$

We are now in a position to find the value of $M(1 + l\,\overline{\theta_1 - \theta})$ for any weight of water at any temperature, and the value of J can then be found by equation (18).

The following Table gives the value of $M(1 + l\,\overline{\theta_1 - \theta})$ for each weight of water at temperatures 15°, 20°, and 25° C.

TABLE XLII.—Value of $M(1 + l\,\overline{\theta_1 - \theta})$ at 15°, 20°, and 25°.

Group.	W.	15°.	20°.	25°.
A	139·776	225·116	225·347	225·578
B	188·065	273·405	273·572	273·739
C	199·674	285·014	285·165	285·317
D	259·500	344·840	344·912	344·984
E	277·931	363·271	363·318	363·366

Substituting the values of T (Table XLI.) and $M(1 + l\,\overline{\theta_1 - \theta})$ in equation (18), we get the following values of J. The first column shows the group of experiments from which each value is derived.

TABLE XLIII.—Values of J given by each Group at different Temperatures.

Group.	15°.	20°.	25°.	Mean.
A	4·1940 × 10^7	4·1940 × 10^7	4·1939 × 10^7	4·1940
B	4·1930 „	4·1941 „	4·1949 „	4·1940
C	4·1939 „	4·1938 „	4·1937 „	4·1938
D	4·1940 „	4·1939 „	4·1940 „	4·1940
E	4·1938 „	4·1940 „	4·1943 „	4·1940
				4·1940

We have in the above Table given the values resulting from the calculation at different temperatures since the limit of our experimental errors is thus clearly indicated; for the values of J ought (in the absence of experimental errors) to be identical at all temperatures. The close agreement between the values from different groups, and from the same group at different temperatures, is a satisfactory proof of the accuracy of our determination of the water equivalent of the calorimeter, and or the changes in it and in the capacity for heat of the water. If we reject Group B (and we have already shown that it has little value) the results are practically identical.

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* Over the range 14° to 26° C.

Hence if we assume—

(1.) The unit of resistance as defined in the 'B.A. Report,' 1892.

(2.) That the E.M.F. of the Cavendish standard Clark cell at 15° C. = 1·4342 volts.[*]

(3.) That the thermal unit = quantity of heat required to raise 1 grm. of water through 1° C. at 15° C.

The most probable value of

$$J = 4{\cdot}1940 \times 10^7.\dagger$$

This, by reduction, gives the following:—

J = 427·45 kilogrammetres in latitude of Greenwich ($g = 981{\cdot}17$).
J = 1402·2 ft.-lbs. per thermal unit C in latitude of Greenwich ($g = 981{\cdot}17$).
J = 778·99 „ „ F „ „ „ „

SECTION XV.—DISCUSSION OF THE RESULTS.

As stated in the Introduction, we proposed to determine the value of J in terms of the thermal unit there defined, viz., the quantity of heat required to raise unit mass of water through 1° C. at 15° C. ROWLAND has preferred to tabulate his results by giving the changes in the numerical value of J caused by changes in the capacity for heat of water. We can, however, deduce from his table the expression for the change in the specific heat of water over our range.

Expressed in the same form as above, it becomes

$$1 - {\cdot}000400\,(\theta_1 - 15).$$

The difference between our results on this point is marked, and it is evident that further investigation is required. ROWLAND finds that the minimum value lies between 30° and 35° C.; we hope to carry the investigation beyond that temperature, and some explanation of the difference in our results may then present itself. The whole question is probably one of thermometry, and possibly our revision of this part of the subject may bring our results into closer agreement. ROWLAND himself (p. 198) points out that the whole matter depends on a small difference which he denotes by the letter *m*. This quantity appears to be of the same nature as that which we term $d\,(=\theta-pt)$ in the platinum-thermometer formulæ. This quantity *m* had to be determined from a difference of the order of 0°·05 C. between the air- and mercury-thermometers. The method adopted for finding the value of the analogous quantity *d* would appear much less liable to error.

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[*] If we assume the E.M.F. of our Clark cells to be the same as that of the Cavendish standard (and we are inclined to think we have over-estimated the difference), we get J = $4{\cdot}1930 \times 10^7$.

† The value obtained by us in 1891 = $(4{\cdot}192 +) \times 10^7$; *supra*, p. 365.

ROWLAND himself was evidently anxious to throw more light on this portion of his investigation, for he performed, with great care, a series of experiments by the method of mixtures. These are given on pp. 125 to 130; he has not, however, tabulated his results. We give, with one exception (No. 10), the results of all these experiments which did not involve temperatures above 30° C.

TABLE XLIV.—Specific Heat of Water by the method of Mixture (ROWLAND).

Experiment.	ROWLAND'S result.	Mean coefficient of decrease.
1	$\frac{\text{Mean sp. heat } 0^\circ-18^\circ}{\text{,, ,, } 18^\circ-27^\circ} = 1{\cdot}0025$	·000185
2	Above our range	—
3	$\frac{\text{Mean sp. heat } 0^\circ-17^\circ}{\text{,, ,, } 17^\circ-29^\circ} = 1{\cdot}0024$	·000165
4	Rejected by ROWLAND	—
5	$\frac{\text{Mean sp. heat } 0^\circ-18^\circ}{\text{,, ,, } 18^\circ-30^\circ} = 1{\cdot}0067$	·000447
6	Rejected by ROWLAND	—
7	$\frac{\text{Mean sp. heat } 0^\circ-16^\circ}{\text{,, ,, } 16^\circ-24^\circ} = 1{\cdot}0010$	·000083
8	$\frac{\text{Mean sp. heat } 0^\circ-17^\circ}{\text{,, ,, } 17^\circ-25^\circ} = 1{\cdot}0027$	·000216
9	$\frac{\text{Mean sp. heat } 0^\circ-21^\circ}{\text{,, ,, } 21^\circ-28^\circ} = 1{\cdot}0045$	·000321
10	Requires explanation, as it gives an increasing specific heat. Also the difference between the mean temperatures (2°·5 C.) is too small to enable us to expect an accurate result. The remaining experiments are above our range	—
	Mean	·000236

It is thus apparent that the results of ROWLAND'S investigation by this method are in closer agreement with our conclusions than they are with his own.* We do not, however, attach much importance to this point, since—as he justly remarks—the method by the mechanical equivalent is of far greater value.

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* Our value ·000266.

The values of J deduced from the two groups of Series I. and the middle group of Series II. are dependent upon the value of the water equivalent deduced from the highest and lowest weights of Series II. The agreement in the results is a sufficient proof of the accuracy of the value assigned to that equivalent.

The coefficient of increase in the specific heat of the calorimeter may appear large. TOMLINSON ('Roy. Soc. Proc.,' 1885), gives the following as the specific heat of copper and zinc:—

$$\begin{aligned} \text{Specific heat of copper*} &= \cdot 09008 + \cdot 000065\ \theta_1, \\ ,, \quad ,, \quad \text{zinc} &= \cdot 09009 + \cdot 000075\ \theta_1, \end{aligned}$$

and he quotes the values given by BÈDE as

$$\begin{aligned} \text{Copper} &= \cdot 0892 + \cdot 000065\ \theta_1, \\ \text{Zinc} &= \cdot 0859 + \cdot 000084\ \theta_1. \end{aligned}$$

Expressing the mean of these results in the same form as that given to the expression for the water equivalent, we get

$$\begin{aligned} \text{Specific heat of copper} &= \cdot 0896\ (1 + \cdot 00071\ \theta_1), \\ ,, \quad ,, \quad \text{zinc} &= \cdot 0876\ (1 + \cdot 00091\ \theta_1). \end{aligned}$$

TOMLINSON states that when reducing his observations, he used REGNAULT'S expression for the capacity for heat of water, viz., $1 + \cdot 00004\ \theta_1 + \cdot 0000009\ \theta_1^2$.

Unfortunately the account of these experiments given in the 'Proceedings' does not supply us with sufficient data to estimate the changes in the coefficient of θ_1 that would be caused by assuming our expression for the specific heat of water. We are, therefore, unable to say how far our rate of increase in the water equivalent agrees with the above determinations. We are, however, enabled to state that the values of the specific heats at 0° C. would be reduced by about 6 parts in 1000, on the assumption that our formula holds over so large a range.

Our calorimeter may be regarded as a mixture of copper, zinc, gold, glass, and mercury, and the coefficient of increase of such a mixture can only be obtained by direct experiment.

ROWLAND, in order to compare his results with those of JOULE, expressed both in kilogrammetres at Baltimore ($g = 980 \cdot 05$), and gave the numerical value of J in terms of water at differing temperatures, assuming his own value of the specific heat of water. In order to make a similar comparison we give our results in the same form. The following Table is extracted from the Appendix to ROWLAND'S paper, with the exception of the last column, which we have added.

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* TOMLINSON'S paper (Table II., p. 497) contains an obvious misprint in the coefficient of θ_1.

TABLE XLV.—Comparative values of J given by JOULE, ROWLAND, and GRIFFITHS.

Date.	JOULE's method.	Temperature of water.	JOULE's value in the metric system. Reduced to ROWLAND's air-thermometer.	ROWLAND, 1880.	GRIFFITHS, 1892.
		° C.			
1847	Friction of water . . .	15	442·8	427·4	427·9
1850	„ „ . . .	14	426·8	427·7	428·1
1850	Friction of mercury . .	9	427·5	428·8	428·6
1850	„ „ . .	9	428·7	428·8	428·6
1850	Friction of iron. . . .	9	429·1	428·8	428·6
1850	„ „	9	428·0	428·8	428·6
1867	Electric heating . . .	18·6	428·0*	426·7	427·6
1878	Friction of water . . .	14·7	425·8	427·6	427·9
1878	„ „ . . .	12·7	427·1	428·0	428·2
1878	„ „ . . .	15·5	426·0	427·3	427·9
1878	„ „ . . .	14·5	422·7	427·5	428·0
1878	„ „ . . .	17·3	426·3	426·9	427·7

ROWLAND remarks as follows : "JOULE rejected quite a number of his results, but I have thought it best to include them, giving them small weight however. In this way we obtain a value of JOULE's experiments of 426·75 at 14°·6 C., my value at this point being 427·52 ; the difference amounts to 1 in 550 only."

Adopting this method of comparison we get (at 14°·6 C., and expressed in kilogrammetres at Baltimore):

JOULE.	ROWLAND.	GRIFFITHS.
426·75	427·52	427·98

Thus our difference from JOULE amounts to 1 in 350,† and our difference from ROWLAND amounts to 1 in 930.‡

The difference between ROWLAND's determinations of the changes in the specific heat of water and ours, would cause the values of J to be identical if expressed in terms of a thermal unit at 11°·5 C., and below that temperature ROWLAND's value would be the greater.

[Notes by E. H. G., added April, 1893.

1. No change in the value of the various units, or constants, involved in our

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* Value deduced by ROWLAND by assuming his own value of the ohm.

† If we attach equal weight to the different values given by JOULE (ROWLAND in obtaining the above numbers attached arbitrary values to different experiments and methods) we obtain J = 428·23 as the mean value at 13°·2 C., *i.e.*, J = 428·08 at 14°·6 C. This *exceeds* our value by 1 in 4281.

‡ If we assume the E.M.F. of our cells to be that of the Cavendish standard, our value would be 427·86 and these differences would become 1 in 386 and 1 in 1260 respectively.

investigation could bring our results into absolute agreement with those obtained by ROWLAND, since, owing to the difference in the expressions for the temperature coefficients of the specific heat of water, it is inevitable that our conclusions should agree at some one temperature, but must necessarily differ when expressed in terms of a thermal unit at any other temperature, and, thus, changes in the values of the units would only alter the temperature of agreement. For example, Dr. GUILLAUME has pointed out to me that the experiments of Commandant DEFFORGES lead to the conclusion that the value of *g*, at Greenwich, should be increased from 981·17 centims. to 981·24 centims. A similar correction would slightly increase ROWLAND's value of J, and thus cause our point of agreement to be about 12° C. instead of 11°·5 C. Again, the value of *g*, assumed by Lord RAYLEIGH ('Phil. Trans.,' A., 1884, p. 427), would have to be slightly increased, and the resulting values of the electro-chemical equivalent of silver, and of the absolute electromotive force of a Clark cell, would require modification, but the only result of any such change would, as before, be to shift the temperature of agreement.

It is, therefore, evident that the difference is chiefly due to errors in thermometry. An error of 0°·01 C., in the value of the range 14° to 25° C., would suffice to explain the discrepancy between the results, and the close agreement in the range value, as obtained from our standard and the Paris nitrogen standard (see p. 430, *supra*), appears to me to warrant a brief criticism of ROWLAND's methods and conclusions.

The science of exact thermometry has made great strides during the past fourteen years, and, no doubt, much of this progress is due to ROWLAND's initiative, for his work undoubtedly marks a distinct advance in this subject. Great, however, as his services have been, he would, I feel sure, be the first to admit that his investigations of 1878-79, by no means exhaust the possibilities of further progress in the science of thermometry. In his paper, he constantly indicates possible causes of error, and, in some cases, makes valuable suggestions as to methods by which they may be eliminated.

In a previous paper ('Phil. Trans.,' A., 1891, p. 155) it was pointed out by Mr. CALLENDAR and myself that "the indications of the air-thermometer cannot be relied on beyond 0°·01 C.... the limit of accuracy of a single reading of the barometer."

On p. 95 of his paper, ROWLAND gives the data on which the following statements are based. A difference of ·03 millim. in the reading of his barometer gave a difference of 0°·01 C. in the air-temperature, and an error of 0°·3 C. in the estimation of the temperature of his barometer-column would produce the same effect. From his remarks, it is evident that he regarded his possible error from this cause alone as ± 0°·01 C.

Again, his temperatures depended on the direct observation of the height of a mercury manometer column by means of a cathetometer, and everyone who has worked with such an instrument knows the difficulty of obtaining measurements accurate to ± ·02 millim. (It is true that *small* differences can be read to a much closer degree of accuracy by means of a microscope fitted with a micrometer scale, but this method is not applicable when large differences have to be read, nor does it appear to have been adopted by Rowland.) On p. 94, Rowland remarks that on repeating the observations upon an accurate scale, they "agreed with the scale to within two or three hundredths of a millimetre, which was as near as I could read on such an object," and 0·03 millim. appears to have been equivalent to 0°·01 C. Also, the temperature of the manometer column was only ascertained by means of a thermometer placed alongside it, and the uncertainty thus introduced might nearly double the resulting error.

A study of Rowland's Tables, XI. to XV., leads to the conclusion that the discrepancy between individual observations, due to the above and other causes, in some cases amounted to as much as 0°·03 to 0°·04 C.

The following extract will, I think, show that Rowland did not rely upon his observations to ± 0°·01 C. In his concluding remarks he says: "I now believe that, with the improvement to the air-thermometer of an artificial atmosphere of constant pressure, we could be reasonably certain of obtaining the temperature at any point up to 50° C., within 0°·01 C. from the mean of two or three observations. I believe that my own thermometers scarcely differ much more than that from the absolute scale at any point up to 40° C., but they represent the mean of eight observations. However, there is an uncertainty of 0°·01 C. at the 20° C. point, owing to uncertainty in the value of m. But, taking $m = \cdot 00015$, I hardly think that the point is uncertain to more than that amount for the thermometers Nos. 6163, 6165, and 6166."

The causes of error above enumerated do not affect the platinum thermometer determinations to the same extent. It is true that the values of d (which corresponds to Rowland's quantity, m), are dependent upon the boiling-points of water and sulphur, which are again dependent on barometer readings, but, on the other hand, an error of as much as 0°·1 C., at those points, would produce so little change in d as to leave the value of our range unaltered, although it would slightly affect the actual elevation; more especially, it should be noticed that any small changes of pressure which may take place *during the comparison of the platinum thermometer with other thermometers* do not affect the results.

The above considerations appear to me to justify the conclusion that the value of Rowland's temperature range, 14° to 25° C., may be in error by as much as 0°·011 C., and, if this is the case, the values of the C.G.S. unit of heat, and of the temperature coefficients of the specific heat of water, as deduced from his experiments, may be identical with ours.

2. I have, since the publication of an abstract of this paper, received several communications criticising our statement (see p. 496, *supra*), that we differ from the results obtained by Joule, by "1 part in 350," and my correspondents have pointed out that the ratio, 772·55 to 778·99, is very different from that of 349 to 350. It would, therefore, appear necessary that I should indicate, more fully than I have

done, the ground upon which we based the above statement. Joule himself, in 1879, made a careful comparison of his thermometer with one which had been standardized by Rowland (Baudin's Therm., No. 6166). The results were published as an appendix to Rowland's paper in the 'Proceedings of the American Academy' for March, 1880. A considerable correction in the values obtained by Joule was then found to be necessary. Again, Joule had assumed Regnault's expression for the specific heat of water, and a correction for the capacity for heat of the calorimeter was also rendered necessary by the changes in the thermometry. As the appendix above referred to appears to be but little known in this country, I give one example (p. 44, *ibid.*) to indicate the nature of the various corrections.

Joule's result in 1878 (friction of water)		774·6
Correction for thermometer	+	3·7
„ „ capacity for heat	+	·2
„ „ latitude (to Baltimore)	+	·9
„ „ vacuum	−	·9
Corrected value (at 12°·7 C.)		778·5

As these corrections were published in 1880, it appears probable that, had they not met with Joule's approval, he would have signified his dissent, but I can find no record of any such expression of opinion on his part.

As previously stated, Rowland attached what appear to me to be curiously arbitrary values to certain of Joule's experiments. If we accept his verdict as to their respective merits, the mean of the selected experiments, as corrected by Rowland (but reduced by me to temperature 15°, and to latitude Greenwich), is 776·75 ft.-lbs.; but if we assign equal value to all Joule's published results, the corrected mean becomes 779·17 ft.-lbs., and it was on these numbers that our statement was based.

It is extraordinary that, although these results (or rather the corrections by which they are deduced) were published as far back as 1880, we still find that our modern books of reference continue to give 772·55 as the number resulting from Joule's own experiments.]

APPENDIX I.

Determination of the "Null Point" and an alternative Method of calculating the Results.

As stated in the introduction we append a full analysis of the "null point," *i.e.*, the point at which the radiation is self-eliminated.

Using a similar notation to that on p. 478, we have as the equation of condition

$$\mathrm{M}\,\frac{d\theta}{dt} = a - \rho\theta \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$

where the temperatures are measured from that of the surrounding envelope.

For the sake of simplicity we can assume that the values of a and M remain constant.

Integrating and putting $\lambda = \rho/\mathrm{M}$, $\mu = a/\mathrm{M}$, and determining the constant from the fact that when $t = 0$, $\theta = -\theta_0$, we obtain the equation

$$e^{\lambda t} = \frac{\frac{\mu}{\lambda} + \theta_0}{\frac{\mu}{\lambda} - \theta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

If there had been no radiation, $\rho = 0$, and the equation of condition would have been $d\theta/dt = \mu$.

Integrating, and using the same constant as before

$$\theta = \mu t - \theta_0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

If we find the points of intersection of (2) and (3) one point $(-\theta_0, 0)$ is that at which the experiment commenced, the other (Θ, T) is the point on (2) at which the radiation is eliminated.

It is more convenient, for experimental work, to obtain an expression involving T rather than Θ; substituting therefore the value of θ given by (3) in (2), we obtain

$$e^{\lambda \mathrm{T}} - \frac{\frac{\mu}{\lambda} + \theta_0}{\mu}\,\frac{e^{\lambda \mathrm{T}} - 1}{\mathrm{T}} = 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4).$$

This equation can be solved for T when the values of λ, μ, θ_0 are known.

In order to obtain λ, μ, we can take the following observations: Commence an experiment at $-\theta_0$, note the time t when $\theta = 0$, and again note the time t_2, when $\theta = +\theta_0$.

Then equation (2) gives the relations

$$e^{\lambda t_1} = \frac{\frac{\mu}{\lambda} + \theta_0}{\frac{\mu}{\lambda}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5),$$

and

$$e^{\lambda t_2} = \frac{\frac{\mu}{\lambda} + \theta_0}{\frac{\mu}{\lambda} - \theta_0} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6).*$$

From this pair of equations we can eliminate $\lambda\theta_0/\mu$ and obtain

$$2\{e^{\lambda(t_2-t_1)} - 1\} - \{e^{\lambda t_2} - 1\} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7),$$

which is the equation from which to determine λ.

So far as we are aware, the solution of these equations cannot be expressed as an algebraical expansion, except on the assumption that λt_1, λt_2, λT are small. Making this assumption, and writing $x = t_2 - 2t_1$, successive approximation leads to

$$\mathrm{T} = 2t_1 + \tfrac{1}{3}x - \tfrac{2}{9}\frac{x^2}{t_1} + \frac{787}{200}\frac{x^3}{t_1^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8).$$

The method which we have found best adapted to the numerical solution of such equations amounts practically to tracing their graph, using, for instance, in equation (7), λ and $f(\lambda)$ as abscissa and ordinate. A rough approximation to the value of λ is soon obtained, and afterwards by means of a table of logarithms as close an approximation as is necessary can be found.

Since μ does not occur as an exponential, its value is at once found from (5) or (6).

Equation (4) can be solved for T in a similar manner to equation (7), using T and f(T) as abscissa and ordinate. A rough approximation to the value of T is given by equation (8),

$$\mathrm{T} = 2t_1 + \tfrac{1}{3}(t_2 - 2t_1).$$

The forms given to equations (4) and (7) are those which we have found most convenient from which to calculate the values of $f(\lambda)$ and f(T).

Besides being thus able to eliminate radiation, we can at the same time eliminate the change in all those quantities whose rate of variation can be expressed as a linear function of the temperature. Thus, on p. 477, we have shown that when the variations in M and a are considered, their full values are

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* Equation (6) shows that θ_0 is not altogether at our disposal, but must be less than μ/λ. For, otherwise, the calorimeter would reach its final state, where the loss by radiation balances the supply of heat, and could never attain the temperature $+\ \theta_0$.

$$a' = a\{1 - k(\theta + \beta)\}$$
$$\mathrm{M}' = \mathrm{M}\{1 + l\theta\},$$

and, hence, equation (1) becomes

$$\mathrm{M}(1 + l\theta)\frac{d\theta}{dt} = a\{1 - k(\theta + \beta)\} - \rho\theta. \quad . \quad . \quad . \quad . \quad . \quad (9).^*$$

By neglecting squares and products of small quantities (9) can be written in the same form as (1), thus,

$$\mathrm{M}\frac{d\theta}{dt} = a(1 - k\beta) - \{\rho + a(k + l)\}\theta,$$

and the values of λ and μ become

$$\lambda = \frac{l + a(k + l)}{\mathrm{M}}, \quad \mu = \frac{a(1 - k\beta)}{\mathrm{M}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

where $a = n^2\mathrm{E}^2/\mathrm{JR}$.

Let λ_{11}, μ_{11} denote the values of λ and μ when a weight of water W_1 and an E.M.F. of n_1 Clark cells are used, the first suffix referring to the weight, the second to the number of Clark cells; further, as before, let W_x be the water equivalent.

Then

$$\mu_{11} = \frac{n_1^2\mathrm{E}^2\{1 - k\beta_1\}}{\mathrm{JR}(\mathrm{W}_1 + \mathrm{W}_x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

(β_1 is written for β as its value is dependent on n).

Similarly

$$\mu_{21} = \frac{n_1^2\mathrm{E}^2\{1 - k\beta_1\}}{\mathrm{JR}(\mathrm{W}_2 + \mathrm{W}_x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Hence, by division of (11) and (12),

$$\frac{\mu_{11}}{\mu_{21}} = \frac{\mathrm{W}_2 + \mathrm{W}_x}{\mathrm{W}_1 + \mathrm{W}_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

Equation (13) gives the means of calculating the water equivalent.

Writing M_1, M_2 for $\mathrm{W}_1 + \mathrm{W}_x$ and $\mathrm{W}_2 + \mathrm{W}_x$ respectively, we have

$$\mu_{12} = \frac{n_2^2\mathrm{E}^2\{1 - k\beta_2\}}{\mathrm{J.R.M}_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

which, with (11), gives

$$\frac{\mu_{12}}{\mu_{11}} = \frac{n_2^2}{n_1^2} \cdot \frac{1 - k\beta_2}{1 - k\beta_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

---

* This does not introduce σ, the heat developed by the stirrer, but this can be found by preliminary experiments and due allowance made for it.

If k, the temperature coefficient of the resistance of the wire, is determined by preliminary observations, and we assume $\beta = p\mathrm{E}^2$ where p is a constant (p. 400), we have with (15) sufficient data to determine the values of β_1, β_2. Equation (11) then gives

$$\mathrm{J} = \frac{n_1^2 \mathrm{E}^2 \{1 - k\beta_1\}}{\mathrm{M}_1.\mathrm{R}.\mu_{11}} \quad \ldots\ldots\ldots \quad (16).$$

Again equation (10) gives

$$\lambda_{11} = \frac{\rho}{\mathrm{M}_1} + \frac{n_1^2 \mathrm{E}^2 (k + l_1)}{\mathrm{J.R.M}_1} \quad \ldots\ldots\ldots \quad (17),$$

and

$$\lambda_{12} = \frac{\rho}{\mathrm{M}_2} + \frac{n_2^2 \mathrm{E}^2 (k + l_1)}{\mathrm{J.R.M}_1} \quad \ldots\ldots\ldots \quad (18),$$

which are two equations to give ρ and l_1.

Similarly, by performing the operations λ_{21} and λ_{22}, we could determine the value of l_2; and since (p. 480)

$$l_1 = \frac{f\mathrm{W}_1 + g\mathrm{W}_x}{\mathrm{W}_1 + \mathrm{W}_x} \qquad \text{and} \qquad l_2 = \frac{f\mathrm{W}_2 + g\mathrm{W}_x}{\mathrm{W}_1 + \mathrm{W}_x},$$

we can find both f and g, the coefficients of increase in specific heat of water and of the calorimeter.

APPENDIX II.

The Regulator for Maintaining the Laboratory at a Constant Temperature.

A narrow glass tube, several feet in length, was fixed in a horizontal position on one of the walls.* This tube contained chloride of silver, and a stream of dry ammonia gas at a low temperature was passed through it until the compound $\mathrm{AgCl,3NH_3}$ was formed.† This compound, as pointed out by ISAMBERT, dissociates at ordinary temperatures, the pressure of the vapour at about 15° C. altering by more than 12 millims. per degree. Care had to be taken to completely saturate the AgCl, for the pressure of the vapour $\mathrm{AgCl,2NH_3}$ changes at ordinary temperatures by a much smaller amount. The horizontal tube communicated with a gas regulator of the ordinary pattern, except that the diameter of the regulator tube could be made of any size, since we are dealing with the pressure of a saturated vapour, not with that of a gas, and thus the changes in *volume* caused by a movement of the mercury column could be disregarded, the diameter of the smallest pipe in our regulator exceeding 1 centim. In order to diminish the effect of changes in the atmospheric pressure, some air was allowed to remain in the horizontal tube. As the column of mercury communicating with the regulator was about 800 millims. in height, this residual air was under a pressure of two atmospheres, and, its volume being very small, and the diameter of the mercury tube large, a small movement sufficed to compensate changes in atmospheric pressure. Thus, an alteration of 20 millims. in the barometer produced a change of only a fraction of a degree in the resulting laboratory temperature. The gas, after passing through the regulator, communicated with a large Doulton ware stove, in which small pilot lights were always kept burning by means of an independent supply. An entrance and an exit tube, closed by glass taps, were so fixed in the vertical tube containing the mercury that, by their means, the instrument could be adjusted.

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* The exterior of this tube was blackened.

† This substance was suggested by Mr. C. T. HEYCOCK.

The action of the apparatus is, in many ways, interesting. On a day when the gas would be burning, if the room was unoccupied, the presence of one person in the room would diminish the supply, whereas the presence of two or three would cause it to be entirely cut off. We were surprised to learn, by this means, the amount of heat communicated to a room by the presence of a human being.

.1.

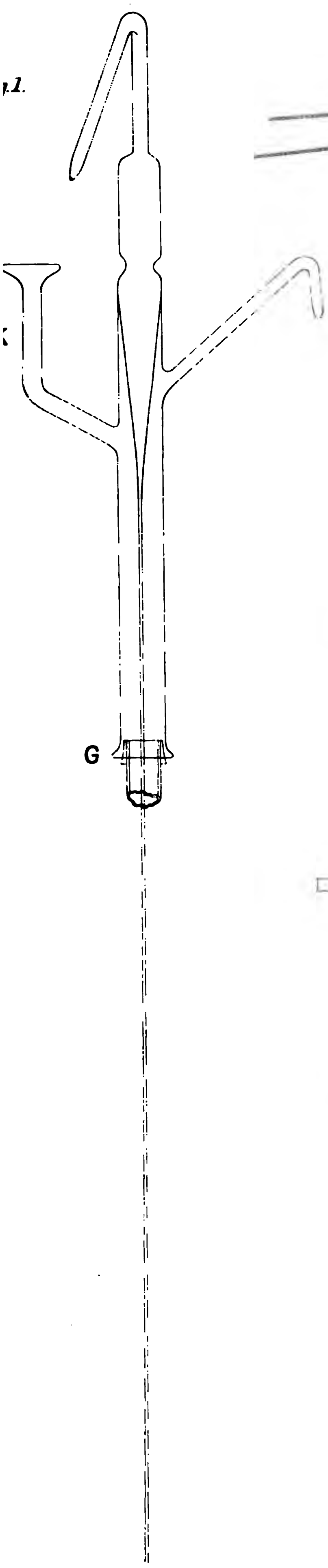
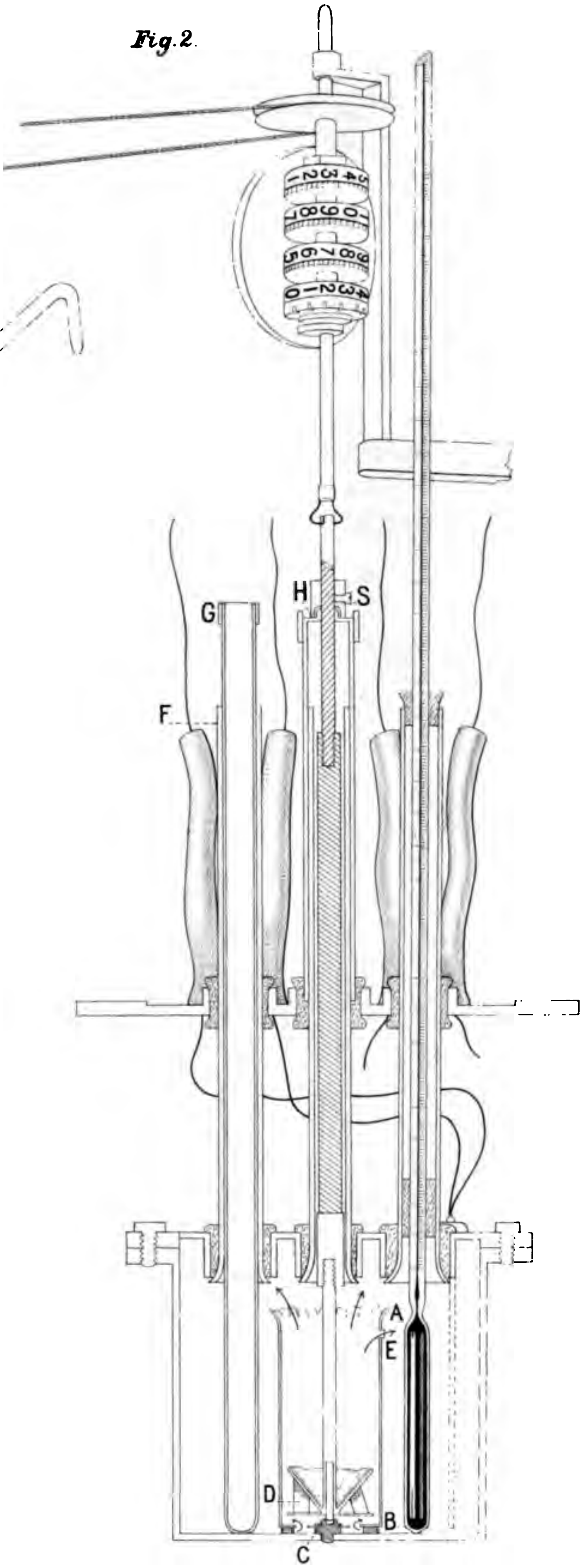


Fig. 2.



Half size

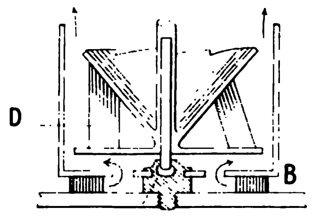
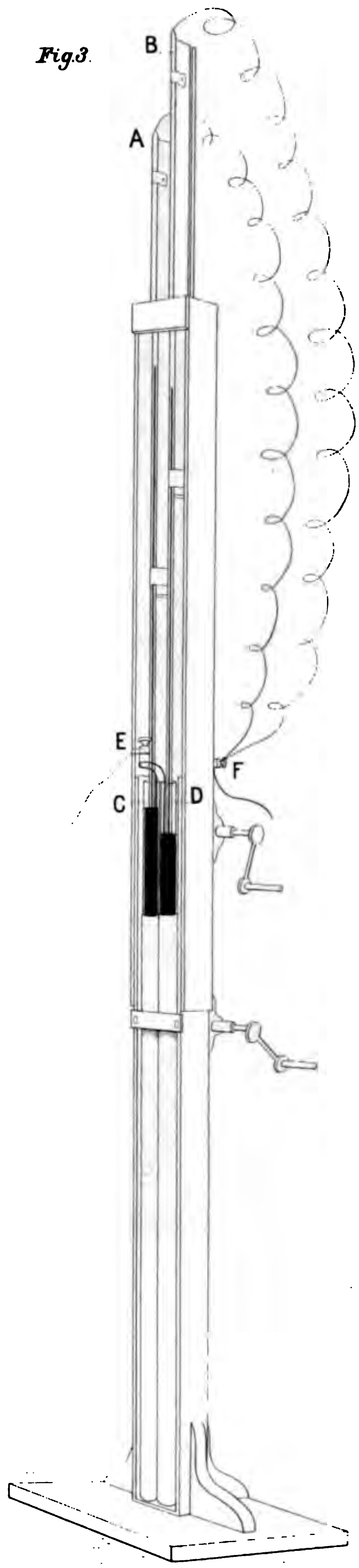
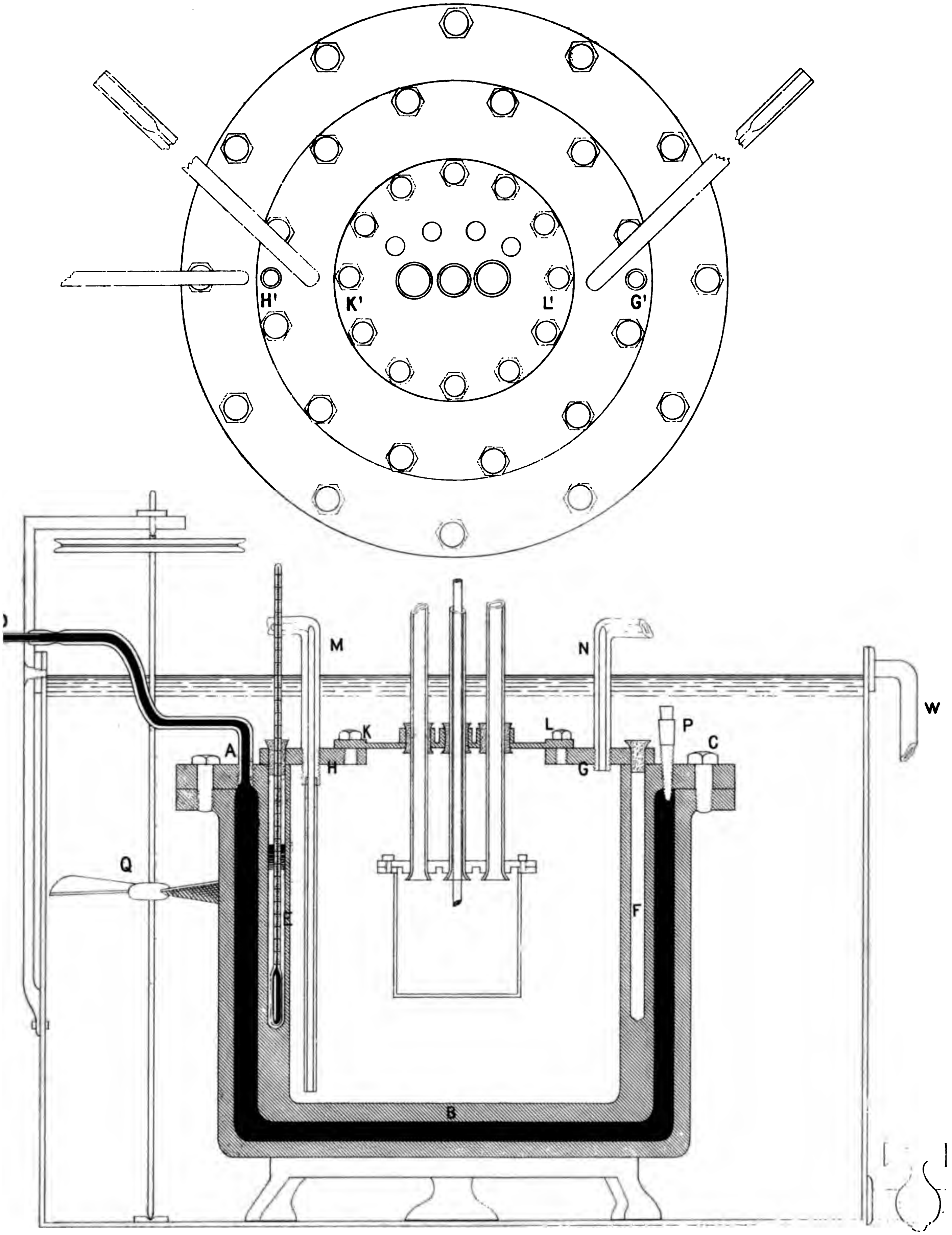


Fig. 3.





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Edwin Wilson Cambr

Fig. 1.

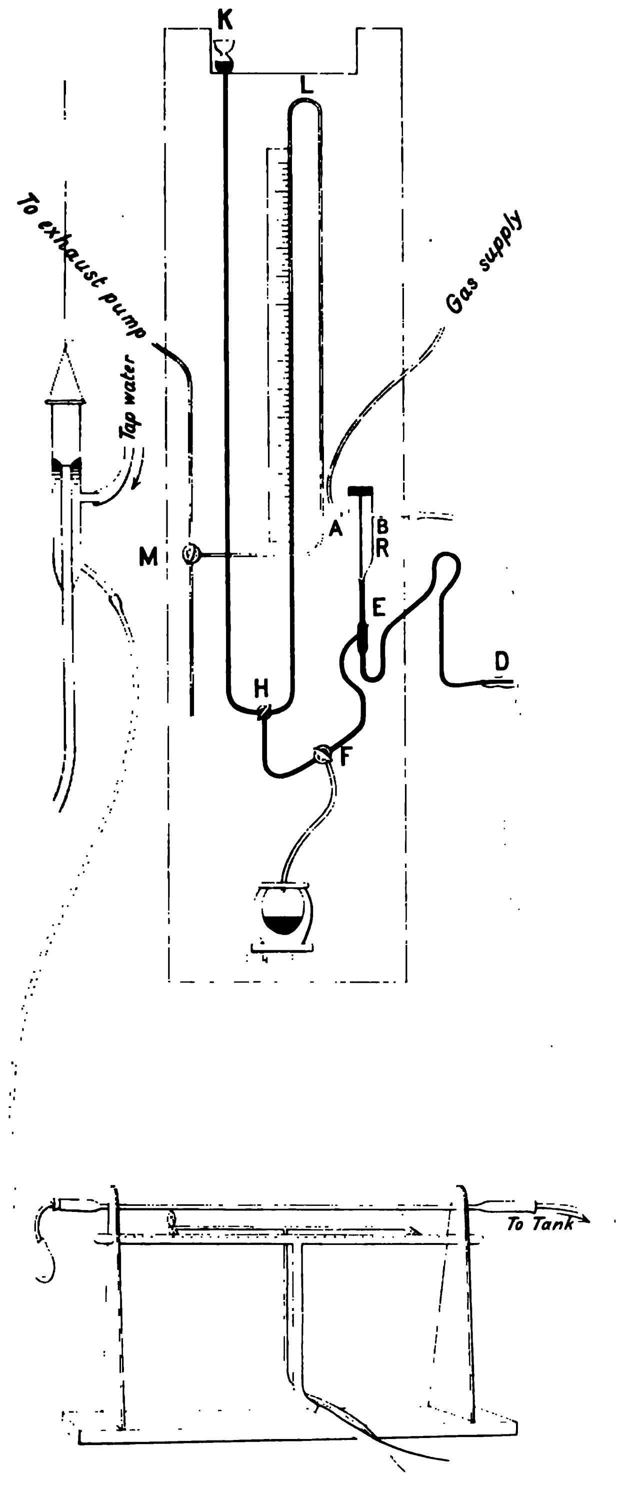
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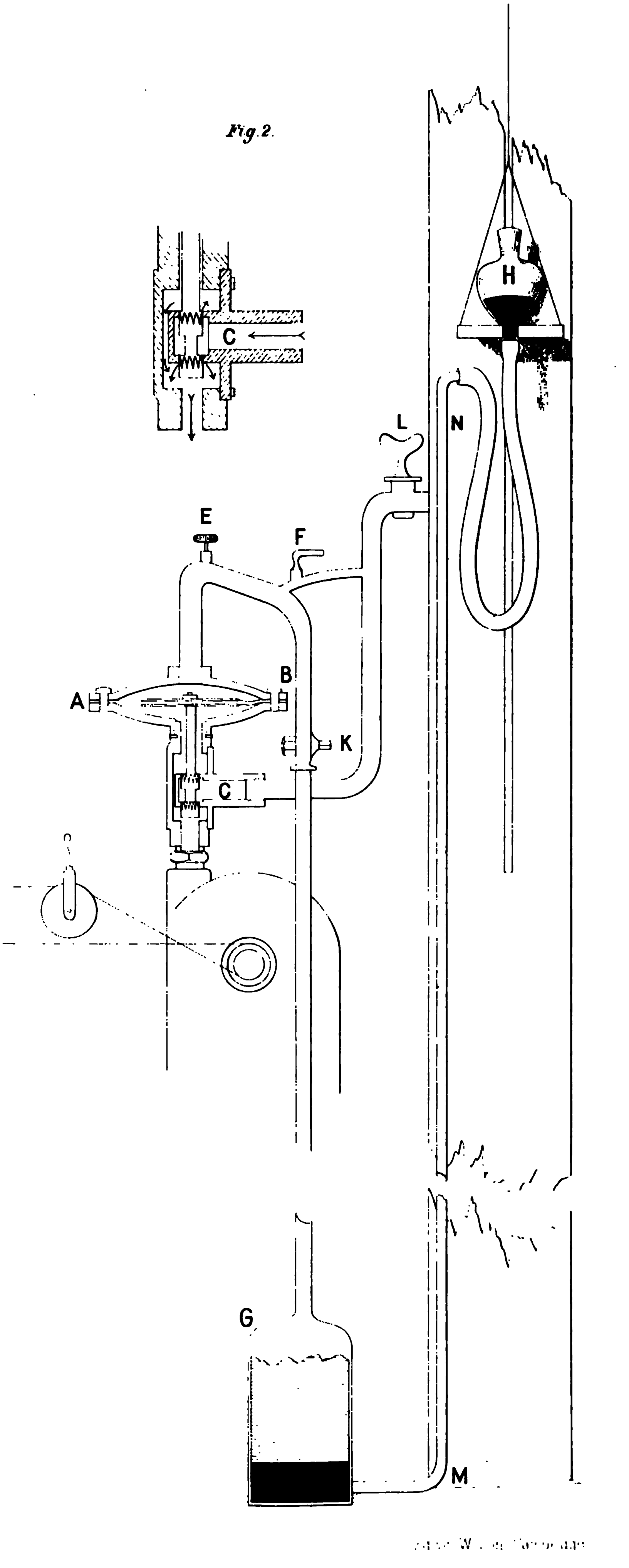
Fig. 2.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